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ON THE HILBERT MATRIX OPERATOR ON HARDY AND WEIGHTED BERGMAN SPACES OF THE UPPER HALF-PLANE

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Abstract. In this paper, we compute the norm and spectrum of the Hilbert matrix operator on Hardy and weighted Bergman spaces of the upper half-plane.

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1. Introduction and preliminaries

Let $\mathbb{C}$ denote the complex plane, and let $z = x + iy = re^{i\theta} \in \mathbb{C}$. Then the set $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ defines the (open) unit disk. Let $dm(z)$ denote the (normalized) area Lebesgue measure on $\mathbb{D}$, that is, $dm(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$, and for $\alpha > -1$, we define the weighted Lebesgue measure on $\mathbb{D}$ by
$dm_\alpha(z) = (1 - |z|^2)^\alpha dm(z)$ for each $z \in \mathbb{D}$.

On the other hand, let $\mathbb{U}= \{w \in \mathbb{C} : \Im(w) > 0\}$ denote the upper half-plane of $\mathbb{C}$, where $\Im(w)$ denotes the imaginary part of w. Let $d\mu(w)$ denote the area measure on $\mathbb{U}$, and for $\alpha > -1$, we define the weighted measure on $\mathbb{U}$ by
$d\mu_\alpha(w) = (\Im(w))^\alpha d\mu(w)$ for $w \in \mathbb{U}$. The Cayley transform, ψ, is a conformal map from the unit disk, $\mathbb{D}$, to the upper half-plane, $\mathbb{U}$, and is given by

$$\psi(z) \;=\; \frac{i(1+z)}{1-z},$$

and it has an inverse. See [2] for more details.

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For an open subset Ω of $\mathbb{C}$, let $\mathcal{H}(\Omega)$ be the space of analytic functions $f : \Omega \to \mathbb{C}$. For $1 \leq p < \infty$, the Hardy space of the unit disk, $H^p(\mathbb{D})$, is defined by

$$H^p(\mathbb{D}) = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{H^p(\mathbb{D})} = \sup_{0<r<1} \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}} < \infty \right\},$$

while the Hardy space of the upper half-plane, $H^p(\mathbb{U})$, is defined by

$$H^p(\mathbb{U}) = \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{H^p(\mathbb{U})} = \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \right)^{\frac{1}{p}} < \infty \right\}. \tag{1.1}$$

It is well noted in literature that H^p - functions can be identified with their boundary values almost everywhere, and therefore we have that

$$\|f\|_{H^p(\mathbb{D})} = \left(\int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}},$$

and

$$\|f\|_{H^p(\mathbb{U})} = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

For $1 \leq p < \infty$, the weighted Bergman space of the unit disk, $L^p_a(\mathbb{D}, m_\alpha)$, is defined by

$$L^p_a(\mathbb{D}, m_\alpha) = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{L^p_a(\mathbb{D},m_\alpha)} = \left(\int_{\mathbb{D}} |f(z)|^p \, dm_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\},$$

while the weighted Bergman space of the upper half-plane, $L^p_a(\mathbb{U}, \mu_\alpha)$, by

$$L^p_a(\mathbb{U}, \mu_\alpha) = \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{L^p_a(\mathbb{U},\mu_\alpha)} = \left(\int_{\mathbb{U}} |f(w)|^p \, d\mu_\alpha(w) \right)^{\frac{1}{p}} < \infty \right\}. \tag{1.2}$$

When $\alpha = 0$ in both cases, we obtain the classical Bergman spaces, of the unit disk, $L^p_a(\mathbb{D}, m)$, and of the upper half-plane, $L^p_a(\mathbb{U}, \mu)$. If we let $X(\mathbb{U})$ denote $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, we can then associate each $X(\mathbb{U})$ with a parameter γ, with $\gamma = \frac{\alpha+2}{p}$, and for $X(\mathbb{U}) = L^p_a(\mathbb{U}, \mu_\alpha)$, $\alpha > -1$, and where $X(\mathbb{U}) = H^p(\mathbb{U})$, $\alpha = -1$. See more details in [2].

The Hilbert matrix, $\mathtt{H}_n$, is a square matrix defined by

$$\mathtt{H}_n = \left(\frac{1}{i+j-1} \right)_{i,j=1,2,3,\ldots,n} = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{pmatrix},$$

where i denotes the row, and j the column.

The Hilbert matrix operator can act discretely or continuously.

(a) Discretely: On the sequence spaces
Considering a sequence $(a_n)_{n\geq 0}$ in l^p spaces as a column vector, and by conducting matrix multiplication with $\mathtt{H}_n$, it follows that

$$\mathbf{H}(a_n)_{n\geq 0} = \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1}\right)_{n\geq 0},$$

for those $(a_n)_{n\geq 0}$ for which the sums are well defined.

