

SERIES OF RECIPROCAL POWERS OF FIGURATE NUMBERS

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ABSTRACT. We derive closed-form expressions for infinite series of reciprocal powers of figurate numbers, including polygonal numbers, centered polygonal numbers, pyramidal numbers, and centered pyramidal numbers. Our approach relies on partial fraction decompositions and analytic properties of the Hurwitz zeta and polygamma functions. In the case of polygonal numbers, we generalize and extend results previously conjectured in the literature. For centered polygonal and pyramidal numbers, we develop systematic methods for evaluating the corresponding reciprocal power sums. Notably, we present what appear to be the first general closed-form formulas for the sums of reciprocal powers of centered pyramidal numbers. Extensive tables of evaluated sums are provided to illustrate the results.

Keywords. Sum of reciprocal powers, figurate numbers, polygonal numbers, pyramidal numbers

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1. INTRODUCTION

In this work, we provide a closed form expression for the series of reciprocal powers of polygonal numbers, thus answering conjectures posited in [5, 3]. The analysis is carried out using partial fraction decomposition and basic properties of the Hurwitz zeta and polygamma functions. A similar analysis is presented for the pyramidal numbers, the centered polygonal numbers, and the centered pyramidal numbers. To the best of our knowledge, closed-form expressions for the sums of reciprocal powers of centered pyramidal numbers have not previously appeared in the literature.

The series we consider, which are of the form

$$\sum_{n=0}^{\infty} ((n+t_1)(n+t_2))^{-s} \quad \text{or} \quad \sum_{n=0}^{\infty} ((n+t_1)(n+t_2)(n+t_3))^{-s},$$

may be viewed as generalizations of the Hurwitz zeta function

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s},$$

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which is itself a generalization of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

The Riemann zeta function, because of its connection to the distribution of primes, carries with it potentially important implications to cybersecurity, see the discussion in [1]. Perhaps the most direct potential application of the work presented here to computer science is in the reduction of computational complexity by providing exact formulas for the computation of reciprocal series with one of the forms given above.

Our starting point is the following partial fraction decomposition formula from Boros and Moll [2, p. 46]:

$$\frac{1}{(n+t_1)^{m+1}(n+t_2)^{m+1}} = \sum_{j=0}^m \binom{m+j}{m} \left(\frac{s^{m+1+j}}{(n+t_2)^{m+1-j}} + \frac{(-s)^{m+1+j}}{(n+t_1)^{m+1-j}} \right), \quad (1.1)$$

where $s = 1/(t_2 - t_1)$. From this we obtain the following theorem:

Theorem 1.1. *For real numbers $t_1, t_2 \neq 0, -1, -2, \dots$ and all positive integers m ,*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(n+t_1)^{m+1}(n+t_2)^{m+1}} \\ = \sum_{j=0}^{m-1} \binom{m+j}{m} \left(s^{m+1+j} \zeta(m+1-j, t_2) + (-s)^{m+1+j} \zeta(m+1-j, t_1) \right) \\ + \binom{2m}{m} s^{2m+1} (\psi(t_2) - \psi(t_1)) \end{aligned}$$

Proof. Applying the partial fraction decomposition (1.1), we find that

$$\begin{aligned} \sum_{n=0}^N \frac{1}{(n+t_1)^{m+1}(n+t_2)^{m+1}} \\ = \sum_{j=0}^{m-1} \left[\binom{m+j}{m} \sum_{n=0}^N \left(\frac{s^{m+1+j}}{(n+t_2)^{m+1-j}} + \frac{(-s)^{m+1+j}}{(n+t_1)^{m+1-j}} \right) \right] \\ + \sum_{n=0}^N \binom{2m}{m} \frac{s^{2m+1}}{(n+t_1)(n+t_2)} \\ = \sum_{j=0}^{m-1} \left[\binom{m+j}{m} \left(s^{m+1+j} \sum_{n=0}^N \frac{1}{(n+t_2)^{m+1-j}} \right. \right. \\ \left. \left. + (-s)^{m+1+j} \sum_{n=0}^N \frac{1}{(n+t_1)^{m+1-j}} \right) \right] \\ + \binom{2m}{m} s^{2m+1} \sum_{n=0}^N \frac{t_2 - t_1}{(n+t_1)(n+t_2)}. \end{aligned}$$

We claim that letting $N \rightarrow \infty$ completes the proof. That the first sum converges to the desired expression follows from the definition of the Hurwitz zeta function. To see the convergence of the last sum, we use the well-known property of the digamma function [4, (5.7.6)]

