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INTEGER PROGRAMMING FORMULATIONS AND PROBABILISTIC BOUNDS FOR SOME DOMINATION PARAMETERS

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Abstract. In this paper, we further study the concepts of hop domination and 2-step domination and introduce the concepts of restrained hop domination, total restrained hop domination, 2-step restrained domination, and total 2-step restrained domination in graphs. We then construct integer programming (IP) formulations and present probabilistic upper bounds for these domination parameters. IP is a powerful optimization technique widely applied in computer science to develop efficient algorithms.

Keywords. hop domination, 2-step domination, restrained domination, IP formulation, probabilistic bounds
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1. Introduction and Preliminaries

Domination in graphs is one of the extensively studied concept in graph theory. This concept has historical roots as early as 1850s when European chess enthusiast studied the problem of dominating queens. The mathematical study of dominating sets has become of interest to numerous authors, in which the concept has also been used for many different applications, such as wireless network topology design [23], wireless sensor network [3], hoc network [22], and many others. Different modifications related to domination in graphs have been studied by several authors, see [1, 2, 16, 17, 19, 20, 21]. The concept of total domination in graphs has also been introduced in [6]. One may refer to [10, 11] for detailed survey on domination parameters and [14] for thorough discussions on total domination.

Domination parameters have also been studied in the context of linear programming because of their fundamental importance in graph theory and their applicability to various optimization problems. These parameters measure the minimum number of vertices required to dominate a graph under specific conditions and are often formulated as integer programming problems. Since accurately

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determining domination numbers is typically NP-hard, linear programming formulations and their relaxations provide a powerful framework for analyzing and approximating these parameters. By framing domination problems as linear programs, researchers can derive useful bounds, develop efficient algorithms, and gain deeper insights into the structural properties of graphs. This approach also enables applications in network design, resource allocation, and other combinatorial optimization areas where domination is critical.

Integer linear programming (ILP or simply IP) on the other hand is a powerful optimization technique widely applied in computer science for solving problems that involve discrete decisions, such as selecting, assigning, or ordering elements under given constraints. In network design, IP can determine the minimum-cost set of links or nodes required to achieve desired connectivity, as in communication or sensor networks. In graph theory, IP models are used to solve problems like the minimum dominating set, vertex cover, graph coloring, and maximum independent set, all of which have applications in areas such as social network analysis, security, and resource placement.

Let $G = (V(G), E(G))$ be a simple undirected graph. A set $S \subseteq V(G)$ is a dominating set of G if every vertex outside S is adjacent to a vertex in S. The domination number of G is the smallest cardinality of a dominating set of G, and is denoted by $\gamma(G)$. If every vertex in G is adjacent to a vertex in S, then we say that S is a total dominating set of G. Similarly, the total domination number of G, denoted by $\gamma_t(G)$, is the smallest cardinality of a total dominating set of G.

In 1999, Domke et al. [7] initiated the study of restrained dominating set. This notion was further examined in [25]. A set $S \subseteq V(G)$ is a restrained dominating set of G if every vertex outside S is adjacent to a vertex in S as well as another vertex outside S. The restrained domination number of G is the smallest cardinality of a restrained dominating set of G, and is denoted by $\gamma_r(G)$. To illustrate the idea of this domination parameter, to maintain safety and support, each prisoner's location $(V(G) \setminus S)$ is monitored by at least a guard's location to maintain security, and each prisoner's location is seen by another prisoner's location to protect the rights of prisoners.

