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NEW CONGRUENCES FOR ℓ-REGULAR BIPARTITIONS FOR $\ell \in \{3, 5, 6, 10, 20\}$

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Abstract. Let $B_\ell(n)$ denote the number of ℓ-regular bipartitions of n. In this paper, we study the Arithmetic properties of $B_\ell(n)$ for $\ell \in \{3, 5, 6, 10, 20\}$ and we establish several infinite families of congruences satisfied by them.

1. Introduction and preliminaries

A partition of a positive integer n is any non-increasing sequence of positive integers whose sum is n. The number of such partitions is denoted by $p(n)$, and the number of partitions where the summands are distinct is denoted by $q(n)$. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n, where an $\ell-$regular partition of n is a partition of n such that none of its parts is divisible by ℓ. The generating function for the number of $\ell-$regular partitions of n is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1},$$

where f_k is defined by $f_k := \prod_{m=1}^{\infty}(1-q^{km})$, k, a positive integer.

In 1997 Gordon and Ono [8] obtained some divisibility properties of $b_\ell(n)$ by powers of certain special primes. This have strongly influenced many authors to study the arithmetic properties of $b_\ell(n)$, its divisibility and the distribution. In fact Andrews, Hirschhon and Sellers [4] derived some infinite families of congruences for $b_4(n)$ modulo 2. Hirschhorn and Sellers [9] obtained many Ramanujan-type congruences for $b_5(n)$ modulo 2. Webb [19] established an infinite family of congruences for $b_{13}(n)$ modulo 3. Xia and Yao [22] established several infinite families of congruences for $b_9(n)$ modulo 2. Cui and Gu [6] derived congruences for $b_\ell(n)$ modulo 2 for certain values of ℓ by employing the p-dissection formulas of Ramanujan's theta functions. In [20], Xia found congruences for $b_4(n)$ modulo 8. Keith [11] obtained the following conjecture which was proved by Xia and Yao in [23]:

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Theorem 1.1. *For $k = 0, 2, 3, 4, \alpha \geq 1$ and $n \geq 0$,*

$$b_9\left(5^{2\alpha}n + \frac{(3k+2)5^{2\alpha-1}-1}{3}\right) \equiv 0 \ (mod\ 3).$$

Dandurand and Penniston [7] gave the exact criteria for the divisibility of $b_\ell(n)$ for $\ell \in \{5, 7, 11\}$. Xia [21] showed that $b_\ell(A(k)n + B(k)) \equiv C(k)b_\ell(n) \pmod{\ell}$, where $A(k)$, $B(k)$ and $C(k)$ are functions in k and $\ell \in \{13, 17, 19\}$ and derived several strange congruences for $b_\ell(n)$ modulo ℓ. Wang [17][18] established several infinite families of congruences modulo powers of 5 for $b_5(n)$. Recently, in [2], Adiga and Ranganatha proved Ramanujan-type congruences modulo powers of 7 for $b_7(n)$ and $b_{49}(n)$.

An ℓ-regular bipartition of n is an ordered pair of ℓ-regular partitions (λ_1, λ_2) such that the sum of all the parts equals n. Denote the number of ℓ-regular bipartitions of n by $B_\ell(n)$. Then, the generating function of $B_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} B_\ell(n)q^n = \frac{f_\ell^2}{f_1^2}. \tag{1.1}$$

Using Ramanujan's two modular equations of degree 7, B. L. S. Lin [12] established an infinite family of congruences for $B_7(n)$ modulo 3: for $\alpha \geq 2$ and $n \geq 0$,

$$B_7\left(3^\alpha n + \frac{5 \times 3^{\alpha-1}-1}{2}\right) \equiv 0 \ (\ mod\ 3\ )$$

and in [13], he established infinite families of similar congruences for $B_{13}(n)$ modulo 3: for $\alpha \geq 2$ and $n \geq 0$,

$$B_{13}\left(3^\alpha n + 2 \times 3^{\alpha-1} - 1\right) \equiv 0 \ (\ mod\ 3\ ).$$