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right) = \gamma + \psi(z).$$

In particular, note that

$$\sum_{n=0}^N \frac{t_2 - t_1}{(n+t_1)(n+t_2)} = \sum_{n=0}^N \left(\frac{1}{n+1} - \frac{1}{n+t_2} \right) - \sum_{n=0}^N \left(\frac{1}{n+1} - \frac{1}{n+t_1} \right).$$

Now, as $N \rightarrow \infty$, we may apply (5.7.6) of [4] to each sum on the right side of the previous equation to find that

$$\sum_{n=0}^{\infty} \frac{t_2 - t_1}{(n+t_1)(n+t_2)} = \psi(t_2) - \psi(t_1).$$

The proof is thus complete. \square

We will also need the following partial fraction decomposition result:

Lemma 1.2. *For all positive integers m ,*

$$\frac{1}{((n-t_1)(n-t_2)(n-t_3))^m} = \sum_{j=1}^m \left(\frac{A_j}{(n-t_1)^j} + \frac{B_j}{(n-t_2)^j} + \frac{C_j}{(n-t_3)^j} \right),$$

where

$$A_j = c_{m-j}(t_1, t_2, t_3, m), \quad B_j = c_{m-j}(t_2, t_1, t_3, m), \quad C_j = c_{m-j}(t_3, t_1, t_2, m),$$

and

$$\begin{aligned} c_k(x, y, z, m) \\ = (-1)^k \sum_{l=0}^k \binom{m+l-1}{l} \binom{m+k-l-1}{k-l} (x-y)^{-m-l} (x-z)^{-m-k+l}. \end{aligned}$$

Proof. To find A_j , multiply both sides of the partial fraction decomposition equation by $(n-t_1)^m$, yielding

$$\frac{1}{((n-t_2)(n-t_3))^m} = \sum_{l=1}^m \left(A_l (n-t_1)^{m-l} + \frac{B_l (n-t_1)^m}{(n-t_2)^l} + \frac{C_l (n-t_1)^m}{(n-t_3)^l} \right).$$

Now differentiate both sides $m - j$ times ($j \leq m$): The $(m - j)$ -th derivative of the left side is found to be

$$\begin{aligned}
 & \frac{d^{m-j}}{dn^{m-j}} \left[\frac{1}{((n-t_2)(n-t_3))} \right] \\
 &= \sum_{l=0}^{m-j} \binom{m-j}{l} \frac{d^l}{dn^l} \left[\frac{1}{(n-t_2)^m} \right] \frac{d^{m-j-l}}{dn^{m-j-l}} \left[\frac{1}{(n-t_3)^m} \right] \\
 &= \sum_{l=0}^{m-j} \binom{m-j}{l} \frac{(-1)^l m^{(l)}}{(n-t_2)^{m+l}} \cdot \frac{(-1)^{m-j-l} m^{(m-j-l)}}{(n-t_3)^{2m-j-l}} \\
 &= (-1)^{m-j} (m-j)! \sum_{l=0}^{m-j} \binom{m+l-1}{l} \binom{2m-j-l-1}{m-j-l} (n-t_2)^{-m-l} (n-t_3)^{-2m+j+l},
 \end{aligned}$$

where the general Leibniz rule was used in the first equality and $m^{(l)}$ denotes the rising factorial. The $(m - j)$ -th derivative of the right side is

$$(m-j)!A_j + h(n),$$

where h is a rational function with a zero of order j at t_1 . Now let $n \rightarrow t_1$ to find that

$$\begin{aligned}
 (m-j)!(-1)^{m-j} \sum_{l=0}^{m-j} \binom{m+l-1}{l} \binom{2m-j-l-1}{m-j-l} (t_1-t_2)^{-m-l} (t_1-t_3)^{-2m+j+l} \\
 = (m-j)!A_j.
 \end{aligned}$$

Dividing both sides of the above equation by $(m-j)!$ gives the desired formula for A_j . The formulas for B_j and C_j follow in a similar fashion. \square

We will also need the following fact:

Lemma 1.3. *If A_1 , B_1 , and C_1 are as defined in Lemma 1.2, then*

$$A_1 + B_1 + C_1 = 0.$$

Proof. This follows from combining the partial fraction decomposition into a single fraction. The numerator is a polynomial with leading coefficient $A_1 + B_1 + C_1$. This polynomial must equal 1, proving the claim. \square

We look at an example to illustrate our approach:

Example 1.4. Consider the infinite sum

$$\sum_{n=1}^{\infty} \frac{36}{n^2(n+1)^2(n+2)^2}.$$

If we can find the partial fraction decomposition

$$\frac{1}{n^2(n+1)^2(n+2)^2} = \frac{A_1}{n} + \frac{A_2}{n^2} + \frac{B_1}{n+1} + \frac{B_2}{(n+1)^2} + \frac{C_1}{n+2} + \frac{C_2}{(n+2)^2},$$

then we can evaluate the series as follows:

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{36}{n^2(n+1)^2(n+2)^2} \\
&= 36 \sum_{n=1}^{\infty} \left(\frac{A_1}{n} + \frac{A_2}{n^2} + \frac{B_1}{n+1} + \frac{B_2}{(n+1)^2} + \frac{C_1}{n+2} + \frac{C_2}{(n+2)^2} \right) \\
&= 36 \left(A_2 \zeta(2) + B_2 \zeta(2, 2) + C_2 \zeta(2, 3) + \sum_{n=1}^{\infty} \frac{(3A_1 + 2B_1 + C_1)n + 2A_1}{n(n+1)(n+2)} \right) \\
&= 36 \left(A_2 \zeta(2) + B_2 \zeta(2, 2) + C_2 \zeta(2, 3) + \sum_{n=1}^{\infty} \left(\frac{2A_1 + B_1}{(n+1)(n+2)} + \frac{2A_1}{n(n+1)(n+2)} \right) \right) \\
&= 36 \left(A_2 \zeta(2) + B_2 \zeta(2, 2) + C_2 \zeta(2, 3) + \frac{3A_1}{2} + \frac{B_1}{2} \right).
\end{aligned}$$

To find A_2, B_2, C_2 , multiply by $n^2, (n+1)^2, (n+2)^2$, and let n approach $0, -1, -2$, respectively:

$$A_2 = \frac{1}{4}, \quad B_2 = 1, \quad C_2 = \frac{1}{4}.$$

To find A_1, B_1, C_1 , multiply by $n^2, (n+1)^2, (n+2)^2$, take the first derivative, and let n approach $0, -1, -2$, respectively:

- For A_1 , the left side becomes

$$\frac{1}{(n+1)^2(n+2)^2} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} c_k x^k$$

where

$$c_k = \frac{(-1)^k}{2^{k+2}} \sum_{j=0}^k (j+1)(k-j+1)2^j$$

The right side becomes

$$A_1 n + A_2 + n^2 \left(\frac{B_1}{n+1} + \frac{B_2}{(n+1)^2} + \frac{C_1}{n+2} + \frac{C_2}{(n+2)^2} \right).$$

In general, the l -th derivative of the left side evaluated at 0 is $l!c_l$. We want the first derivative evaluated at 0, which is

$$c_1 = -\frac{1}{8} \sum_{j=0}^1 (j+1)(2-j)2^j = -\frac{1}{8}(2+4) = -\frac{3}{4}$$

The derivative of the right side evaluated at 0 is A_1 . So $A_1 = -3/4$. For B_1 and C_1 , we follow the same approach, except we evaluate the derivatives at -1 and -2 , respectively:

- For B_1 ,

$$c_k = \sum_{j=0}^k (j+1)(k-j+1)(-1)^{k+j}.$$

So, we should find

$$B_1 = c_1 = \sum_{j=0}^1 (j+1)(2-j)(-1)^{1+j} = -2 + 2 = 0.$$

• For C_1 ,

$$c_k = \sum_{j=0}^k (j+1)(k-j+1)2^{-2-j}.$$

So, we should find

$$C_1 = c_1 = \sum_{j=0}^1 (j+1)(2-j)2^{-2-j} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

Putting this all together, we find the exact value of the series to be

$$\sum_{n=1}^{\infty} \frac{36}{n^2(n+1)^2(n+2)^2} = 9\pi^2 - \frac{351}{4}.$$

We conclude this section by recalling that $\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}B_{2k}}{2(2k)!}$ for $k \in \mathbb{Z}^+$, though we will continue to write $\zeta(2k)$ throughout.