A new domination parameter called 2-step domination in graphs was introduced by Chartrand et al. [5] and further investigated in [4, 8, 15, 24]. Analogous to 2-step domination, the concept of hop domination in graphs was introduced by Natarajan and Ayyaswamy [18], which was further studied by some authors, see [12, 13]. For any two vertices u and v of G, the symbol $d_G(u, v)$ or simply $d(u, v)$, is the length of a shortest path connecting vertices u and v in G, which is also referred to as the distance between u and v. The degree of a vertex i in a graph G, denoted by $\deg_G(i)$ or $d_G(i)$, is the number of vertices adjacent to i and the smallest degree among the vertices of G is denoted by $\delta(G)$, or simply δ. The set of vertices adjacent to i in G is denoted by $N(i)$ and the set of vertices at a distance 2 from i in G is denoted by $N_2(i)$. The hop degree of a vertex i in a graph G, denoted by $\deg_h(i)$ or $d_h(i)$, is the number of vertices at distance 2 from i in G. The smallest hop-degree among the vertices of G is denoted by $\delta_h(G)$, or simply δ_h. A set $S \subseteq V(G)$ is a hop dominating set of G if for every $i \in V(G) \setminus S$, there exists $j \in S$ such that $d_G(i, j) = 2$. The hop domination number of G is

the smallest cardinality of a hop dominating set of G and is denoted by $\gamma_h(G)$. A set $S \subseteq V(G)$ is a 2-step dominating set of G if for every $i \in V(G)$, there exists $j \in S$ such that $d_G(i,j) = 2$. The 2-step domination number of G, denoted by $\gamma_{2\text{step}}(G)$, is the smallest cardinality of a 2-step dominating set of G.

In this paper, the concept of restrained hop dominating set, total restrained hop dominating set, 2-step restrained dominating set, and total 2-step restrained dominating set in graphs will be introduced. An integer programming (IP) formulation will be constructed for these newly defined domination parameters and sharp upper bounds will be provided using probabilistic methods.

2. IP Formulation

2.1. **Hop domination and 2-step domination problems.** From now onwards, let $G = (V, E)$ be a simple undirected graph with $V = \{1, 2, \ldots, n\}$. Decision variable x_i indicates whether vertex i belongs to a hop dominating set $S \subseteq V$, i.e.,

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } d(i,j) = 2 \\ 0 & \text{otherwise.} \end{cases}$$

An IP formulation for the hop dominating set problem (HDP) can be constructed as:

$$\min \sum_{i=1}^{n} x_i \tag{2.1}$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \geq 1, \qquad \forall i \in V \tag{2.2}$$

$$x_i \in \{0, 1\}, \qquad \forall i \in V \tag{2.3}$$

Theorem 2.1. *The optimal solution of the IP formulation for the HDP is equal to the hop domination number of G.*

Next, we construct an IP formulation for the 2-step dominating set problem (2SDP). Decision variable x_i indicates whether vertex i belongs to a 2-step dominating set. Since a 2-step dominating set is a special case of a hop dominating set, we just replace constraint (2.2) with constraint (2.5). An IP formulation for the 2SDP can be constructed as:

$$\min \sum_{i=1}^{n} x_i \tag{2.4}$$

subject to:

$$\sum_{j=1,\ j\neq i}^{n} a_{ij}x_j \geq 1, \qquad \forall i \in V \tag{2.5}$$

$$x_i \in \{0,1\}, \qquad \forall i \in V \tag{2.6}$$

Theorem 2.2. *The optimal solution of the IP formulation for the 2SDP is equal to the 2-step domination number of G.*

2.2. **Restrained hop and total restrained hop domination problems.** A hop dominating set $S \subseteq V(G)$ is a *restrained hop dominating set* of G if for every $u \in V(G) \setminus S$, there exists $v \in V(G) \setminus S$ such that $d(u,v) = 1$. We need to explicitly state here that a restrained hop dominating set of G has to satisfy two independent properties: (1) it is a hop dominating set and (2) the subgraph induced by $V(G) \setminus S$ has no isolated vertices. For the construction of an IP formulation for restrained hop dominating set problem (RHDP), decision variable x_i indicates whether vertex i belongs to a restrained hop dominating set. Define

$$b_{ij} = \begin{cases} -1 & \text{if } i = j \\ 1 & \text{if } (i,j) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Following the techniques employed in [9], an IP formulation for the RHDP can be constructed as:

$$\min \sum_{i=1}^{n} x_i \tag{2.7}$$

subject to:

$$\sum_{j=1}^{n} a_{ij}x_j \geq 1, \qquad \forall i \in V \tag{2.8}$$

$$\sum_{j=1}^{n} b_{ij}x_j < \deg_G(i), \qquad \forall i \in V \tag{2.9}$$

$$x_i \in \{0,1\}, \qquad \forall i \in V \tag{2.10}$$

Theorem 2.3. *The optimal solution of the IP formulation for the RHDP is equal to the restrained hop domination number of G.*