In [10], Kathiravan and Fathima proved several infinite families of congruences satisfied by $B_\ell(n)$ for $\ell \in \{5, 7, 13\}$. They showed that for all $\alpha > 0$,

$$B_5\left(4^\alpha n + \frac{5 \times 4^\alpha - 2}{6}\right) \equiv 0 \ (mod\ 5),$$

$$B_7\left(5^{8\alpha}n + \frac{5^{8\alpha}-1}{2}\right) \equiv 3^\alpha B_7(n) \ (mod\ 7)$$

and

$$B_{13}\left(5^{12\alpha}n + 5^{12\alpha} - 1\right) \equiv B_{13}(n) \ (mod\ 13).$$

Wang [18] established some Ramanujan-type congruences for $B_5(n)$ and found numerous congruences for $B_5(n)$ modulo higher powers of 5, such as,

$$B_5\left(5^{2\alpha-1}n + \frac{2 \times 5^{2\alpha-1}-1}{3}\right) \equiv 0 \ (mod\ 5^\alpha),$$

for $\alpha > 0$ and $n \geq 0$. In particular, he showed that $B_5(4n+3)$ and $B_5(5n+2)$ are congruent to 0 modulo 5. Ranganatha [15] proved Ramanujan-type congruences for $B_{25}(n)$ modulo powers of 5. In particular, he proved that for $j \geq 0$ and $n \geq 0$, $B_{25}(5^j n + 5^j - 2)$ is congruent to 0 modulo 5^j.

In [16], Somashekara and the author established several infinite families of congruences satisfied by $B_\ell(n)$ for $\ell \in \{2, 4, 7\}$. and also established a relation between $b_9(2n)$ and $B_3(n)$.

The main goal of this paper is to study the arithmetic properties of $B_\ell(n)$ for $\ell \in \{3, 5, 6, 10, 20\}$ using elementary approaches and p-dissection of $(q;q)_\infty, \psi(q), (q;q)_\infty^3, \psi(q^2)(q;q)_\infty^2$

In the next section, we list some standard identities of basic hypergeometric series, we define Ramanujans theta functions and present some preliminary lemmas that we use while we prove ours results. In Section 3, we prove our main results.

2. Some Standard Identities of Basic Hypergeometric Series and Preliminary Lemmas

In this Section, we list some of the standard identities that we use while proving our results.
Jacobi's triple product identity is given by

$$\sum_{m=-\infty}^{\infty} (-1)^m z^m q^{(3m+1)/2} = (zq^2, q/z, q^3; q^3)_\infty. \tag{2.1}$$

Euler's Pentagonal Number Theorem:

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{k(3k+1)/2} = (q;q)_\infty. \tag{2.2}$$

$$\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty} = 1 - 3a(q) + 9a(q^3), \tag{2.3}$$

where

$$a(q) = \sum_{n=1}^{\infty} \binom{n}{k} \frac{q^n}{1-q^n}.$$

Ramanujan's [5][14] general theta function $f(a,b)$ is given by

$$f(a,b) := 1 + \sum_{n=1}^{\infty} (ab)^{n(n-1)/2}(a^n + b^n)$$

$$= \sum_{-\infty}^{\infty} (ab)^{n(n+1)/2} (b)^{n(n-1)/2}, \ |ab| < 1.$$

The special cases of $f(a,b)$ are given by

$$\varphi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty}, \tag{2.4}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.5}$$

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = (q;q)_\infty, \tag{2.6}$$

$$\chi(q) := (-q;q^2)_\infty. \tag{2.7}$$

Cui and Gu[6] found the following p-dissections of $\psi(q)$ and $f(-q)$.

Lemma 2.1. *(Cui and Gu*[6, Theorem(2.1)]*)For any odd prime p,*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}), \tag{2.8}$$

where $\frac{k^2+k}{2}$ and $\frac{p^2-1}{8}$ are not in the same residue class modulo p for $0 \le k \le (p-3)/2$.