(b) Continuously: On spaces of analytic functions
(i) On the unit disk
For $f \in \mathcal{H}(\mathbb{D})$, the Hilbert matrix operator is defined by

$$\mathbf{H}(f)(z) = \int_0^1 f(t)\frac{1}{1-tz}dt, \qquad \text{for } z \in \mathbb{D}.$$

(ii) On the upper half-plane
For $f \in \mathcal{H}(\mathbb{U})$, the Hilbert matrix operator is defined by

$$\mathbf{H}(f)(z) = \int_0^{\infty} \frac{f(\zeta)}{\zeta + z}\, d\zeta, \qquad \text{for } z \in \mathbb{U}. \tag{1.3}$$

See more details in [3, 8].

Let X be a Banach space over $\mathbb{C}$, and let T be a closed operator on X. The resolvent set of T is given by $\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible}\}$, and the spectrum of T is given by $\sigma(T) = \mathbb{C}\backslash\rho(T)$. The point spectrum of T is given by $\sigma_p(T) = \{\lambda \in \mathbb{C} : T(x) = \lambda x \text{ for some } 0 \neq x \in X\}$. The resolvent operator of T or the resolvent of T at $\lambda \in \rho(T)$ is given by $R(\lambda, T) = (\lambda I - T)^{-1}$. The spectral radius of T is defined as $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$, and is characterized by $r(T) \leq \|T\|$. More details can be found in [5, 6].

A family $(T_t)_{t\geq 0}$ of bounded linear operators on X, is said to be a one parameter semigroup if it satisfies the following properties,

$$\begin{aligned} &\text{(i)} \quad T_{t+s} = T_t \circ T_s \quad \text{for } t, s \geq 0,\\ &\text{(ii)} \quad T_0 = I, \ \text{where } I \text{ is the identity operator on } X. \end{aligned}$$

The family $(T_t)_{t\geq 0}$ is said to be strongly continuous if

$$\lim_{t\to 0^+} \|T_t(x) - x\| = 0 \qquad \text{for every } x \in X.$$

If $(T_t)_{t\geq 0}$ is a strongly continuous semigroup, then its infinitesimal generator, G, (an unbounded operator), is defined by

$$Gx = \lim_{t\to 0^+} \frac{T_t(x) - x}{t} = \frac{\partial}{\partial t} T_t(x)\bigg|_{t=0} \qquad \text{for each } x \in dom(G),$$

where $dom(G)$ is the domain of the infinitesimal generator, and is given by

$$dom(G) = \left\{x \in X \,\middle|\, \lim_{t\to 0^+} \frac{T_t(x) - x}{t} \text{ exists}\right\}.$$

See [6, 7] for more details.

Let φ be a self analytic map on Ω, then the composition operator $\mathcal{C}_\varphi$ induced by φ is defined by $\mathcal{C}_\varphi(f) = f \circ \varphi$ for all $f \in \mathcal{H}(\Omega)$. Let φ_t for $t \geq 0$ be a family of self analytic maps on Ω, then the composition semigroup $\mathcal{C}_{\varphi_t}$ is defined by $\mathcal{C}_{\varphi_t}(f) = f \circ \varphi_t$ for all $f \in \mathcal{H}(\Omega)$. The weighted composition semigroup is given by $\mathcal{S}_{\varphi_t}(f) = (\varphi_t')^\gamma (f \circ \varphi_t)$, where $\gamma = \frac{\alpha+2}{p}$ for $1 \leq p < \infty$. For more details, see [2].

Let $m, n \in \mathbb{C}$. Then the gamma function, $\Gamma(m)$, is given by

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1}\, dx, \text{ for } \Re(m) > 0,$$

while the beta function, $\beta(m,n)$, for $\Re(m) > 0$ and $\Re(n) > 0$ is given by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx, \tag{1.4}$$

or equivalently, by

$$\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt. \tag{1.5}$$

The beta and gamma relationship is given by

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{1.6}$$

Let $m \in \mathbb{C} \setminus \mathbb{Z}$, then the Euler reflection formula is given by

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin \pi m}.$$

See [1, 4] for more details.

Over the years, there has been extensive research on the Hilbert matrix operator acting on $\mathcal{H}(\mathbb{D})$. The research on the Hilbert matrix operator acting on $\mathcal{H}(\mathbb{U})$ is minimal. In 2021, Siskakis [8] initiated the study of the Hilbert matrix operator acting on $H^p(\mathbb{U})$. Nevertheless, the scope of research on the Hilbert matrix operator on $L^p_a(\mathbb{U}, \mu_\alpha)$ remains limited.

In this paper, we study the Hilbert matrix operator on $H^p(\mathbb{U})$ and $L^p_a(\mathbb{U}, \mu_\alpha)$, with a focus on determining its operator norm and spectrum. These results contribute to a deeper understanding of how the operator behaves on $\mathcal{H}(\mathbb{U})$, where stability and boundedness are central questions. Beyond pure mathematics, the findings can also support areas of computer science that rely on functional analysis, such as numerical algorithms and computational methods for approximation and stability analysis.