A 2-step dominating set $S \subseteq V(G)$ is a *total restrained hop dominating set* of G if for every $u \in V(G)\setminus S$, there exists $v \in V(G)\setminus S$ such that $d(u,v) = 1$. Since the total restrained hop dominating set is a special case for restrained hop dominating set, we just replace constraint (2.8) by constraint (2.12). Decision variable x_i indicates whether vertex i belongs to a total restrained hop dominating set. An IP formulation for the total restrained hop dominating set problem (TRHDP) can be constructed as:

$$\min \sum_{i=1}^{n} x_i \tag{2.11}$$

subject to:

$$\sum_{j=1,\ j\neq i}^{n} a_{ij}x_j \geq 1, \qquad \forall i \in V \tag{2.12}$$

$$\sum_{j=1}^{n} b_{ij}x_j < \deg_G(i), \qquad \forall i \in V \tag{2.13}$$

$$x_i \in \{0,1\}, \qquad \forall i \in V \tag{2.14}$$

Theorem 2.4. *The optimal solution of the IP formulation for the TRHDP is equal to the total restrained hop domination number of G.*

2.3. **2-step restrained and total 2-step restrained domination problems.** Finally, we construct an IP formulation for the 2-step restrained dominating set problem (2SRDP). If G is a graph, then we denote by $\mathrm{Dist}(G;2)$ the graph consisting of the vertex set V and edge set $\{uv : d_G(u,v) = 2\}$.

A hop dominating set $S \subseteq V(G)$ is a *2-step restrained dominating set* of G if for every $u \in V(G) \setminus S$, there exists $v \in V(G) \setminus S$ such that $d(u,v) = 2$. Decision variables x_i indicates whether a vertex i belongs to a 2-step restrained dominating set. Define

$$c_{ij} = \begin{cases} -1 & \text{if } i = j \\ 1 & \text{if } d(i,j) = 2 \\ 0 & \text{otherwise.} \end{cases}$$

An IP formulation for the 2SRDP can be constructed as:

$$\min \sum_{i=1}^{n} x_i \tag{2.15}$$

subject to:

$$\sum_{j=1}^{n} a_{ij}x_j \geq 1, \qquad \forall i \in V \tag{2.16}$$

$$\sum_{j=1}^{n} c_{ij}x_j < \deg_{\mathrm{Dist}(G;2)}(i), \qquad \forall i \in V \tag{2.17}$$

$$x_i \in \{0,1\}, \qquad \forall i \in V \tag{2.18}$$

Theorem 2.5. *The optimal solution of the IP formulation for the 2SRDP is equal to the 2-step restrained domination number of G.*

A 2-step dominating set $S \subseteq V(G)$ is a *total 2-step restrained dominating set* of G if for every $u \in V(G) \setminus S$, there exists $v \in V(G) \setminus S$ such that $d(u,v) = 2$. Decision variable x_i indicates whether a vertex i belongs to a total 2-step restrained dominating set. Replacing constraint (2.16) with constraint (2.20), an IP formulation for the total 2-step restrained dominating set problem (T2SRDP) can be constructed as:

$$\min \sum_{i=1}^{n} x_i \tag{2.19}$$

subject to:

$$\sum_{j=1,\ j\neq i}^{n} a_{ij}x_j \geq 1, \qquad \forall i \in V \tag{2.20}$$

$$\sum_{j=1}^{n} c_{ij}x_j < \deg_{\mathrm{Dist}(G;2)}(i), \qquad \forall i \in V \tag{2.21}$$

$$x_i \in \{0,1\}, \qquad \forall i \in V \tag{2.22}$$

Theorem 2.6. *The optimal solution of the IP formulation for the T2SRDP is equal to the total 2-step restrained domination number of G.*

3. Probabilistic Bounds

In this section, we present probabilistic upper bounds for $\gamma_{2\mathrm{step}}(G)$, $\gamma_{rh}(G)$, $\gamma_{trh}(G)$, $\gamma_{2sr}(G)$, and $\gamma_{t2sr}(G)$.