Lemma 2.2. *(Cui and Gu*[6, Theorem(2.2)]*) If $p \ge 5$ is a prime and,*

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, \ if p \equiv 1 \ (\ mod\ 6\ ) \\ \\ \frac{-p-1}{6}, \ if p \equiv -1 \ (\ mod\ 6\ ), \end{cases}$$

then

$$(q;q)_\infty = \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2-(6k+1)p}{2}}, -q^{\frac{3p^2+(6k+1)p}{2}}) + (-1)^{\pm\frac{p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2};q^{p^2})_\infty, \tag{2.9}$$

where $\pm$ depends on the conditions that $(\pm p-1)/6$ should be an integer. Moreover, note that $(3k^2+k)/2 \not\equiv (p^2-1)/24 \ (\ mod\ p\ )$ as k runs through the range of the summation.

Ahmed and Baruah [3] found the following p-dissections of $(q;q)_\infty^3$ and $\psi(q^2)(q;q)_\infty^2$.

Lemma 2.3. *(Ahmed and Baruah* [3, Lemma(2.3)]*) If $p \ge 3$ is a prime,then*

$$(q;q)_\infty^3 = \sum_{\substack{k=0 \\ k\neq\frac{p-1}{2}}}^{p-1} (-1)^k q^{k(k+1)/2} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn\cdot\frac{pn+2k+1}{2}} + p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} (q^{p^2};q^{p^2})_\infty^3. \tag{2.10}$$

Furthermore, if $k \neq \frac{p-1}{2}$, $0 \leq k \leq p-1$, *then* $\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8}$ (*mod* p).

Lemma 2.4. *(Ahmed and Baruah* [3, Lemma(2.4)]*) If* $p \geq 5$ *is a prime, and*

$$\frac{\pm p-1}{3} := \begin{cases} \frac{p-1}{3}, & if\, p \equiv 1 \ (\ mod\ 3\ ) \\ \\ \frac{-p-1}{3}, & if\, p \equiv -1 \ (\ mod\ 3\ ), \end{cases}$$

then

$$\psi(q^2)(q;q)_\infty^2 = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{3}}}^{\frac{p-1}{2}} q^{3k^2+2k} \sum_{n=0}^{\infty} (3pn+3k+1) q^{pn\cdot(3pn+6k+2)} \pm pq^{\frac{p^2-1}{3}} \psi(q^{2p^2})(q^{p^2};q^{p^2})_\infty^2. \quad (2.11)$$

Furthermore, if $k \neq \frac{\pm p-1}{3}$, $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$, *then* $3k^2+2k \not\equiv \frac{p^2-1}{3}$ (*mod* p).

3. Main Results

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty} B_3(n) q^{4(6n+1)} \equiv \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{k+m} (3k+1) q^{(6k+1)^2+3(6m+1)^2} \ (mod\ 9). \quad (3.1)$$

Proof. By (2.3), we have

$$(q^3;q^3)_\infty \equiv (q;q)_\infty^3 (1+3a(q)) \ (\ mod\ 9\ ). \quad (3.2)$$

Logarithmic differentiation of (2.1) with respect to z gives

$$\frac{\sum_{m=-\infty}^{\infty} (-1)^m m z^{m-1} q^{m(3m+1)/2}}{(zq^2, q/z, q^3; q^3)_\infty} = \sum_{n=o}^{\infty} \left(\frac{z^{-2} q^{3n+1}}{1-z^{-1}q^{3n+1}} - \frac{q^{3n+2}}{1-zq^{3n+2}} \right). \quad (3.3)$$

Setting $z=1$ in (3.3) and using Euler's pentagonal number theorem, we have

$$\begin{aligned} \frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2} &\equiv (q;q)_\infty (q^3;q^3)_\infty (1+3a(q)) \\ &\equiv \left(\sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k+1)/2} (1+3k) \right) \sum_{m=-\infty}^{\infty} (-1)^m q^{3m(3m+1)/2} \\ &\equiv \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{k+m} (3k+1) q^{k(3k+1)/2+3m(3m+1)/2} \ (\ mod\ 9\ ), \end{aligned}$$

which gives (3.1).

