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# IMMERSION OF THE RICCI-RECURRENT NORMAL ALMOST CONTACT METRIC MANIFOLDS

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Abstract. In this paper, we investigate the immersion of three-dimensional Ricci-recurrent normal almost contact metric manifolds into four-dimensional Riemannian manifolds with constant curvature 1. A relationship between the shape operator $A$ and the structure tensor defining the normal almost contact structure is derived. We explore the geometric implications of structure tensor under specific conditions and establish connections between the scalar functions connected with the almost normal contact metric manifold that are associated with the covariant derivative of the Reeb vector field. Additionally, the study characterizes the behavior of the vector field associated with the Ricci-recurrence 1-form, emphasizing its impact on the manifold's curvature and immersion properties.

*Keywords.* Riemannian geometry, Differential geometry, Manifolds, Curvature, Metric tensor, Kähler manifolds, Sasakian manifolds, Immersions, Submanifolds
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## 1. Introduction and preliminaries

The study of almost contact structures can be traced back to the late 1950s. In 1959, J. W. Gray investigated differentiable manifolds with both contact and almost contact structures, focusing on their global properties from a topological perspective [10]. This early exploration set the stage for further developments in the field. Shigeo Sasaki significantly advanced this area of study in 1960 by formalizing the concept of almost contact structures in his seminal paper [19]. Sasaki's work provided the foundation for the development of geometric structures closely related to contact geometry. After Sasaki's formalization of almost contact structures $(\phi, \xi, \eta)$, the concept of normality within this framework gained prominence. A normal almost contact structure is one in which the Nijenhuis tensor of the structure tensor field $\phi$ vanishes, which is analogous to the integrability condition in almost complex structures. This condition ensures that the almost contact structure is well-behaved from a geometric standpoint,

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much like integrable complex structures on even-dimensional manifolds. When an almost contact structure is equipped with a compatible Riemannian metric $g$, the manifold becomes a normal almost contact metric manifold. These structures have been extensively studied for their curvature properties and their role in higher-dimensional geometry, particularly through the works of mathematicians like Zbigniew Olszak [30], who explored their classification and curvature characteristics. His work has laid the foundation for subsequent studies on curvature properties and the geometric behavior of such structures. In three dimensions, normal almost contact metric manifolds play a special role, as their structure and curvature properties can be fully characterized in a relatively simpler manner than in higher dimensions. Uday Chand De and Abul Kalam Mondal explored the $\xi$-projectively flat and $\phi$-projectively flat three-dimensional normal almost contact metric manifolds [22], contributing significantly to understanding the geometry of these specialized cases [24]. U.C. De, Ahmet Yildiz and A. Funda Yaliniz studied about the three dimensional normal almost contact metric manifolds when it is locally $\phi$-symmetric [23] and also studied when it satisfies certain curvature conditions [24]. U.C. De and S. Ghosh have studied about the D-homethetic deformation of normal almost contact metric manifolds [27].

The relationship between Ricci solitons and normal almost contact metric manifolds has also been a subject of interest. Mohd Danish Siddiqi and collaborators studied the existence of $\eta$-Ricci solitons in such manifolds, extending the theory of Ricci solitons to the framework of contact geometry [14]. Similarly, K. Mandal et al. examined $*$-Ricci solitons in three-dimensional normal almost contact metric manifolds, further contributing to the understanding of Ricci solitons in these geometrical structures [11].

Curvature conditions have remained a central theme in the study of normal almost contact metric manifolds. V. Venkatesha and co-authors investigated the curvature properties of these manifolds, focusing on $\phi$-projective flatness and the interaction between the metric and the structure tensors [28]. Their findings have shed light on how these curvature conditions influence the geometry of the manifold.

Additionally, the behavior of curves within normal almost contact metric geometry has been explored extensively. Abdullah Yildirim examined the geometry of curves, such as Legendre and slant curves, within these manifolds [2]. Other researchers, such as Constantin Calin and Jun-Ichi Inoguchi, concentrated on slant curves and their geometric properties in three-dimensional normal almost contact manifolds [9, 4].