2. SERIES OF RECIPROCAL POWERS OF POLYGONAL NUMBERS

The n -th polygonal number is given by

$$P(r, n) = \frac{(r-2)n^2 - (r-4)n}{2}, \quad n \geq 1, \quad r \geq 3,$$

so that

$$\frac{1}{P(r, n)} = \frac{b}{n(n-a)},$$

where $b = 2/(r-2)$ and $a = (r-4)/(r-2) = 1 - b$. Thus,

$$\frac{1}{P(r, n)^m} = \frac{b^m}{n^m(n-a)^m}.$$

Applying Theorem 1.1, we obtain a closed form expression for

$$\sum_{n=1}^{\infty} \frac{1}{P(r, n)^{m+1}},$$

thus answering conjectures on such a formula [3, 5]:

Theorem 2.1. *For every integer $m \geq 0$ and $r \neq 4$,*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{P(r, n)^{m+1}} &= (-b)^{m+1} \sum_{j=0}^{m-1} \left(\binom{m+j}{m} \frac{\psi^{(m-j)}(b) + (-1)^{m+1+j} \psi^{(m-j)}(1)}{a^{m+1+j} (m-j)!} \right) \\ &\quad + b^{m+1} \binom{2m}{m} (-1)^{m+1} a^{-2m-1} (\psi(b) + \gamma). \end{aligned}$$

In the case $r = 4$, we have

$$\sum_{n=1}^{\infty} \frac{1}{P(4, n)^{m+1}} = \zeta(2m+2).$$

Proof of Theorem 2.1. Assume $r \geq 3$ and $r \neq 4$. Begin by noting that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{P(r, n)^{m+1}} &= b^{m+1} \sum_{n=1}^{\infty} \frac{1}{n^{m+1}(n-a)^{m+1}} \\ &= b^{m+1} \sum_{n=0}^{\infty} \frac{1}{(n+1)^{m+1}(n+1-a)^{m+1}} \\ &= b^{m+1} \sum_{n=0}^{\infty} \frac{1}{(n+1)^{m+1}(n+b)^{m+1}}. \end{aligned}$$

Now apply Theorem 1.1 with $t_1 = 1$ and $t_2 = b$ to find that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{P(r, n)^{m+1}} &= (-b)^{m+1} \sum_{j=0}^{m-1} \left(\binom{m+j}{m} \left((-a)^{-m-1-j} \zeta(m+1-j, b) + a^{-m-1-j} \zeta(m+1-j, 1) \right) \right. \\ &\quad \left. + b^{m+1} \binom{2m}{m} (-1)^{m+1} a^{-2m-1} (\psi(b) + \gamma) \right). \end{aligned}$$

Converting the Hurwitz zeta functions to polygamma functions via Eq. 25.11.12 from [4], we find that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{P(r, n)^{m+1}} &= (-b)^{m+1} \sum_{j=0}^{m-1} \left(\binom{m+j}{m} \frac{\psi^{(m-j)}(b) + (-1)^{m+1+j} \psi^{(m-j)}(1)}{a^{m+1+j} (m-j)!} \right) \\ &\quad + b^{m+1} \binom{2m}{m} (-1)^{m+1} a^{-2m-1} (\psi(b) + \gamma). \end{aligned}$$

If $r = 4$ then $P(r, n) = n^2$, and so the claim follows. \square

3. SERIES OF RECIPROCAL POWERS OF CENTERED POLYGONAL NUMBERS

The n -th centered k -gonal number is given by

$$C(k, n) = \frac{kn(n+1) + 2}{2}, \quad n \geq 0, \quad k \geq 3.$$

Thus, we have

$$\frac{1}{C(k, n)} = \frac{2}{k} \frac{1}{(n+a)(n+b)},$$

where

$$a = \frac{k - \sqrt{k^2 - 8k}}{2k}, \quad b = \frac{k + \sqrt{k^2 - 8k}}{2k}.$$

Theorem 3.1. For every integer $m \geq 0$ and every positive integer $k \neq 8$,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{C(k, n)^{m+1}} \\ &= \left(\frac{2}{k}\right)^{m+1} \sum_{j=0}^{m-1} \binom{m+j}{m} \left(s^{m+1+j} \zeta(m+1-j, b) + (-s)^{m+1+j} \zeta(m+1-j, a) \right) \\ & \quad + \binom{2m}{m} s^{2m+1} (\psi(b) - \psi(a)) \end{aligned}$$

where

$$s = \frac{1}{b-a} = \sqrt{\frac{k}{k-8}}.$$

In the case $k = 8$,

$$\sum_{n=0}^{\infty} \frac{1}{C(k, n)^{m+1}} = \frac{2^{2m+2} - 1}{2^{2m+2}} \zeta(2m+2).$$

Proof. Assume $k \neq 8$. Apply Theorem 1.1 with $t_1 = a$ and $t_2 = b$ to find that