□

Theorem 3.2. *If $p \equiv 5 \pmod 6$, n_o is a positive integer less than p and $4(6n_o+1) \equiv 0 \pmod p$, then*

$$\sum_{n=0}^{\infty} B_3(pn+n_o)q^{4(6pn+6n_o+1)} \equiv \sum_{6k+1=px} \sum_{6m+1=py} (-1)^{k+m}(3k+1)q^{p^2(x^2+3y^2)} \pmod 9. \tag{3.4}$$

Proof. Since $p \equiv 5 \pmod 6$, $(6k+1)^2+3(6m+1)^2$ is divisible by p if and only if both $(6k+1)$ and $(6m+1)$ are congruent to 0 (mod p). Extracting those terms in (3.1) in which the powers of q are congruent to 0 (mod p) gives (3.4).

□

Remark: Note that if $4(6pn+6n_0+1)$ is not divisible by p^2, then we have $B_3(pn+n_0) \equiv 0 \pmod 9$).

Theorem 3.3. *We have*

$$\sum_{n=0}^{\infty} B_5(n)q^{8(3n+1)} \equiv \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{k+m}(2k+1)q^{3(2k+1)^2+5(6m+1)^2} \pmod 5. \tag{3.5}$$

Proof. By the Binomial theorem, we have

$$\sum_{n=0}^{\infty} B_5(n)q^n = \frac{(q^5;q^5)_\infty^2}{(q;q)_\infty^2} \equiv (q;q)_\infty^8 \text{ (} \pmod 5 \text{)}$$

$$\equiv \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2} \sum_{m=-\infty}^{\infty} (-1)^m q^{5m(3m+1)/2} \text{ (} \pmod 5 \text{)},$$

which is nothing but (3.5).

□

Theorem 3.4. *If $p \equiv 7, 11, 13, 29 \pmod{30}$, n_o is a positive integer less than p and $8(3n_o+1) \equiv 0 \pmod p$, then*

$$\sum_{n=0}^{\infty} B_5(pn+n_o)q^{8(3pn+3n_o+1)} \equiv \sum_{2k+1=px} \sum_{6m+1=py} (-1)^{k+m}(2k+1)q^{p^2(3x^2+5y^2)} \pmod 5. \tag{3.6}$$

Proof. Since $p \equiv 7, 11, 13, 29 \pmod{30}$, $3(2k+1)^2+5(6m+1)^2$ is divisible by p if and only if both $2k+1$ and $6m+1$ are congruent to each other (mod p). Extracting those terms on both the sides of (3.5) in which the powers of q are congruent to 0 (mod p), we get (3.6).

□

Remark: Note that if $8(3n+3n_0+1)$ is not divisible by p^2, then we have $B_5(pn+n_0) \equiv 0 \pmod 5$).

Corollary 3.5. *If* $24n+1 \equiv -1(\ mod\ 3\ )$, *then* $B_3(n) \equiv 0(\ mod\ 9\ )$.

Proof. If $24n+1 \equiv -1(\ mod\ 3\ )$, then there are no integers k, m such that $4(6n+1) = (6k+1)^2 + 3(6m+1)^2$. Therefore, by (3.1), we have that $B_3(n) \equiv 0(\ mod\ 9\ )$.

□

Corollary 3.6. *If* $8n+1 \equiv -1(\ mod\ 5\ )$, *then* $B_5(n) \equiv 0(\ mod\ 5\ )$.

Proof. If $8n+1 \equiv -1(\ mod\ 5\ )$, then there are no integers k, m such that $4(6n+2) = 3(2k+1)^2 + 5(6m+1)^2$. Therefore, by (3.5), we have that $B_5(n) \equiv 0(\ mod\ 5\ )$.

□

Theorem 3.7. *If* p *is a prime such that* $\left(\frac{-6}{p}\right) = -1$, *then for all* $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} B_6\left(p^{2\alpha}\cdot n + 10\cdot\frac{p^{2\alpha}-1}{24}\right)q^n = (-1)^{\alpha\cdot\frac{\pm p-1}{6}}\psi^2(q)(q^2;q^2)_\infty^2\ (\ mod\ 3\ ). \tag{3.7}$$