Another interesting area of research is the study of pseudo-Hermitian magnetic curves. Ji-Eun Lee explored these magnetic curves in three-dimensional normal almost contact metric manifolds, contributing new insights into the geometric behavior of such curves and their interaction with the underlying structure [8].

In summary, normal almost contact metric manifolds are a rich field of study with deep connections to curvature properties, Ricci solitons, and the geometry of curves. The foundational work of Zbigniew Olszak, U. C. De, and their collaborators continues to shape the direction of research in this area, providing insights into both theoretical and applicative aspects of these manifolds.

Beyond pure mathematics, recent research shows that ideas from contact geometry—such as those underlying normal almost contact metric manifolds—are finding applications in computer science and robotics. For example, frameworks like Geometric Contact Flows [1] employ the structure of contact manifolds to model dynamical systems in robotics. In the context of this study, immersing Ricci-recurrent normal almost contact metric manifolds into four-dimensional Riemannian manifolds of constant curvature 1 provides not only new geometric insights but also concepts that can inspire structure-preserving approaches in control systems, simulation, and computational learning.

In this paper, we study the Ricci-recurrent three dimensional almost contact metric manifold and its isometric immersion in a four dimensional Riemannian manifold of constant curvature 1. We begin by reviewing the preliminaries required for our study.

### 1.1. Preliminaries.

Let $M^3$ be a three-dimensional connected differentiable manifold endowed with an almost contact metric structure [30] $(\phi, \xi, \eta)$, such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{1.01}$$

where, $\phi$ is a structure tensor field of type (1,1), $\xi$ is a tensor field of type $(1, 0)$, called the *Reeb vector field*, and $\eta$ is a *1-form*. If $M^3 \times \mathbb{R}$ is the product manifold [30], then for a tangent vector on the product manifold, we have:

$$J(X, \lambda \frac{d}{dt}) = (\phi X - \lambda\xi, \eta(X)\frac{d}{dt}), \tag{1.02}$$

where, $J$ is the *almost complex structure* on $M^3 \times \mathbb{R}$ [30], and the pair $(X, \lambda\frac{d}{dt})$ denotes the *tangent vector* on $M^3 \times \mathbb{R}$, with $X$ and $\lambda\frac{d}{dt}$ tangent to $M^3$ and $\mathbb{R}$, respectively. The normality condition for the structure $(\phi, \xi, \eta)$ is expressed using the *Nijenhuis tensor* [16] [17] [30]:

$$[\phi, \phi] + 2d\eta \otimes \xi = 0. \tag{1.03}$$

This equation provides the necessary and sufficient condition for $(\phi, \xi, \eta)$ to be a *normal almost contact structure*, where the *Nijenhuis tensor* $[\phi, \phi]$ is defined as:

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \tag{1.04}$$

$\forall\, X, Y \in TM^3$. From this point forward, we shall assume that $(\phi, \xi, \eta)$ represents a *normal almost contact structure* [30] on $M^3$. This structure satisfies the defining conditions given by equations (1.01), (1.02), (1.03), and (1.04), ensuring the consistency of the *almost contact metric structure* with normality. The Riemannian metric $g$ on $M^3$ is said to be *compatible* with the almost contact structure $(\phi, \xi, \eta)$ if it satisfies the condition:

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.05}$$

$\forall\, X, Y \in TM^3$. When this compatibility holds, $(\phi, \xi, \eta, g)$ forms a *normal almost contact metric structure* on $M^3$, the manifold is called a *normal almost contact metric manifold* [30]. On such a manifold, the following relation also holds [30]:

$$\eta(X) = g(X, \xi), \tag{1.06}$$

for all $X \in TM^3$. Additionally, the 2-form $\Omega$ associated with this structure is defined by:

$$\Omega(X,Y) = g(X,\phi Y), \tag{1.07}$$

$\forall\ X,Y \in TM^3$. Two metrics $g$ and $g'$ on $M^3$, equipped with a normal almost contact metric structure, are said to be *compatible* if and only if:

$$g' = \sigma g + (1-\sigma)\eta \otimes \eta,$$

where, $\sigma$ is a scalar function. For a normal almost contact metric structure $(\phi,\xi,\eta,g)$ on $M^3$, the covariant derivative of $\phi$, for any $X,Y \in TM^3$, is given by:

$$(\nabla_X \phi)(Y) = g(\phi\nabla_X\xi, Y) - \eta(Y)\phi(\nabla_X\xi), \tag{1.08}$$

and the covariant derivative of $\xi$, for any $X \in TM^3$, is:

$$\nabla_X\xi = \alpha(X - \eta(X)\xi) - \beta\phi X, \tag{1.09}$$

where, $2\alpha = \mathrm{div}(\xi)$ and $2\beta = \mathrm{tr}(\phi\nabla\xi)$. By substituting equation (1.09) into equation (1.08), an alternative form of the covariant derivative of $\phi$ is obtained as:

$$(\nabla_X\phi)(Y) = \alpha\left(g(\phi X,Y)\xi - \eta(Y)\phi X\right) + \beta\left(g(X,Y)\xi - \eta(Y)X\right). \tag{1.10}$$

Moreover, from (1.06), (1.01), (1.09), (1.07), we have

$$(\nabla_X\eta)(Y) = \alpha g(\phi Y,\phi X) - \beta g(X,\phi Y) = \alpha g(\phi Y,\phi X) - \beta\Omega(X,Y). \tag{1.11}$$

The Riemannian curvature tensor is a fundamental object in Riemannian geometry, which measures the extent to which the metric of a manifold deviates from being flat. For any differentiable manifold with a Riemannian metric, the Riemannian curvature tensor [5] $R$ is a $(1,3)$-tensor defined as:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

where $X,Y,Z$ are vector fields on the manifold, and $\nabla$ denotes the Levi-Civita connection associated with the metric. If $R(X,Y,Z,W) = g(R(X,Y)Z,W)$, then $R$ satisfies the following properties:

$$R(X,Y,Z,W) = -R(Y,X,Z,W);\, R(X,Y,Z,W) = -R(X,Y,W,Z) \tag{1.12}$$

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0 \tag{1.13}$$

$$(\nabla_X R)(Y,Z,W,U) + (\nabla_Y R)(Z,X,W,U) + (\nabla_Z R)(X,Y,W,U) = 0 \tag{1.14}$$

In the context of a three dimensional normal almost contact metric manifold $M^3$, the Riemannian curvature tensor involving the Reeb vector field $\xi$ takes the following specific form [30]. For all $X,Y \in TM^3$, the expression is given by:

$$\begin{aligned} R(X,Y)\xi = {} & (Y\alpha + (\alpha^2-\beta^2)\eta(Y))\phi^2 X - (X\alpha + (\alpha^2-\beta^2)\eta(X))\phi^2 Y \\ & + (Y\beta + 2\alpha\beta\eta(Y))\phi X - (X\beta + 2\alpha\beta\eta(X))\phi Y, \end{aligned} \tag{1.15}$$

where, $\alpha$ and $\beta$ are scalar functions related to the structure of the manifold as defined earlier. From (1.15), using (1.12), we get,

$$R(\xi,Y,Z,\xi) = -(\xi\alpha + \alpha^2 - \beta^2)g(\phi X,\phi Y) \tag{1.16}$$

and

$$\xi\beta + 2\alpha\beta = 0. \tag{1.17}$$

The Ricci tensor involving the Reeb vector field $\xi$ [30] in a three dimensional normal almost contact metric manifold is given by:

$$S(X,\xi) = -X\alpha - (\phi X)\beta - (\xi\alpha + 2(\alpha^2 - \beta^2))\eta(X), \tag{1.18}$$

where, $S$ denotes the Ricci tensor.
It is also well known that the curvature tensor in a three-dimensional manifold satisfies [18]

$$\begin{aligned} R(X,Y,Z,W) = g(X,W)S(Y,Z) &- S(X,Z)g(Y,W) \\ &+ S(X,W)g(Y,Z) - g(X,Z)S(Y,W) \\ &- \frac{r}{2}(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)). \end{aligned} \tag{1.19}$$