3.1. 2-step domination number.

Before we present the probabilistic upper bound for $\gamma_{2\mathrm{step}}(G)$, we shall consider first the following two known results:

Theorem 3.1. [14] *If G is a graph with minimum degree δ, then*

$$\gamma_t(G) \leq \frac{\ln \delta + 1}{\delta} n.$$

Observation 3.2. [12, p.915] *If G is a graph, then the following hold:*

(i) $\gamma_h(G) = \gamma(\mathrm{Dist}(G;2))$.
(ii) $\gamma_{2step}(G) = \gamma_t(\mathrm{Dist}(G;2))$.

In view of Theorem 3.1 and Observation 3.2, we have the following result.

Theorem 3.3. *If G is a graph of order n with $\delta_h := \delta_h(G) \geq 1$, then*

$$\gamma_{2step}(G) \leq \frac{\ln \delta_h + 1}{\delta_h} n.$$

Proof. Note that

$$\begin{aligned} \gamma_{2\mathrm{step}}(G) &= \gamma_t(\mathrm{Dist}(G;2)) \\ &\leq \frac{\ln \delta(\mathrm{Dist}(G;2)) + 1}{\delta(\mathrm{Dist}(G;2))} n \\ &\leq \frac{\ln \delta_h + 1}{\delta_h} n. \end{aligned}$$

This completes the proof of the theorem. □

3.2. Restrained hop and total restrained hop domination numbers.

Given a graph G, a *matching* M in G is a set of pairwise non-adjacent edges, that is, no two edges share common vertices. A *maximum matching*, also known as *maximum-cardinality matching*, is a matching that contains the largest possible number of edges. There may be many maximum matchings. The *matching number* of a graph G, denoted by $\nu(G)$ is the size of a maximum matching. This number is also called the *edge independence number*.

A *perfect matching*, also known as *complete matching*, is a matching that matches all vertices of the graph, that is, a matching is perfect if every vertex of the graph is incident to an edge of the matching.

A *near-perfect matching*, or *near-complete matching*, is a matching in which exactly one vertex is unmatched. It is not difficult to see that a graph can only contain a near-perfect matching when the graph has an odd number of vertices.

The proofs of the following three results are analogous in [26].

Theorem 3.4. *If G is a graph of order n with $\delta_h \geq 1$ and $\nu := \nu(G) \geq \gamma_h(G)$, then*

$$\gamma_{rh}(G) \leq \frac{2\ln(\delta_h + 1) + \delta_h + 3}{\delta_h + 1} n - 2\nu$$

and

$$\gamma_{trh}(G) \leq \frac{2\ln(\delta_h) + \delta_h + 2}{\delta_h} n - 2\nu.$$

Proof. Let D be a minimum hop dominating set of G. Then $|D| = \gamma_h(G) =: \gamma_h$. Let M be a maximum matching of G. Then $\nu := \nu(G) = |M|$. Let $M = \{e_1, e_2, \ldots, e_\nu\}$. If all of the end vertices of the edges in M are not in D, we may replace an edge of M with an edge such that one of the end vertex is in D. Hence, we may assume that the first k edges of M have at least one end vertex contained in D. It follows that $\nu - k$ edges in M have both end vertices contained in $V(G) \setminus D$. Since $k \leq |D| = \gamma_h(G)$, $\nu - \gamma_h \leq \nu - k$. Since $\nu \geq \gamma_h$, at least $\nu - \gamma_h$ edges in M have both end vertices contained in $V(G) \setminus D$. Let $D' = \{u \in V(G) \setminus D : u \text{ is not an end vertex of the } \nu - k \text{ edges in } M\}$. Let $S := D \cup D'$. Then S is a restrained hop dominating set of G. Now,

$$\begin{aligned} \gamma_{rh}(G) &\leq |S| \\ &= |D \cup D'| \\ &= n - 2(\nu - k) \\ &\leq n - 2(\nu - \gamma_h) \\ &= n + 2\gamma_h - 2\nu. \end{aligned}$$

In view of [12, Theorem 12, p.926], $\gamma_h \leq \dfrac{\ln(\delta_h + 1) + 1}{\delta_h + 1} n$. Hence,

$$\gamma_{rh}(G) \leq \frac{2\ln(\delta_h + 1) + \delta_h + 3}{\delta_h + 1} n - 2\nu.$$

This proves the first part of the theorem.