Proof. We prove the result by induction on α. By definition of $B_\ell(n)$, we have

$$\sum_{n=0}^{\infty} B_6(n)q^n = \frac{(q^6;q^6)_\infty^2}{(q;q)_\infty^2}.$$

Since we have $(q;q)_\infty^3 \equiv (q^3;q^3)_\infty\ (\ mod\ 3\ )$, we see that

$$\sum_{n=0}^{\infty} B_6(n)q^n \equiv \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^2} \equiv \psi^2(q)(q^2;q^2)_\infty^2\ (\ mod\ 3\ ),$$

which is the $\alpha = 0$ case of (3.7). Suppose (3.7) holds for some $\alpha \geq 0$. Then we have

$$\sum_{n=0}^{\infty} B_6\left(p^{2\alpha}\cdot n + 10\cdot\frac{p^{2\alpha}-1}{24}\right)q^n = (-1)^{\alpha\cdot\frac{\pm p-1}{6}}\psi^2(q)(q^2;q^2)_\infty^2\ (\ mod\ 3\ ). \tag{3.8}$$

Now we prove the case for $\alpha+1$. Squaring both the sides of (2.8), changing q to q^2 in (2.9) and then squaring, substituting the resulting identities to the right side of (3.8) extracting those terms in which the power of q is congruent to $10\cdot\frac{p^2-1}{24}$

modulo p in the resulting identity and then changing q^p to q, we obtain

$$\sum_{n=0}^{\infty} B_6\left(p^{2\alpha}\left(pn+10\frac{p^2-1}{24}\right)+10\cdot\frac{p^{2\alpha}-1}{24}\right)q^n$$
$$=(-1)^{(\alpha+1)\cdot\frac{\pm p-1}{6}}\psi^2(q^p)(q^{2p};q^{2p})_\infty^2 \ (\ mod\ 3\ ).$$

Changing n to pn and then changing q^p to q in the above identity, we obtain

$$\sum_{n=0}^{\infty} B_6\left(p^{2(\alpha+1)}n+10\cdot\frac{p^{2(\alpha+1)}-1}{24}\right)q^n$$
$$=(-1)^{(\alpha+1)\cdot\frac{\pm p-1}{6}}\psi^2(q)(q^2;q^2)_\infty^2 \ (\ mod\ 3\ ).$$

Therefore, the result is true for $\alpha+1$ and hence for all $\alpha\geq 0$. □

Theorem 3.8. *If p is a prime such that $p\equiv 5$ or 7 (mod 8) and $\alpha\geq 0$, then*

$$\sum_{n=0}^{\infty} B_{10}\left(p^{2\alpha}n+6\cdot\frac{p^{2\alpha}-1}{8}\right)q^n=p^\alpha(-1)^{\alpha\cdot\frac{p-1}{2}}\psi^2(q)\cdot(q^2;q^2)_\infty^6 \ (\ mod\ 5\ ).$$
(3.9)

Proof. We prove the result by induction on α. By definition of $B_\ell(n)$, we have

$$\sum_{n=0}^{\infty} B_{10}(n)q^n=\frac{(q^{10};q^{10})_\infty^2}{(q;q)_\infty^2}.$$

Since we have $(q;q)_\infty^p\equiv(q^p;q^p)_\infty \ (\ mod\ p\ )$ for any prime p, we see that

$$\sum_{n=0}^{\infty} B_{10}(n)q^n\equiv\frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^2} \ (\ mod\ 5\ )\equiv\psi^2(q)(q^2;q^2)_\infty^6 \ (\ mod\ 5\ ),$$

which is the $\alpha=0$ case of (3.9). Suppose (3.9) holds for some $\alpha\geq 0$. Then we have

$$\sum_{n=0}^{\infty} B_{10}\left(p^{2\alpha}n+6\cdot\frac{p^{2\alpha}-1}{8}\right)q^n=p^\alpha(-1)^{\alpha\cdot\frac{p-1}{2}}\psi^2(q)\cdot(q^2;q^2)_\infty^6 \ (\ mod\ 5\ ).$$
(3.10)

Now we prove the case for $\alpha+1$. Squaring both the sides of (2.8), changing q to q^2 in (2.10) and then squaring, substituting the resulting identities to the right side of (3.10) extracting those terms in which the power of q is congruent to