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{C(k, n)^{m+1}} \\ &= \left(\frac{2}{k}\right)^{m+1} \sum_{j=0}^{m-1} \binom{m+j}{m} \left(s^{m+1+j} \zeta(m+1-j, b) + (-s)^{m+1+j} \zeta(m+1-j, a) \right) \\ & \quad + \binom{2m}{m} s^{2m+1} (\psi(b) - \psi(a)), \end{aligned}$$

where

$$s = \frac{1}{b-a} = -\sqrt{\frac{k}{k-8}}.$$

For the case $k = 8$, observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{C(k, n)^{m+1}} &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2m+2}} \\ &= \zeta(2m+2) - \sum_{n=1}^{\infty} \frac{1}{(2n)^{2m+2}} \\ &= \zeta(2m+2) - \frac{1}{2^{2m+2}} \zeta(2m+2) = \left(\frac{2^{2m+2} - 1}{2^{2m+2}} \right) \zeta(2m+2). \quad \square \end{aligned}$$

4. SERIES OF RECIPROCAL POWERS OF PYRAMIDAL NUMBERS

The n -th pyramidal number with r -sided base is given by

$$P(r, n) = \frac{n(n+1)((r-2)n - r + 5)}{6}.$$

The reciprocal can be written as

$$\frac{1}{P(r, n)} = \frac{2b}{n(n+1)(n-a)},$$

where $a = (r-5)/(r-2)$ and $b = 3/(r-2)$, so that $a = 1 - b$.

Theorem 4.1. *For all $m \geq 1$,*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{P(r, n)^m} &= (2b)^m \left(\sum_{j=2}^m \left(A_j \zeta(j) + B_j \zeta(j, 2) + C_j \zeta(j, b) \right) \right. \\ &\quad \left. - ((1-a)A_1 - aB_1 + C_1) \frac{\psi^{(0)}(1-a) + \gamma - 1}{a+1} \right. \\ &\quad \left. + aA_1 \frac{\psi^{(0)}(1-a) + \gamma + a}{a^2 + a} \right) \end{aligned}$$

Proof. By Lemma 1.2, the coefficients of the partial fraction decomposition satisfy

$$A_i = c_{m-i}(0, -1, a, m), \quad B_i = c_{m-i}(-1, 0, a, m), \quad C_i = c_{m-i}(a, -1, 0, m),$$

where

$$\begin{aligned} c_k(t_1, t_2, t_3, m) &= (-1)^k \sum_{j=0}^k \binom{m+j-1}{j} \binom{m+k-j-1}{k-j} (t_1 - t_2)^{-m-j} (t_1 - t_3)^{-m-k+j}. \end{aligned}$$

Then

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{P(r, n)^m} &= (2b)^m \sum_{n=1}^{\infty} \sum_{j=1}^m \left(\frac{A_j}{n^j} + \frac{B_j}{(n+1)^j} + \frac{C_j}{(n-a)^j} \right) \\ &= (2b)^m \sum_{j=2}^m \left(\sum_{n=1}^{\infty} \frac{A_j}{n^j} + \sum_{n=1}^{\infty} \frac{B_j}{(n+1)^j} + \sum_{n=1}^{\infty} \frac{C_j}{(n-a)^j} \right) \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{A_1}{n} + \frac{B_1}{(n+1)} + \frac{C_1}{(n-a)} \right) \\ &= (2b)^m \left(\sum_{j=2}^m (A_j \zeta(j) + B_j \zeta(j, 2) + C_j \zeta(j, b)) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left(\frac{A_1}{n} + \frac{B_1}{(n+1)} + \frac{C_1}{(n-a)} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= (2b)^m \left(\sum_{j=2}^m (A_j \zeta(j) + B_j \zeta(j, 2) + C_j \zeta(j, b)) \right. \\
 &\quad \left. + \sum_{n=1}^{\infty} \frac{(A_1 + B_1 + C_1)n}{(n+1)(n-a)} + \sum_{n=1}^{\infty} \frac{(1-a)A_1 - aB_1 + C_1}{(n+1)(n-a)} \right. \\
 &\quad \left. - \sum_{n=1}^{\infty} \frac{aA_1}{n(n+1)(n-a)} \right) \\
 &= (2b)^m \left(\sum_{j=2}^m (A_j \zeta(j) + B_j \zeta(j, 2) + C_j \zeta(j, b)) \right. \\
 &\quad \left. - ((1-a)A_1 - aB_1 + C_1) \frac{\psi^{(0)}(1-a) + \gamma - 1}{a+1} \right. \\
 &\quad \left. + aA_1 \frac{a + \psi^{(0)}(1-a) + \gamma + a}{a^2 + a} \right). \quad \square
 \end{aligned}$$

5. SERIES OF RECIPROCAL POWERS OF CENTERED PYRAMIDAL NUMBERS

The n -th centered pyramidal number with r -sided base is given by

$$C(r, n) = \frac{n((r-1)n^2 - r + 7)}{6} = \frac{n(n^2 - b)}{a} = \frac{n(n - \sqrt{b})(n + \sqrt{b})}{a},$$

where $a = 6/(r-1)$ and $b = (r-7)/(r-1) = 1-a$.