2. Norm and Spectrum of the Hilbert matrix operator on $X(\mathbb{U})$

Theorem 2.1. *The Hilbert matrix operator on $\mathcal{H}(\mathbb{U})$ can be expressed as*

$$\mathbf{H}(f)(z) = \int_0^1 \frac{1}{1-s} f\left(\frac{s}{1-s}z\right) ds,$$

which is the average of weighted composition operators.

Proof. We know from equation (1.3) that for all $f \in \mathcal{H}(\mathbb{U})$,

$$\mathbf{H}(f)(z) = \int_0^\infty \frac{f(\zeta)}{\zeta + z} d\zeta.$$

We then choose the path of integration to be a straight line from 0 to ∞ passing through z, and parametrizing it as $\psi(s) = \frac{s}{1-s}z$ for $0 < s < 1$.

We let $\psi(s) = \frac{s}{1-s}z = \zeta$ and $d\zeta = \frac{z}{(1-s)^2}ds$. Substituting, we get that

$$\begin{aligned}\mathbf{H}(f)(z) &= \int_0^1 \frac{f(\frac{s}{1-s}z)}{(\frac{s}{1-s}z + z)} \cdot \frac{z}{(1-s)^2} ds \\ &= \int_0^1 \frac{1}{1-s} f\left(\frac{s}{1-s}z\right) ds,\end{aligned}$$

as desired. □

Remark 2.2. By the definition of the weighted composition semigroup, we define a family of self analytic maps on $\mathbb{U}$ as $\varphi_t(z) = e^t z$ where $t \in \mathbb{R},\ z \in \mathbb{U}$, therefore the corresponding group of weighted composition operators on $X(\mathbb{U})$ is given by $S_{\varphi_t}(f)(z) = e^{\gamma t} f(e^t z)$. The properties of a similar group, $(S_{\varphi_{-t}})_{t\in\mathbb{R}}$, were fully analyzed in [2], where a family of self analytic maps $\varphi_t = e^{-t}z$ on $\mathbb{U}$ was considered.

Following [2], it can be easily deduced that $(S_{\varphi_t})_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on $X(\mathbb{U})$, whose infinitesimal generator, G, is given by

$$Gf(z) = \gamma f(z) + z f'(z),$$

with domain, $dom(G) = \{f \in X(\mathbb{U}) : zf'(z) \in X(\mathbb{U})\}$. Moreover, the point spectrum is $\sigma_p(G) = \emptyset$, while the spectrum is $\sigma(G) = i\mathbb{R}$, indicating that G is unbounded.

In the next result, we determine the upper norm bound of the Hilbert matrix operator, $\mathbf{H}$,

Theorem 2.3. *Let $X(\mathbb{U})$ denote either $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, and let $\mathbf{H}$ be the Hilbert matrix operator acting on $X(\mathbb{U})$. Then,*

$$\|\mathbf{H}\|_{X(\mathbb{U})\to X(\mathbb{U})} \le \frac{\pi}{\sin(\pi\gamma)},$$

for non-integral values of γ.

Proof. When $X(\mathbb{U}) = H^p(\mathbb{U})$, as remarked in the introduction, we use the fact that H^p-functions can be identified by their boundary values. Therefore,

$$\begin{aligned}
\|\mathbf{H}(f)\|_{H^p(\mathbb{U})} &= \left(\int_{-\infty}^{\infty} |\mathbf{H}(f)(x)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_{-\infty}^{\infty} \left|\int_0^1 \frac{1}{1-s} f\left(\frac{s}{1-s}x\right) ds\right|^p dx\right)^{\frac{1}{p}} \\
&\quad \text{by the definition} \\
&\leq \left(\int_{-\infty}^{\infty} \int_0^1 \left|\frac{1}{1-s} f\left(\frac{s}{1-s}x\right)\right|^p ds^p dx\right)^{\frac{1}{p}} \\
&\quad \text{by triangle inequality,}
\end{aligned}$$

and by Fubini's Theorem, we finally get

$$\begin{aligned}
\|\mathbf{H}(f)\|_{H^p(\mathbb{U})} &\leq \int_0^1 \left(\int_{-\infty}^{\infty} \left|\frac{1}{1-s} f\left(\frac{s}{1-s}x\right)\right|^p dx\right)^{\frac{1}{p}} ds \\
&= \int_0^1 \frac{1}{1-s}\left(\int_{-\infty}^{\infty} \left|f\left(\frac{s}{1-s}x\right)\right|^p dx\right)^{\frac{1}{p}} ds.
\end{aligned}$$