Replacing $X$, $W$ by $\xi$ in (1.19) and using (1.16), (1.01), (1.18), we get

$$\begin{aligned} S(Y,Z) = (\frac{r}{2} &+ \xi\alpha + \alpha^2 - \beta^2)g(\phi Y, \phi Z) \\ &- \eta(Z)(Y\alpha + (\phi Y)\beta) - \eta(Y)(Z\alpha + (\phi Z)\beta) \\ &- 2(\alpha^2 - \beta^2)\eta(Y)\eta(Z). \end{aligned} \tag{1.20}$$

From this point forward, we shall consider $\alpha$ and $\beta$ as constants. Then equation (1.20) becomes

$$S(Y,Z) = (\frac{r}{2} + \alpha^2 - \beta^2)g(\phi Y, \phi Z) - 2(\alpha^2 - \beta^2)\eta(Y)\eta(Z). \tag{1.21}$$

Since, $\alpha$ and $\beta$ are constants, it follows from (1.17) that the three-dimensional normal almost contact metric manifold $M^3$ is either $\alpha$-Kenmotsu when $\beta = 0$, or $\beta$-Sasakian when $\alpha = 0$ [refer, [23], [24], [18]]. Consequently, from (1.21), we obtain

$$S(\xi,\xi) = -2(\alpha^2 - \beta^2) \neq 0, \tag{1.22}$$

which holds on $M^3$. Note that (1.17) makes it evident that the case $\alpha = \beta = 0$ does not occur, ensuring that $M^3$ is not co-symplectic. Note here that, since either $\alpha$ or $\beta$ are the non-zero constants, this study can be reduced to either $\alpha$-Kenmotsu or $\beta$-Sasakian. Note here that the 1-Kenmostu is the Kenmotsu manifold and 1-Sasakian is the Sasakian manifold [7]. In the next chapter, we will examine the characteristics of Ricci-recurrent normal almost contact metric manifolds through their immersion in a four-dimensional Riemannian manifold of constant curvature 1.

## 2. Ricci-Recurrent Normal Almost Contact Metric Manifolds

The notion of Ricci-recurrence was introduced by E. M. Patterson [6], in the year 1952. Several authors have studied on Ricci-recurrent manifolds [29] [13] [15], [20]. A Riemannian Manifold is said to be Ricci-Recurrent [26] [6] if its Ricci tensor $S$ satisfies

$$(\nabla_U S)(Y,Z) = \omega(U)S(Y,Z), \tag{2.01}$$

where, $\omega(U) = g(\rho, U)$ is not identically zero and $\rho$ is the vector field associated with the 1-form $\omega$. From this point forward, we shall refer this 1-form $\omega$ as Ricci-recurrence 1-form. This leads to the following theorem:

**Theorem 2.1.** *Suppose $M^3$ be a three-dimensional Ricci-recurrent normal almost contact metric manifold. Then the following relationship holds:*

$$\begin{aligned}\omega(U)S(Y,Z) = {} & \frac{Ur}{2} g(\phi Y, \phi Z) \\ & - (\frac{r}{2} + 3(\alpha^2 - \beta^2))(\eta(Y)(\alpha g(\phi U, \phi Z) + \beta g(U, \phi Z)) \\ & + \eta(Z)(\alpha g(\phi U, \phi Y) + \beta g(U, \phi Y))), \end{aligned} \tag{2.02}$$

*where, $S$ is the Ricci tensor, $\omega$ is the Ricci-recurrence 1-form, $\phi$ is the structure tensor field of type (1,1), $r$ is the scalar curvature and $\alpha, \beta$ are constants.*

*Proof.* Differentiating (1.21) covariantly with respect to $U \in TM^3$, we get:

$$\begin{aligned}(\nabla_U S)(Y,Z) = {} & \frac{Ur}{2} g(\phi Y, \phi Z) \\ & + (\frac{r}{2} + \alpha^2 - \beta^2)\,[g((\nabla_U \phi)Y, \phi Z) + g((\nabla_U \phi)Z, \phi Y)] \\ & - 2\left(\alpha^2 - \beta^2\right)[\eta(Z)(\nabla_U \eta)Y + \eta(Y)(\nabla_U \eta)Z], \end{aligned} \tag{2.03}$$