$6 \cdot \dfrac{p^2-1}{8}$ modulo p in the resulting identity and then changing q^p to q, we obtain

$$\sum_{n=0}^{\infty} B_{10}\left(p^{2\alpha}\left(pn + 6\cdot\frac{p^2-1}{8}\right) + 6\cdot\frac{p^{2\alpha}-1}{8}\right) q^n$$
$$= p^{\alpha+1}(-1)^{(\alpha+1)\cdot\frac{p-1}{2}}\psi^2(q^p)\cdot(q^{2p};q^{2p})_\infty^6 \ (\ mod\ 5\ ).$$

Changing n to pn and then changing q^p to q in the above identity, we obtain

$$\sum_{n=0}^{\infty} B_{10}\left(p^{2\alpha}\left(p^2n + 6\cdot\frac{p^2-1}{8}\right) + 6\cdot\frac{p^{2\alpha}-1}{8}\right) q^n$$
$$= \sum_{n=0}^{\infty} B_{10}\left(p^{2(\alpha+1)}n + 6\cdot\frac{p^{2(\alpha+1)}-1}{8}\right) q^n$$
$$= p^{\alpha+1}(-1)^{(\alpha+1)\cdot\frac{p-1}{2}}\psi^2(q)\cdot(q^2;q^2)_\infty^6 \ (\ mod\ 5\ ).$$

Therefore, the result is true for $\alpha+1$ and hence for all $\alpha \geq 0$. □

Theorem 3.9. *If p is a prime such that $p \equiv -1\ (\ mod\ 6\ )$, then for all $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} B_{20}\left(p^{2\alpha}\cdot n + 38\cdot\frac{p^{2\alpha}-1}{24}\right) q^n = (-p)^\alpha\psi^2(q)\psi^2(q^4)(-q^2;-q^2)_\infty^4 \ (\ mod\ 5\ ).$$

(3.11)

Proof. We prove the result by induction on α. By definition of $B_\ell(n)$, we have

$$\sum_{n=0}^{\infty} B_{20}(n)q^n = \frac{(q^{20};q^{20})_\infty^2}{(q;q)_\infty^2}.$$

Since we have $(q;q)_\infty^5 \equiv (q^5;q^5)_\infty\ (\ mod\ 5\ )$, we see that

$$\sum_{n=0}^{\infty} B_{20}(n)q^n \equiv \frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^2} \equiv \psi^2(q)\psi^2(q^4)(-q^2;-q^2)_\infty^4 \ (\ mod\ 5\ ),$$

which is the $\alpha = 0$ case of (3.11). Suppose (3.11) holds for some $\alpha \geq 0$. Then we have

$$\sum_{n=0}^{\infty} B_{20}\left(p^{2\alpha}\cdot n + 38\cdot\frac{p^{2\alpha}-1}{24}\right) q^n = (-p)^\alpha\psi^2(q)\psi^2(q^4)(-q^2;-q^2)_\infty^4 \ (\ mod\ 5\ ).$$

(3.12)

Now we prove the case for $\alpha+1$. Squaring both the sides of (2.8), changing q to $-q^2$ in (2.11) and then squaring, substituting the resulting identities to the

right side of (3.12) extracting those terms in which the power of q is congruent to $38 \cdot \dfrac{p^2-1}{24}$ modulo p in the resulting identity and then changing q^p to q, we obtain

$$\sum_{n=0}^{\infty} B_{20}\left(p^{2\alpha}\left(pn + 38 \cdot \frac{p^2-1}{24}\right) + 38 \cdot \frac{p^{2\alpha}-1}{24}\right) q^n$$
$$= (-p)^{\alpha+1}\psi^2(q^p)\psi^2(q^{4p})(-q^{2p};-q^{2p})_\infty^4 \ (\ mod\ 5\ ).$$

Changing n to pn and then changing q^p to q in the above identity, we obtain

$$\sum_{n=0}^{\infty} B_{20}\left(p^{2(\alpha+1)}n + 38 \cdot \frac{p^{2(\alpha+1)}-1}{24}\right) q^n$$
$$= (-p)^{\alpha+1}\psi^2(q)\psi^2(q^4)(-q^2;-q^2)_\infty^4 \ (\ mod\ 5\ ).$$

Therefore, the result is true for $\alpha+1$ and hence for all $\alpha \geq 0$. □

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