Let $u = \frac{s}{1-s}x$, then $dx = \frac{1-s}{s}\,du$. Substituting, we get

$$\begin{aligned}
\|\mathbf{H}(f)\|_{H^p(\mathbb{U})} &\leq \int_0^1 \frac{1}{1-s}\left(\int_{-\infty}^{\infty} |f(u)|^p \left(\frac{1-s}{s}\right) du\right)^{\frac{1}{p}} ds \\
&= \int_0^1 \frac{1}{1-s}\left(\frac{1-s}{s}\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |f(u)|^p\, du\right)^{\frac{1}{p}} ds \\
&= \int_0^1 s^{-\frac{1}{p}}(1-s)^{(\frac{1}{p}-1)} \left(\int_{-\infty}^{\infty} |f(u)|^p\, du\right)^{\frac{1}{p}} ds,
\end{aligned}$$

which is equal to

$$\|\mathbf{H}(f)\|_{H^p(\mathbb{U})} \leq \|f\|_{H^p(\mathbb{U})} \int_0^1 s^{-\frac{1}{p}}(1-s)^{(\frac{1}{p}-1)}\, ds. \tag{2.1}$$

By the definition of $\beta(m, n)$ given by equation (1.4), we get that

$$m = 1 - \tfrac{1}{p} \quad , \quad n = \tfrac{1}{p},$$

making it $\beta(1 - \frac{1}{p}, \frac{1}{p})$, and by using the beta and gamma relationship given by equation (1.6), we get that

$$\beta\left(1 - \frac{1}{p}, \frac{1}{p}\right) = \Gamma\left(1 - \frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right),$$

which by Euler reflection formula results in

$$\Gamma\left(1 - \frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right) = \frac{\pi}{\sin(\frac{\pi}{p})}. \tag{2.2}$$

Finally, replacing with results obtained in equation (2.2) in equation (2.1), we get

$$\|\mathbf{H}(f)\|_{H^p(\mathbb{U})} \le \frac{\pi}{\sin(\frac{\pi}{p})}\|f\|_{H^p(\mathbb{U})},$$

and by taking supremum over all $f \in H^p(\mathbb{U})$ with $\|f\|_{H^p(\mathbb{U})} \le 1$, we get that

$$\|\mathbf{H}\|_{H^p(\mathbb{U})\to H^p(\mathbb{U})} \le \frac{\pi}{\sin(\frac{\pi}{p})}, \quad \text{where } 1 < p < \infty. \tag{2.3}$$

The same result was obtained by Siskakis [8] using the usual norm definition given by equation (1.1).

When $X(\mathbb{U}) = L^p_a(\mathbb{U}, \mu_\alpha)$, we use the norm definition of $L^p_a(\mathbb{U}, \mu_\alpha)$ given by equation (1.2) to get

$$\begin{aligned}
\|\mathbf{H}(f)\|_{L^p_a(\mathbb{U},\mu_\alpha)} &= \left(\int_{\mathbb{U}} |\mathbf{H}(f)(z)|^p d\mu_\alpha(z)\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{U}} \left|\int_0^1 \frac{1}{1-s} f\left(\frac{s}{1-s}z\right) ds\right|^p d\mu_\alpha(z)\right)^{\frac{1}{p}} \\
&\quad \text{by substitution} \\
&= \left(\int_{\mathbb{U}} \left|\int_0^1 \frac{1}{1-s} f\left(\frac{s}{1-s}z\right) ds\right|^p (\Im(z))^\alpha d\mu(z)\right)^{\frac{1}{p}} \\
&\quad \text{by the definition} \\
&\le \left(\int_{\mathbb{U}} \int_0^1 \left|\frac{1}{1-s} f\left(\frac{s}{1-s}z\right)\right|^p ds^p (\Im(z))^\alpha d\mu(z)\right)^{\frac{1}{p}} \\
&\quad \text{by triangle inequality,}
\end{aligned}$$

and by Fubini's Theorem, we finally get

$$\|\mathbf{H}(f)\|_{L^p_a(\mathbb{U},\mu_\alpha)} \le \int_0^1 \left(\int_{\mathbb{U}} \left|\frac{1}{1-s} f\left(\frac{s}{1-s}z\right)\right|^p (\Im(z))^\alpha d\mu(z)\right)^{\frac{1}{p}} ds.$$

We let $z = x + iy$ which results in

$$\begin{aligned}
\|\mathbf{H}(f)\|_{L^p_a(\mathbb{U},\mu_\alpha)} &\le \int_0^1 \left(\int_{-\infty}^{\infty} \int_0^{\infty} \left|\frac{1}{1-s} f\left(\frac{s}{1-s}(x+iy)\right)\right|^p y^\alpha dy dx\right)^{\frac{1}{p}} ds \\
&= \int_0^1 \frac{1}{1-s} \left(\int_{-\infty}^{\infty} \int_0^{\infty} \left|f\left(\frac{s}{1-s}(x+iy)\right)\right|^p y^\alpha dy dx\right)^{\frac{1}{p}} ds \\
&= \int_0^1 \frac{1}{1-s} \left(\int_{-\infty}^{\infty} \int_0^{\infty} \left|f\left(\frac{s}{1-s}x + i\frac{s}{1-s}y\right)\right|^p y^\alpha dy dx\right)^{\frac{1}{p}} ds.
\end{aligned}$$