Theorem 5.1. *For all $m \geq 1$, if $r \neq 7$ then*

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{1}{C(r, n)^m} &= a^m \left(\sum_{j=2}^m (A_j \zeta(j) + B_j \zeta(j, 1 - \sqrt{b}) + C_j \zeta(j, 1 + \sqrt{b})) \right. \\
 &\quad \left. + (B_1 - C_1) \frac{1 - \pi\sqrt{b} \cot(\pi\sqrt{b})}{2\sqrt{b}} \right. \\
 &\quad \left. + A_1 \frac{\psi^{(0)}(1 - \sqrt{b}) + \psi^{(0)}(1 + \sqrt{b}) + 2\gamma}{2} \right).
 \end{aligned}$$

In the case $r = 7$, we have

$$\sum_{n=1}^{\infty} \frac{1}{C(7, n)^m} = \zeta(3m).$$

Proof. Assume $r \neq 7$. Apply the Lemmas 1.2 and 1.3 to get

$$\begin{aligned}
 &\sum_{n=1}^{\infty} \frac{1}{C(r, n)^m} \\
 &= a^m \sum_{n=1}^{\infty} \sum_{j=1}^m \left(\frac{A_j}{n^j} + \frac{B_j}{(n - \sqrt{b})^j} + \frac{C_j}{(n + \sqrt{b})^j} \right)
 \end{aligned}$$

$$\begin{aligned}
&= a^m \left(\sum_{j=2}^m \left(\sum_{n=1}^{\infty} \frac{A_j}{n^j} + \sum_{n=1}^{\infty} \frac{B_j}{(n - \sqrt{b})^j} + \sum_{n=1}^{\infty} \frac{C_j}{(n + \sqrt{b})^j} \right) \right. \\
&\quad \left. + \sum_{n=1}^{\infty} \left(\frac{A_1}{n} + \frac{B_1}{n - \sqrt{b}} + \frac{C_1}{n + \sqrt{b}} \right) \right) \\
&= a^m \left(\sum_{j=2}^m (A_j \zeta(j) + B_j \zeta(j, 1 - \sqrt{b}) + C_j \zeta(j, 1 + \sqrt{b})) \right. \\
&\quad \left. + \sum_{n=1}^{\infty} \left(\frac{(A_1 + B_1 + C_1)n}{(n^2 - b)} + \frac{B_1 \sqrt{b} - C_1 \sqrt{b}}{(n^2 - b)} - \frac{A_1 b}{n(n^2 - b)} \right) \right) \\
&= a^m \left(\sum_{j=2}^m (A_j \zeta(j) + B_j \zeta(j, 1 - \sqrt{b}) + C_j \zeta(j, 1 + \sqrt{b})) \right. \\
&\quad \left. + \sum_{n=1}^{\infty} \left(\frac{(B_1 - C_1) \sqrt{b}}{(n^2 - b)} - \frac{A_1 b}{n(n^2 - b)} \right) \right) \\
&= a^m \left(\sum_{j=2}^m (A_j \zeta(j) + B_j \zeta(j, 1 - \sqrt{b}) + C_j \zeta(j, 1 + \sqrt{b})) \right. \\
&\quad \left. + (B_1 - C_1) \frac{1 - \pi \sqrt{b} \cot(\pi \sqrt{b})}{2\sqrt{b}} + A_1 \frac{\psi^{(0)}(1 - \sqrt{b}) + \psi^{(0)}(1 + \sqrt{b}) + 2\gamma}{2} \right).
\end{aligned}$$

In the case $r = 7$, we see that $C(7, n) = n^3$, so that

$$\sum_{n=1}^{\infty} \frac{1}{C(7, n)^m} = \sum_{n=1}^{\infty} \frac{1}{n^{3m}} = \zeta(3m). \quad \square$$

6. TABLE OF SUMS

We present in this section the first several cases ($3 \leq r, k \leq 10$) for each class of figurate numbers considered. Table 1 looks at the series $\sum_{n=1}^{\infty} P(r, n)^{-5}$, Table 2 at the series $\sum_{n=1}^{\infty} P(r, n)^{-2}$, Table 3 at the series $\sum_{n=0}^{\infty} C(k, n)^{-3}$, and Table 4 at the series $\sum_{n=1}^{\infty} C(r, n)^{-1}$.