Using (1.10), (1.11) in (2.03) and simplifying using (1.01), we get:

$$\begin{aligned}(\nabla_U S)(Y,Z) = {} & \frac{Ur}{2} g(\phi Y, \phi Z) \\ & - (\frac{r}{2} + 3(\alpha^2 - \beta^2))(\eta(Y)(\alpha g(\phi U, \phi Z) + \beta g(U, \phi Z)) \\ & + \eta(Z)(\alpha g(\phi U, \phi Y) + \beta g(U, \phi Y))), \end{aligned} \tag{2.04}$$

Now, substituting (2.01) in (2.04), we get (2.02). Hence the theorem. □

From (2.01), it follows that a Ricci-recurrent manifold can exhibit Ricci-symmetry, but only along certain specific directions. Substituting $Y$ and $Z$ with $\xi$ in (2.02), and using (1.22), we observe that the non-zero Ricci curvature of $M^3$ along the direction of the Reeb vector field $\xi$ necessitates the orthogonality of the vector field $U$ to the Ricci-recurrence vector field $\rho$. Consequently, from (2.01), we deduce that $M^3$ is Ricci-symmetric and non-flat along the Reeb vector field $\xi$. Furthermore, the vector field $\rho$, associated with the Ricci-recurrence 1-form $\omega$, remains orthogonal to the Ricci-recurrence direction defined by the vector field $U$ along $\xi$. This leads us to the following theorem:

**Theorem 2.2.** *In a three-dimensional Ricci-recurrent normal almost contact metric manifold $M^3$, the non-zero Ricci curvature along with Reeb vector field $\xi$ enforces that $M^3$ is Ricci-symmetric and non-flat along $\xi$, with the vector field $\rho$, that is associated with the Ricci-recurrence 1-form $\omega$, is everywhere orthogonal to the Ricci-recurrence direction given by the vector field $U$.*

Notice here that, from (2.01), (2.02) and theorem 2.2, $M^3$ is a proper Ricci recurrent manifold, only when no two of $Y$, $Z$ and $U$ assumes $\xi$. In view of this, we state the upcoming theorems for the vector fields in $TM^3 \setminus \{\xi\}$.

**Theorem 2.3.** *Suppose $(M^3, g)$ is a three-dimensional Ricci-recurrent normal almost contact metric manifold with Ricci-recurrence 1-form $\omega \not\equiv 0$. Then, Ricci tensor $S$ of $M^3$ satisfies the condition:*

$$2\omega(\xi)S(Y,Z) = g(\phi Y, \phi Z)(\xi r), \tag{2.05}$$

*$\forall\ Y, Z \in TM^3 \setminus \{\xi\}$, where $\xi$ is the Reeb vector field, $r$ is the scalar curvature and $\phi$ is the structure tensor field of type (1,1) in $M^3$.*

*Proof.* Replacing $Z$ by $\xi$ in (2.02) and using (1.01), we get (2.05). Hence the theorem. □

**Theorem 2.4.** *Suppose $(M^3, g)$ be a three-dimensional Ricci-recurrent normal almost contact metric manifold with Ricci-recurrence 1-form $\omega \not\equiv 0$. Then, Ricci tensor $S$ of $M^3$ satisfies the condition:*

$$\omega(U)S(Y,\xi) = (\frac{r}{2} + 3(\alpha^2 - \beta^2))(\alpha g(\phi^2 Y, U) - \beta g(\phi Y, U)), \tag{2.06}$$

*$\forall\ U, Y \in TM^3 \setminus \{\xi\}$, where $\xi$ is the Reeb vector field, $r$ is the scalar curvature, $\phi$ is the structure tensor field of type (1,1) in $M^3$ and $\alpha, \beta$ are the constants.*

*Proof.* In (2.02), replacing $Z$ by $\xi$, we get:

$$\omega(U)S(Y,\xi) = -(\frac{r}{2} + 3(\alpha^2 - \beta^2))(\eta(Z)(\alpha g(\phi U, \phi Y) + \beta g(U, \phi Y))). \tag{2.07}$$

Since

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.08}$$

in a three-dimensional Ricci-recurrent normal almost contact metric manifold, using this in (2.07), we get (2.06). Hence the theorem. □