Let $u = \frac{s}{1-s}x$ and $v = \frac{s}{1-s}y$, then $y = \frac{1-s}{s}v$, and $dydx = (\frac{1-s}{s})^2\, dvdu$. Substituting, we get

$$\begin{aligned}
\|\mathbf{H}(f)\|_{L^p_a(\mathbb{U},\mu_\alpha)} &\leq \int_0^1 \frac{1}{1-s}\left(\int_{-\infty}^{\infty}\int_0^{\infty} |f(u+iv)|^p \left(\frac{1-s}{s}v\right)^{\alpha}\left(\frac{1-s}{s}\right)^2 dvdu\right)^{\frac{1}{p}} ds \\
&= \int_0^1 \frac{1}{1-s}\left(\int_{-\infty}^{\infty}\int_0^{\infty} |f(u+iv)|^p \left(\frac{1-s}{s}\right)^{\alpha+2} v^{\alpha} dvdu\right)^{\frac{1}{p}} ds \\
&= \int_0^1 (1-s)^{-1}\left(\frac{1-s}{s}\right)^{\frac{\alpha+2}{p}}\left(\int_{-\infty}^{\infty}\int_0^{\infty} |f(u+iv)|^p v^{\alpha} dvdu\right)^{\frac{1}{p}} ds \\
&= \int_0^1 s^{-(\frac{\alpha+2}{p})}(1-s)^{(\frac{\alpha+2}{p}-1)}\left(\int_{-\infty}^{\infty}\int_0^{\infty} |f(u+iv)|^p v^{\alpha} dvdu\right)^{\frac{1}{p}} ds.
\end{aligned}$$

However, we know that $\gamma = \frac{\alpha+2}{p}$. Substituting, we obtain

$$\|\mathbf{H}(f)\|_{L^p_a(\mathbb{U},\mu_\alpha)} \leq \int_0^1 s^{-\gamma}(1-s)^{\gamma-1}\left(\int_{-\infty}^{\infty}\int_0^{\infty} |f(u+iv)|^p v^{\alpha} dvdu\right)^{\frac{1}{p}} ds.$$

If we let $w = u + iv$, then,

$$\|\mathbf{H}(f)\|_{L^p_a(\mathbb{U},\mu_\alpha)} \leq \int_0^1 s^{-\gamma}(1-s)^{\gamma-1}\left(\int_{\mathbb{U}} |f(w)|^p (\Im(w))^{\alpha}\, d\mu(w)\right)^{\frac{1}{p}} ds,$$

which is equal to

$$\|\mathbf{H}(f)\|_{L^p_a(\mathbb{U},\mu_\alpha)} \leq \|f\|_{L^p_a(\mathbb{U},\mu_\alpha)} \int_0^1 s^{-\gamma}(1-s)^{\gamma-1}\, ds. \tag{2.4}$$

By the definition of $\beta(m,n)$ given by equation (1.4), we get that

$$m = 1-\gamma \quad , \quad n = \gamma,$$

making it $\beta(1-\gamma,\gamma)$, and by using the beta and gamma relationship given by equation (1.6), we get that

$$\beta(1-\gamma,\gamma) = \Gamma(1-\gamma)\Gamma(\gamma),$$

which by Euler reflection formula results in

$$\Gamma(1-\gamma)\Gamma(\gamma) = \frac{\pi}{\sin(\pi\gamma)}. \tag{2.5}$$

Finally, replacing with results obtained in equation (2.5) in equation (2.4), we get

$$\|\mathbf{H}(f)\|_{L^p_a(\mathbb{U},\mu_\alpha)} \leq \frac{\pi}{\sin(\pi\gamma)}\|f\|_{L^p_a(\mathbb{U},\mu_\alpha)},$$

and by taking supremum over all $f \in L^p_a(\mathbb{U},\mu_\alpha)$ with $\|f\|_{L^p_a(\mathbb{U},\mu_\alpha)} \leq 1$, we get that

$$\|\mathbf{H}\|_{L^p_a(\mathbb{U},\mu_\alpha)\to L^p_a(\mathbb{U},\mu_\alpha)} \leq \frac{\pi}{\sin(\pi\gamma)}, \tag{2.6}$$

for non-integral values of γ.

From the result obtained in equation (2.6), we can also obtain the result given in equation (2.3). To show this, we know that $\alpha = -1$ on $H^p(\mathbb{U})$, resulting in $\gamma = 1/p$, which is a non-integral value provided $p > 1$. Substituting, we get

$$\|\mathbf{H}\|_{H^p(\mathbb{U})\to H^p(\mathbb{U})} \leq \frac{\pi}{\sin(\frac{\pi}{p})}, \quad \text{where } 1 < p < \infty,$$

which is the same as our result in equation (2.3).