Next, let $D^{(t)}$ be a minimum 2-step dominating set of G. Then $|D^{(t)}| = \gamma_{2\text{step}}(G) := \gamma_{2\text{step}}$. Using the same technique employed above, we can construct a total restrained hop dominating set $S_t := D^{(t)} \cup D'$ of G so that

$$\gamma_{trh}(G) \leq n + 2\gamma_{2\text{step}} - 2\nu.$$

By Theorem 3.3, $\gamma_{2\text{step}} \leq \dfrac{\ln \delta_h + 1}{\delta_h} n$. Hence,

$$\gamma_{trh}(G) \leq \frac{2\ln \delta_h + \delta_h + 2}{\delta_h} n - 2\nu.$$

This completes the proof of the theorem. □

Corollary 3.5. *If a graph G of order n has a perfect matching with $\nu(G) \geq \gamma_h(G)$, then*

$$\gamma_{rh}(G) \leq \frac{\ln(\delta_h + 1) + 1}{\delta_h + 1} 2n$$

and

$$\gamma_{trh}(G) \leq \frac{\ln \delta_h + 1}{\delta_h} 2n.$$

Corollary 3.6. *If a graph G of order n has a near-perfect matching with $\nu(G) \geq \gamma_h(G)$, then*

$$\gamma_{rh}(G) \leq \frac{\ln(\delta_h + 1) + 1}{\delta_h + 1} 2n + 1$$

and

$$\gamma_{trh}(G) \leq \frac{\ln \delta_h + 1}{\delta_h} 2n + 1$$

Next, we improve the assumption in Corollary 3.5. From now onwards, let $\mathbb{R}$ be the set of real numbers and let $[0,1]^n = \{\boldsymbol{p} = (p_1, \ldots, p_n) : p_i \in \mathbb{R}, 0 \leq p_i \leq 1\}$.

Let $f_{rh} : [0,1]^n \to \mathbb{R}$ be the function defined by

$$\begin{aligned} f_{rh}(\boldsymbol{p}) &= \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} (1 - p_i) \prod_{j \in N_2(i)} (1 - p_j) \\ &\quad + \sum_{i=1}^{n} (1 - p_i) \left(1 - (1 - p_i) \prod_{j \in N_2(i)} (1 - p_j) \right) \\ &\quad \times \prod_{j \in N(i)} \left[p_j + (1 - p_j) \prod_{k \in N_2(j)} (1 - p_k) \right]. \end{aligned}$$

Theorem 3.7. *If G is a graph of order n, then*

$$\gamma_{rh}(G) = \min_{\boldsymbol{p} \in [0,1]^n} f_{rh}(\boldsymbol{p})$$

Proof. Let G be a graph with vertex set $V = \{1, 2, \ldots, n\}$. We pick randomly and independently each vertex $i \in V$ with probability p_i, where $0 \leq p_i \leq 1$, to form a set $X \subseteq V$, that is, $\mathbb{P}(i \in X) := \mathbb{P}(\{i \in V : i \in X\}) = p_i$. Let $Z = \{i \notin X : N_2(i) \cap X = \varnothing\}$ and $Y = \{i \notin X \cup Z : N(i) \subseteq X \cup Z\}$. Consider $D = X \cup Z \cup Y$.