TABLE 1. Sums of reciprocal powers of polygonal numbers, $m = 4$.

r	Exact value of $\sum_{n=1}^{\infty} P(r, n)^{-(m+1)}$
3	$4032 - \frac{32}{9}\pi^4 - \frac{1120}{3}\pi^2$
4	$\frac{\pi^{10}}{93555}$ (Not possible with current method; must be treated separately.)
5	$272160 \log(3) + 51840\zeta(3) + 3840\zeta(5) - 30240\pi\sqrt{3} - 960\sqrt{3}\pi^3 - \frac{64\sqrt{3}\pi^5}{9} - 30240\Psi\left(\frac{2}{3}\right) - 80\Psi^{(3)}\left(\frac{2}{3}\right) - 5040\pi^2 - \frac{16\pi^4}{3}$
6	$2240 \log(2) + 360\zeta(3) + 30\zeta(5) - \frac{560\pi^2}{3} - \frac{16\pi^4}{9}$
7	$\frac{-1400000(\gamma + \Psi(\frac{2}{5}))}{19683} - \frac{4000\zeta(3)}{729} - \frac{32\zeta(5)}{729} - \frac{140000\Psi'(\frac{2}{5})}{729} - \frac{243}{2000\Psi''(\frac{2}{5})} - \frac{6561}{400\Psi^{(3)}(\frac{2}{5})} - \frac{4\Psi^{(4)}(\frac{2}{5})}{729} - \frac{70000\pi^2}{19683} - \frac{80\pi^4}{6561}$
8	$\frac{8505 \log(3)}{512} + \frac{405\zeta(3)}{32} + \frac{15\zeta(5)}{4} + \frac{945\pi\sqrt{3}}{512} + \frac{15\sqrt{3}\pi^3}{512} + \frac{64}{\sqrt{3}\pi^5} - \frac{945\Psi'(\frac{1}{3})}{256} - \frac{5\Psi^{(3)}(\frac{1}{3})}{128} - \frac{315\pi^2}{512} - \frac{\pi^4}{384}$
9	$\frac{1075648(\gamma + \Psi(\frac{2}{7}))}{390625} - \frac{4704\zeta(3)}{15625} - \frac{32\zeta(5)}{15625} - \frac{76832\Psi'(\frac{2}{7})}{15625} - \frac{3125}{2352\Psi''(\frac{2}{7})} - \frac{78125}{112\Psi^{(3)}(\frac{2}{7})} - \frac{4\Psi^{(4)}(\frac{2}{7})}{9375} - \frac{38416\pi^2}{234375} - \frac{112\pi^4}{140625}$
10	$\frac{8960\pi}{19683} + \frac{17920(\log(2) - G)}{6561} + \frac{80\zeta(3)}{27} + \frac{55\zeta(5)}{27} + \frac{10\Psi^{(3)}(\frac{1}{4})}{2187} - \frac{7840\pi^2}{19683} + \frac{80\pi^3}{729} - \frac{2\pi^4}{6561} + \frac{5\pi^5}{729}$

TABLE 2. Sums of reciprocal powers of pyramidal numbers, $m = 2$.

r	Exact Value of $\sum_{n=1}^{\infty} P(r, n)^{-m}$
3	$9\pi^2 - \frac{351}{4}$
4	$84\pi^2 - 828$
5	$-20 + \frac{2\pi^4}{45} - 8\zeta(3) + \frac{8\pi^2}{3}$
6	$\frac{576\Psi(1, \frac{3}{4})}{25} - \frac{82944 \ln(2)}{125} + \frac{13824\pi}{125} + \frac{156\pi^2}{25} - \frac{828}{125}$
7	$\frac{225\Psi(1, \frac{3}{5})}{49} + \frac{10125\Psi(\frac{3}{5})}{343} + \frac{159\pi^2}{98} + \frac{10125\gamma}{343} - \frac{1116}{343}$
8	$\frac{44\pi^2}{27} - \frac{512 \ln(2)}{27} - \frac{52}{27}$
9	$\frac{441\Psi(1, \frac{3}{7})}{484} + \frac{46305\Psi(\frac{3}{7})}{10648} + \frac{411\pi^2}{968} + \frac{46305\gamma}{10648} - \frac{1692}{1331}$
10	$\frac{2304\Psi(1, \frac{3}{8})}{4225} + \frac{663552\Psi(\frac{3}{8})}{274625} + \frac{1164\pi^2}{4225} + \frac{663552\gamma}{274625} - \frac{1980}{2197}$