Therefore, we can now generalize our result to be, let $X(\mathbb{U})$ denote either $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, then

$$\|\mathbf{H}\|_{X(\mathbb{U})\to X(\mathbb{U})} \leq \frac{\pi}{\sin(\pi\gamma)}, \tag{2.7}$$

for non-integral values of γ. This completes the proof. □

Theorem 2.4. *Let E be a Banach space over $\mathbb{C}$, and let $L(E)$ denote the Banach algebra of bounded linear operators on E endowed with multiplication $\circ$. Let $(T_t)_{t\in\mathbb{R}} \subseteq L(E)$ be a strongly continuous group of isometries on E, and $\phi \in L^1(\mathbb{R})$. Define the operator T_ϕ by*

$$T_\phi f = \int_{-\infty}^{\infty} \phi(t) T_t f \, dt \quad \text{for } f \in E,$$

then $T_\phi \in L(E)$, and by using convolution $*$, $T_{\phi*\varphi} = T_\phi \circ T_\varphi$ for $\phi, \varphi \in L^1(\mathbb{R})$.

Proof. By the definition of linearity, we get that if we let $f, g \in E$ and $\alpha, \beta \in \mathbb{R}$, then,

$$\begin{aligned} T_\phi(\alpha f + \beta g) &= \int_{-\infty}^{\infty} \phi(t) T_t(\alpha f + \beta g) \, dt \\ &= \int_{-\infty}^{\infty} \phi(t)(\alpha T_t f + \beta T_t g) \, dt \\ &= \alpha T_\phi f + \beta T_\phi g. \end{aligned}$$

Therefore T_ϕ is linear.

By the norm definition,

$$\begin{aligned} \|T_\phi f\|_E &\leq \left| \int_{-\infty}^{\infty} \phi(t) T_t f \, dt \right| \\ &\leq \int_{-\infty}^{\infty} |\phi(t) T_t f| \, dt \\ &\leq \|T_t f\|_E \int_{-\infty}^{\infty} |\phi(t)| \, dt. \end{aligned}$$

Since $(T_t)_{t\in\mathbb{R}}$ is an isometry, we can therefore replace to get

$$\begin{aligned} \|T_\phi f\|_E &\leq \|f\|_E \int_{-\infty}^{\infty} |\phi(t)| \, dt \\ &= \|f\|_E \|\phi\|_{L^1(\mathbb{R})}, \end{aligned}$$

and by taking supremum over all $f \in E$ with $\|f\|_E \leq 1$, we get that

$$\|T_\phi\|_{E\to E} \leq \|\phi\|_{L^1(\mathbb{R})}.$$

Therefore T_ϕ is bounded, implying that $T_\phi \in L(E)$.

By the definition of convolution, we get

$$\begin{aligned} T_{\phi*\varphi} f &= \int_{-\infty}^{\infty} (\phi * \varphi)(t) T_t f\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \phi(s)\varphi(t-s)\, ds \right) T_t f\, dt, \quad \text{for } t, s \in \mathbb{R}. \end{aligned}$$

Then by Fubini's Theorem, we get that

$$T_{\phi*\varphi} f = \int_{-\infty}^{\infty} \phi(s) \left(\int_{-\infty}^{\infty} \varphi(t-s) T_t f\, dt \right) ds.$$

We let $w = t - s$, therefore $t = s + w$ and $dt = dw$. Substituting, we get

$$\begin{aligned} T_{\phi*\varphi} f &= \int_{-\infty}^{\infty} \phi(s) \left(\int_{-\infty}^{\infty} \varphi(w) T_{s+w} f\, dw \right) ds \\ &= \int_{-\infty}^{\infty} \phi(s) \left(\int_{-\infty}^{\infty} \varphi(w) (T_s \circ T_w) f\, dw \right) ds \\ &= \int_{-\infty}^{\infty} \phi(s) T_s \left(\int_{-\infty}^{\infty} \varphi(w) T_w f\, dw \right) ds. \end{aligned}$$

Next, we let

$$T_\varphi f = \int_{-\infty}^{\infty} \varphi(w) T_w f\, dw \quad \text{for } f \in E.$$

Substituting, we get that

$$\begin{aligned} T_{\phi*\varphi} f &= \int_{-\infty}^{\infty} \phi(s) T_s (T_\varphi f)\, ds \\ &= (T_\phi \circ T_\varphi) f, \end{aligned}$$

thus $T : L^1(\mathbb{R}) \longrightarrow L(E)$ is an algebra homomorphism between $L^1(\mathbb{R})$ endowed with convolution $*$, and $L(E)$ endowed with multiplication $\circ$. □

In the next result, we apply Theorem 2.4 to give a new representation of the Hilbert matrix operator, $\mathbf{H}$,