First, we show that D is a restrained hop dominating set of G. Let $u \in V \setminus D$. Then $u \notin X \cup Z \cup Y$ so that $N_2(u) \cap X \neq \varnothing$. This means that there exists $v \in X \subset D$ such that $d_G(u, v) = 2$. Also, $N(u) \nsubseteq X \cup Z$. It follows that there exists $s \in N(u)$ such that $s \notin X \cup Z$. If $s \in Y$, then $N(s) \subseteq X \cup Z$, which implies that $u \in N(s) \subseteq X \cup Z$, a contradiction. Hence, $s \notin Y$. Thus, $s \in V \setminus D$. Accordingly, D is a restrained hop dominating set of G.

Define $X_i := X(i)$ by

$$X_i = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, define $Z_i := Z(i)$ (resp., $Y_i := Y(i)$) by

$$Z_i \text{ (resp., } Y_i) = \begin{cases} 1 & \text{if } i \in Z \text{ (resp., } i \in Y) \\ 0 & \text{otherwise.} \end{cases}$$

Then $|X| = \sum_{i=1}^{n} X_i$, $|Z| = \sum_{i=1}^{n} Z_i$, and $|Y| = \sum_{i=1}^{n} Y_i$. Note that

$$\mathbb{E}[X_i] = \mathbb{P}(i \in X) = p_i,$$

$$\mathbb{E}[Z_i] = \mathbb{P}(i \in Z) = (1 - p_i) \prod_{j \in N_2(i)} (1 - p_j),$$

and

$$\begin{aligned} & \mathbb{E}[Y_i] \\ = \ & \mathbb{P}(i \in Y) \\ = \ & (1 - p_i) \left(1 - (1 - p_i) \prod_{j \in N_2(i)} (1 - p_j) \right) \prod_{j \in N(i)} \left[p_j + (1 - p_j) \prod_{k \in N_2(j)} (1 - p_k) \right]. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}\left[|D|\right] &= \mathbb{E}\left[|X|\right] + \mathbb{E}\left[|Z|\right] + + \mathbb{E}\left[|Y|\right] \\ &= \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} (1 - p_i) \prod_{j \in N_2(i)} (1 - p_j) \\ &\quad + \sum_{i=1}^{n} (1 - p_i) \left(1 - (1 - p_i) \prod_{j \in N_2(i)} (1 - p_j) \right) \\ &\quad \times \prod_{j \in N(i)} \left[p_j + (1 - p_j) \prod_{k \in N_2(j)} (1 - p_k) \right] \\ &= f_{rh}(p_1, p_2, \ldots, p_n). \end{aligned}$$

This means that there exists a restrained hop dominating set of G of cardinality at most $\mathbb{E}[|D|]$. Thus, $\gamma_{rh}(G) \leq \min_{\boldsymbol{p} \in [0,1]^n} f_{rh}(\boldsymbol{p})$.

Next, let D' be a minimum restrained hop dominating set of G. Then $|D'| = \gamma_{rh}(G)$. Let $\boldsymbol{p}' = (p'_1, p'_2, \ldots, p'_n)$, where

$$p'_i = \begin{cases} 1 & \text{if } i \in D' \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$f_{rh}(\boldsymbol{p}') = \sum_{i=1}^{n} p'_i = |D'| = \gamma_{rh}(G).$$

Accordingly, $\gamma_{rh}(G) = \min_{\boldsymbol{p}\in[0,1]^n} f_{rh}(\boldsymbol{p})$. □

Theorem 3.8. *If G is a graph of order n with $\delta_h \geq 1$, then*

$$\gamma_{rh}(G) \leq \frac{\ln(\delta_h+1)+1}{\delta_h+1}2n.$$

Proof. Let $\boldsymbol{p} = (p, p, \ldots, p) \in [0,1]^n$. For each $i \in \{1, 2, \ldots, n\}$, $\delta_h \leq d_h(i)$ and $1 - x \leq e^{-x}$ for $x \in \mathbb{R}$, and we have

$$\begin{aligned}
f_{rh}(\boldsymbol{p}) &= \sum_{i=1}^{n} p + \sum_{i=1}^{n}(1-p)^{d_h(i)+1} \\
&\quad + \sum_{i=1}^{n}(1-p)\left(1-(1-p)^{d_h(i)+1}\right)\left(p+(1-p)^{d_h(i)+1}\right)^{\deg(i)} \\
&\leq np + n(1-p)^{\delta_h+1} + n(1-p)\left(p+(1-p)^{\delta_h+1}\right)^{\delta} \\
&\leq np + n(1-p)^{\delta_h+1} + n(1-p)\left(p+(1-p)^{\delta_h+1}\right) \\
&\leq np + n(1-p)^{\delta_h+1} + np(1-p) + n(1-p)^{\delta_h+2} \\
&\leq 2np + 2n(1-p)^{\delta_h+1} \\
&\leq 2np + 2ne^{-p(\delta_h+1)}.
\end{aligned}$$