TABLE 3. Sums of reciprocal powers of centered polygonal numbers, $m = 3$.

k	Exact Value of $\sum_{n=0}^{\infty} C(k, n)^{-m}$
3	$-\frac{1}{225 \cosh\left(\frac{\pi\sqrt{15}}{6}\right)^3} \left[8\sqrt{15} \sinh\left(\frac{\pi\sqrt{15}}{6}\right) \pi^2 \left(\frac{18\sqrt{15} \sinh\left(\frac{\pi\sqrt{15}}{6}\right) \cosh\left(\frac{\pi\sqrt{15}}{6}\right)^2 + 9 \cosh\left(\frac{\pi\sqrt{15}}{6}\right)^3 \right) \pi \right]$
4	hypergeom $\left(\left[1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2} \right], \left[\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right], 1 \right)$
5	$-\frac{1}{225 \cosh\left(\frac{\pi\sqrt{15}}{10}\right)^3} \left[8\pi \left(-10\sqrt{15} \sinh\left(\frac{\pi\sqrt{15}}{10}\right) \cosh\left(\frac{\pi\sqrt{15}}{10}\right)^2 \right. \right. \\ \left. \left. + \sinh\left(\frac{\pi\sqrt{15}}{10}\right) \sqrt{15}\pi^2 + 15 \cosh\left(\frac{\pi\sqrt{15}}{10}\right) \pi \right) \right]$
6	$\frac{1}{9} \left[\left(\sqrt{3}\pi^2 \tanh\left(\frac{\pi\sqrt{3}}{6}\right) + 9 \tanh\left(\frac{\pi\sqrt{3}}{6}\right)^2 \right) \pi + (-\pi^2 + 18)\sqrt{3} \tanh\left(\frac{\pi\sqrt{3}}{6}\right) - 9\pi \right]$
7	$-\frac{1}{49 \cosh\left(\frac{\pi\sqrt{7}}{14}\right)^3} \left[8\pi \left(-42\sqrt{7} \sinh\left(\frac{\pi\sqrt{7}}{14}\right) \cosh\left(\frac{\pi\sqrt{7}}{14}\right)^2 + \sqrt{7}\pi^2 \sinh\left(\frac{\pi\sqrt{7}}{14}\right) + 21 \cosh\left(\frac{\pi\sqrt{7}}{14}\right) \pi \right) \right]$
8	$\frac{\pi^6}{960}$ (Not possible with current method; must be treated separately.)
9	$\frac{(32\pi^3 + 1296\pi)\sqrt{3}}{243} - \frac{32\pi^2}{9}$
10	$\frac{\pi(30\sqrt{5} \sin\left(\frac{\pi\sqrt{5}}{10}\right) \cos\left(\frac{\pi\sqrt{5}}{10}\right) + \sqrt{5}\pi^2 \sin\left(\frac{\pi\sqrt{5}}{10}\right) - 15\pi \cos\left(\frac{\pi\sqrt{5}}{10}\right))}{25 \cos\left(\frac{\pi\sqrt{5}}{10}\right)^3}$

TABLE 4. Sums of reciprocal powers of centered pyramidal numbers, $m = 1$

r	Exact value of $\sum_{n=1}^{\infty} C(r, n)^{-m}$
3	$\frac{3\gamma}{2} + \frac{3}{4}\Psi\left(1 - \sqrt{2}i\right) + \frac{3}{4}\Psi\left(1 + \sqrt{2}i\right)$
4	$2\gamma + \Psi(1 - i) + \Psi(1 + i)$
5	$3\gamma + \frac{3}{2}\Psi\left(1 - \frac{\sqrt{2}i}{2}\right) + \frac{3}{2}\Psi\left(1 + \frac{\sqrt{2}i}{2}\right)$
6	$6\gamma + 3\Psi\left(1 - \frac{\sqrt{5}i}{5}\right) + 3\Psi\left(1 + \frac{\sqrt{5}i}{5}\right)$
7	$\zeta(3)$ (Not possible with current method; must be treated separately.)
8	$-6\gamma - 3\Psi\left(1 - \frac{\sqrt{7}}{7}\right) - 3\Psi\left(\frac{\sqrt{7}}{7} + 1\right)$
9	$6 \log(2) - 3$
10	$-2\gamma - \Psi\left(1 - \frac{\sqrt{3}}{3}\right) - \Psi\left(\frac{\sqrt{3}}{3} + 1\right)$

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