Proposition 2.5. *Let $X(\mathbb{U})$ denote $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $(S_{\varphi_t})_{t\in\mathbb{R}}$ be a strongly continuous group of isometries on $X(\mathbb{U})$, and $\phi_\gamma \in L^1(\mathbb{R})$, then the Hilbert matrix operator takes the form*

$$\mathbf{H}(f)(z) = \int_{-\infty}^{\infty} \phi_\gamma(t) S_{\varphi_t}(f)(z) dt,$$

where

$$\phi_\gamma(t) = \frac{e^{(1-\gamma)t}}{e^t + 1}, \tag{2.8}$$

and

$$S_{\varphi_t}(f)(z) = e^{\gamma t} f(e^t z).$$

Proof. Using

$$\mathbf{H}(f)(z) = \int_0^\infty \frac{f(\zeta)}{\zeta + z}\, d\zeta,$$

we choose the path of integration to be a straight line through z, parametrizing it as $\varphi(t) = e^t z$ for $-\infty < t < \infty$. We let $\varphi(t) = e^t z = \zeta$, then $d\zeta = e^t z dt$, and substituting, we get

$$\begin{aligned} \mathbf{H}(f)(z) &= \int_{-\infty}^{\infty} \frac{f(e^t z)}{e^t z + z} \cdot e^t z dt \\ &= \int_{-\infty}^{\infty} \frac{e^t}{e^t + 1} \cdot f(e^t z) dt. \end{aligned}$$

We then introduce γ and obtain

$$\mathbf{H}(f)(z) = \int_{-\infty}^{\infty} \frac{e^{(1-\gamma)t}}{e^t + 1} \cdot e^{\gamma t} f(e^t z) dt,$$

which can be written as

$$\mathbf{H}(f)(z) = \int_{-\infty}^{\infty} \phi_\gamma(t) S_{\varphi_t}(f)(z) dt,$$

as claimed. □

Proposition 2.6. *Let ϕ_γ be defined as in equation (2.8), then,*

$$\int_{-\infty}^{\infty} \phi_\gamma(t) dt = \frac{\pi}{\sin(\pi\gamma)},$$

for non-integral values of γ.

Proof. Integrating $\phi_\gamma(t)$ given by equation (2.8) with respect to $-\infty < t < \infty$, we get that

$$\begin{aligned} \int_{-\infty}^{\infty} \phi_\gamma(t) dt &= \int_{-\infty}^{\infty} \frac{e^{(1-\gamma)t}}{e^t + 1} dt \\ &= \int_{-\infty}^{\infty} \frac{e^t \cdot e^{-\gamma t}}{e^t(1 + e^{-t})} dt \\ &= \int_{-\infty}^{\infty} \frac{e^{-\gamma t}}{(1 + e^{-t})} dt. \end{aligned}$$

We let $u = e^{-t}$, then $dt = \frac{du}{-u}$. We also get that when $t = -\infty$, $u = \infty$, and when $t = \infty$, $u = 0$, which gives us that

$$\begin{aligned} \int_{-\infty}^{\infty} \phi_\gamma(t) dt &= \int_{\infty}^{0} \frac{u^\gamma}{(1 + u)} \frac{du}{-u} \\ &= \int_0^\infty \frac{u^{\gamma - 1}}{(1 + u)} du. \end{aligned}$$

By the definition of $\beta(m, n)$ given by equation (1.5), we get that

$$m = \gamma \text{ and } n = 1 - \gamma,$$

making it $\beta(\gamma, 1-\gamma)$, and by using the beta and gamma relationship given by equation (1.6), we get

$$\beta(\gamma, 1-\gamma) = \Gamma(\gamma)\Gamma(1-\gamma),$$

which by Euler reflection formula gives us that

$$\Gamma(\gamma)\Gamma(1-\gamma) = \frac{\pi}{\sin(\pi\gamma)},$$

for non-integral values of γ. □

Theorem 2.7. *Let ϕ_γ be defined as in equation (2.8), then its Fourier transform, $\hat{\phi}_\gamma$, is given by*

$$\hat{\phi}_\gamma(s) = \frac{\pi}{\sin(\pi\gamma + i\pi s)}.$$

Proof. By using the definition of the Fourier transform, we get that

$$\begin{aligned} \hat{\phi}_\gamma(s) &= \int_{-\infty}^{\infty} e^{-ist} \frac{e^{(1-\gamma)t}}{e^t + 1} dt \\ &= \int_{-\infty}^{\infty} e^{-ist} \frac{e^t \cdot e^{-\gamma t}}{e^t(1 + e^{-t})} dt \\ &= \int_{-\infty}^{\infty} e^{-ist} \frac{e^{-\gamma t}}{(1 + e^{-t})} dt. \end{aligned}$$