## 3. IMMERSSION OF THE RICCI-RECURRENT NORMAL ALMOST CONTACT METRIC MANIFOLD OF DIMENSION THREE

Let $M$ and $M'$ be smooth manifolds of dimension $m$ and $m'$, respectively. Consider a smooth map $f : M \to M'$. For each point $x \in M$, the tangential map at $x$, denoted by $f_{*x} : T_x(M) \to T_{f(x)}(M')$, maps the tangent space at $x$ in $M$ to the tangent space at $f(x)$ in $M'$. The map $f$ is called an *immersion* if $f_{*x}$ is injective at each point $x \in M$. Let $M$ and $M'$ be Riemannian manifolds equipped with Riemannian metrics $g$ on $M$ and $g'$ on $M'$. An immersion $f : M \to M'$ between Riemannian manifolds $(M, g)$ and $(M', g')$ is called an *isometric immersion* if it preserves the Riemannian metric, meaning that the pullback of $g'$ by $f$ is equal to $g$ on $M$. Specifically, for each point $x \in M$, the tangential map $f_* : T_xM \to T_{f(x)}M'$ satisfies:

$$g(X, Y) = g'(f_*X, f_*Y)$$

for all tangent vectors $X, Y \in T_xM$. Isometric immersions are of particular interest because they allow the study of a lower-dimensional manifold $M$ within a higher-dimensional manifold $M'$ without altering its intrinsic geometry.

Some special type of immersions of contact and almost contact metric manifolds into higher-dimensional Riemannian spaces, particularly of constant curvature, were notably studied by **T. Takahashi and S. Tanno** [21], who focused on

*K-contact Riemannian manifolds*. Building on this, **Sujit Ghosh** [18] and **U. C. De, Ahmet Yildiz, and Avijit Sarkar** [25] studied on such types of immersions. The Gauss Formula in submanifold theory [12] [3] is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y), \tag{3.01}$$

where, $X$, $Y$ are any two vector field on $M$ which is immersed in $M'$, $B$ is the second fundamental form and $\widetilde{\nabla}$, $\nabla$ denote the covariant derivative with respect to Levi-Civita connection in $M'$ and $M$ respectively. From this point forward, we shall regard $M$ as $M^3$ and $M'$ as $N$, where $M^3$ is the three-dimensional Ricci recurrent normal almost contact metric manifold immersed in the four dimensional manifold $N$ of constant curvature 1. Also the Gauss equation for the immersion of $M^3$ in $N$ is given by

$$\begin{aligned} R(X,Y,Z,W) &= (g(X,W)g(Y,Z) - g(X,Z)g(Y,W)) \\ &\quad + (g(AX,W)g(AY,Z) - g(AX,Z)g(AY,W)) \end{aligned} \tag{3.02}$$

where, the $(1,1)$ tensor field $A$ is related to the second fundamental form $B$ by $B(X,Y) = g(AX,Y)$. Replacing $X$, $W$ by $e_i$ in (3.02) and summing up from $i = 1$ to 3, we get,

$$S(Y,Z) = 2g(Y,Z) + 3Hg(AY,Z) - g(A^2Y,Z) \tag{3.03}$$

By virtue of (3.03) we state and prove the following theorems:

**Theorem 3.1.** *Let $M^3$ be a three-dimensional Ricci-recurrent normal almost contact metric manifold isometrically immersed in a four-dimensional Riemannian manifold $N^4$ of constant curvature 1. Then, $\forall\ Y \in TM^3 \setminus \{\xi\}$, the following relationship holds:*

$$2\omega(\xi)(A^2 - 3HA - 2I)Y = (\xi r)\,(\phi^2 Y),$$

*where, $\omega \not\equiv 0$, $A$ is the shape operator, $H$ is the mean curvature, $I$ is the identity operator, $\xi$ is the Reeb vector field and $r$ is the scalar curvature.*