We let $u = e^{-t}$, then $dt = \frac{du}{-u}$. We also get that when $t = -\infty$, $u = \infty$, and when $t = \infty$, $u = 0$, which gives us that

$$\begin{aligned} \hat{\phi}_\gamma(s) &= \int_{\infty}^{0} u^{is} \frac{u^\gamma}{(1+u)} \frac{du}{-u} \\ &= \int_{0}^{\infty} \frac{u^{(\gamma+is)-1}}{(1+u)} du. \end{aligned}$$

By the definition of $\beta(m, n)$ given by equation (1.5), we get that

$$m = (\gamma + is) \text{ and } n = 1 - (\gamma + is),$$

making it $\beta(\gamma + is, 1 - (\gamma + is))$, and by using the beta and gamma relationship given by equation (1.6), we get

$$\beta(\gamma + is, 1 - (\gamma + is)) = \Gamma(\gamma + is)\Gamma(1 - (\gamma + is)),$$

which by Euler reflection formula gives us that

$$\Gamma(\gamma + is)\Gamma(1 - (\gamma + is)) = \frac{\pi}{\sin(\pi\gamma + i\pi s)},$$

as claimed. □

Theorem 2.8. *The spectrum of* $\mathbf{H}$ *is given by*

$$\sigma(\mathbf{H}) = \overline{\left\{\frac{\pi}{\sin(\pi\gamma - \pi q)} : q \in \mathbb{R}\right\}},$$

and consequently,

$$\|\mathbf{H}\|_{X(\mathbb{U})\to X(\mathbb{U})} = \frac{\pi}{\sin(\pi\gamma)},$$

for non-integral values of γ*.*

Proof. As noted in [8], we have that

$$\sigma(\mathbf{H}) = \overline{\hat{\phi}(\sigma(G))}. \tag{2.9}$$

Using equation (2.9), we obtain

$$\begin{aligned}\sigma(\mathbf{H}) &= \overline{\left\{\frac{\pi}{\sin(\pi\gamma + i\pi s)} : s \in i\mathbb{R}\right\}} \\ &= \overline{\left\{\frac{\pi}{\sin(\pi\gamma - \pi q)} : q \in \mathbb{R}\right\}}.\end{aligned}$$

When we take $q = 0$, it means that $\frac{\pi}{\sin(\pi\gamma)} \in \sigma(\mathbf{H})$. Since the quantity $\frac{\pi}{\sin(\pi\gamma)}$ is a real number contained in $\sigma(\mathbf{H})$, it must be dominated by the spectral radius of $\mathbf{H}$, $r(\mathbf{H})$. That is,

$$\frac{\pi}{\sin(\pi\gamma)} \leq r(\mathbf{H}).$$

Now, by using the spectral radius inequality, it follows that

$$\frac{\pi}{\sin(\pi\gamma)} \leq r(\mathbf{H}) \leq \|\mathbf{H}\|,$$

for non-integral values of γ.

By combining these results with the ones obtained on the upper norm bound of $\mathbf{H}$ on $X(\mathbb{U})$ given in equation (2.7), we get that

$$\|\mathbf{H}\|_{X(\mathbb{U})\to X(\mathbb{U})} = \frac{\pi}{\sin(\pi\gamma)},$$

for non-integral values of γ, as desired. □

Corollary 2.9. *If* $X(\mathbb{U}) = H^p(\mathbb{U})$*, then,*

$$\|\mathbf{H}\|_{H^p(\mathbb{U})\to H^p(\mathbb{U})} = \frac{\pi}{\sin(\frac{\pi}{p})}, \quad \textit{where } 1 < p < \infty.$$

Proof. Taking $\alpha = -1$ yields $\gamma = 1/p$ by the definition of γ. By substitution, we get that

$$\|\mathbf{H}\|_{H^p(\mathbb{U})\to H^p(\mathbb{U})} = \frac{\pi}{\sin(\frac{\pi}{p})}, \quad \text{where } 1 < p < \infty. \tag{2.10}$$

The results shown in equation (2.10) coincide with what was obtained by Siskakis [8]. □

Corollary 2.10. *If $X(\mathbb{U}) = L_a^p(\mathbb{U}, \mu)$, then,*

$$\|\mathbf{H}\|_{L_a^p(\mathbb{U},\mu)\to L_a^p(\mathbb{U},\mu)} = \frac{\pi}{\sin(\frac{2\pi}{p})}, \quad \textit{where } 1 < p < 2 \textit{ and } 2 < p < \infty.$$

Proof. Taking $\alpha = 0$ yields $\gamma = 2/p$ by the definition of γ. By substitution, we get that

$$\|\mathbf{H}\|_{L_a^p(\mathbb{U},\mu)\to L_a^p(\mathbb{U},\mu)} = \frac{\pi}{\sin(\frac{2\pi}{p})}, \quad \text{for } 1 < p < 2 \text{ and } 2 < p < \infty,$$

as claimed. □

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