*Proof.* Multiplying (3.03) by $2\omega(\xi)$ and simplifying, we have

$$2\omega(\xi)S(Y,Z) = 2\omega(\xi)\left(g((2 + 3HA - A^2)Y,Z)\right).$$

Using (2.05) and (2.08), we get:

$$g(\phi^2Y,Z)\xi r = 2\omega(\xi)\left(g((A^2 - 3HA - 2)Y,Z)\right).$$

Hence the theorem. □

**Theorem 3.2.** *Suppose $M^3$ is a three-dimensional Ricci-recurrent normal almost contact metric manifold isometrically immersed in a four-dimensional Riemannian manifold $N^4$ of constant curvature 1. Then, $\forall\ U \in TM^3 \setminus \{\xi\}$, the following relationship holds:*

$$\omega(U)((2I + 2HA - A^2)\xi) = \left(\frac{r}{2} + 3(\alpha^2 - \beta^2)\right)(\alpha\phi^2 + \beta\phi)U, \tag{3.04}$$

*where $\omega \not\equiv 0$, $A$ is the shape operator, $H$ is the mean curvature, $I$ is the identity operator, $\xi$ is the Reeb vector field, $r$ is the scalar curvature, $\phi$ is the (1,1) tensor field and $\alpha$, $\beta$ are the constants.*

*Proof.* Multiplying (3.03) by $\omega(U)$, replacing $Z$ by $\xi$, using (2.06) and since $A$ is symmetric, we get (3.04). Hence the theorem. $\square$

From (3.04), we state the following corollary:

**Corollary 3.3.** *Suppose $M^3$ is a three-dimensional Ricci-recurrent normal almost contact metric manifold isometrically immersed in a four-dimensional Riemannian manifold $N^4$ of constant curvature 1. If the immersion is minimal, then $\forall$ $U \in TM^3 \setminus \{\xi\}$, the following relationship holds:*

$$\omega(U)((2I - A^2)\xi) = \left(\frac{r}{2} + 3(\alpha^2 - \beta^2)\right)(\alpha\phi^2 + \beta\phi)U, \tag{3.05}$$

*where $\omega \not\equiv 0$, $A$ is the shape operator, $I$ is the identity operator, $\xi$ is the Reeb vector field, $r$ is the scalar curvature, $\phi$ is the (1,1) tensor field and $\alpha$, $\beta$ are the constants.*

Notice that the operator $2I + 3HA - A^2$ that arises due to immersion plays a crucial role. The following theorem gives an expression for this operator.

**Theorem 3.4.** *Suppose a three-dimensional Ricci-Recurrent normal almost contact metric manifold $M^3$ is isometrically immersed in a four dimensional Riemannian manifold $N^4$ of constant curvature 1. If $\rho$ is the vector field associated with the Ricci-recurrence 1-form $\omega$, then $\forall$ $U \in TM^3 \setminus \{\xi\}$,*

$$2I + 3HA - A^2 = \frac{Ur}{2\omega(U)} \tag{3.06}$$

*where, $\omega \not\equiv 0$, $H$ is the mean curvature function, $A$ is the shape operator, $r$ is the scalar curvature and $I$ is the identity operator in $M^3$.*

*Proof.* Replacing $Y$ by $\phi Y$ and $Z$ by $\phi Z$ in (2.02), simplifying by using (1.01) and (3.03), we get

$$\frac{Ur}{2}(\phi^2 Y) = \omega(U)\phi(2I + 3HA - A^2)\phi Y.$$

Since, $A$ is symmetric, using (1.01), we notice that $g(\phi A\xi, \xi) = g(A\xi, \phi\xi) = g(\xi, A\phi\xi)$. By using this property, we get (3.06). Hence the theorem: $\square$

## CONCLUSION

In this study, we have examined the geometric and curvature properties of three-dimensional Ricci-recurrent normal almost contact metric manifolds and their isometric immersion in four-dimensional Riemannian manifolds of constant curvature 1. The derived relationships between the structure tensors and the scalar functions $\alpha$ and $\beta$ highlight the intrinsic geometry of these manifolds. Through the immersion framework, we demonstrated that the Reeb vector field $\xi$ plays a central role in determining the Ricci-symmetry and orthogonality of the Ricci-recurrence vector fields. Moreover, the connections between the second fundamental form, shape operator, and Ricci tensor were used to establish theorems about the curvature behavior of these manifolds during immersion. These findings contribute to a deeper understanding of the interaction between contact geometry and Riemannian submanifold theory, with potential implications for further exploration of higher-dimensional contact and metric structures.

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