Note that the function $g(p) = 2np + 2ne^{-p(\delta_h+1)}$, $p \in [0,1]$, has an absolute minimum value at $p = \frac{\ln(\delta_h+1)}{\delta_h+1}$. Observe that $0 < \frac{\ln(\delta_h+1)}{\delta_h+1} < 1$. Let $\boldsymbol{p}' = (p', p', \ldots, p') \in [0,1]^n$, where $p' = \frac{\ln(\delta_h+1)}{\delta_h+1}$. Then by Theorem 3.7,

$$\gamma_{rh}(G) \leq f(\boldsymbol{p}') \leq 2np' + 2ne^{-p'(\delta_h+1)} = \frac{\ln(\delta_h+1)+1}{\delta_h+1}2n.$$

This completes the proof of the theorem. □

Next, we improve the upper bound in Theorem 3.8 but we need to put additional assumption. This technique is also used in [26].

Theorem 3.9. *If G be a graph of order n with $\delta, \delta_h \geq 1$ and $n < \frac{\delta\delta_h}{\ln\delta_h+1}$, then*

$$\gamma_{rh}(G) \leq \frac{\ln(\delta_h+1)+1}{\delta_h+1}n$$

and

$$\gamma_{trh}(G) \leq \frac{\ln\delta_h+1}{\delta_h}n$$

Proof. In view of [12, Theorem 12, p.926], let D be a hop dominating set of G with

$$|D| \leq \frac{\ln(\delta_h+1)+1}{\delta_h+1}n.$$

Since $n < \dfrac{\delta\delta_h}{\ln\delta_h+1}$, $\delta > \dfrac{\ln\delta_h+1}{\delta_h}n$. Let $u \in V(G)\setminus D$. Then

$$\deg_G(u) \geq \delta > \frac{\ln\delta_h+1}{\delta_h}n > \frac{\ln(\delta_h+1)+1}{\delta_h+1}n = |D|.$$

This means that there exists $v \in V(G)\setminus D$ such that $uv \in E(G)$. Thus, D is a restrained hop dominating set of G. Accordingly,

$$\gamma_{rh}(G) \leq \frac{\ln(\delta_h+1)+1}{\delta_h+1}n.$$

This proves the first part of the theorem.

Next, in view of Theorem 3.3, let $D^{(t)}$ be a 2-step dominating set of G with

$$|D^{(t)}| \leq \frac{\ln\delta_h+1}{\delta_h}n.$$

Let $u \in V(G)\setminus D$. Then

$$\deg_G(u) \geq \delta > \frac{\ln\delta_h+1}{\delta_h}n = |D^{(t)}|.$$

This means that there exists $v \in V(G)\setminus D$ such that $uv \in E(G)$. Thus, $D^{(t)}$ is a total restrained hop dominating set of G. Accordingly,

$$\gamma_{trh}(G) \leq \frac{\ln\delta_h+1}{\delta_h}n.$$

This completes the proof. □

3.3. **2-step restrained and total 2-step restrained domination numbers.** Given a graph G, a *hop matching* H in G is a set of paths of size two such that no two paths share a common end vertex. A *maximum hop matching* is a hop matching that contains the largest possible number of paths of size two. There may be many maximum hop matchings. The *hop matching number* of a graph G, denoted by $\nu_h(G)$, is the cardinality of a maximum hop matching.

A *perfect hop matching* or *complete hop matching*, is a hop matching such that every vertex of the graph is an end vertex of an element of the hop matching.

A *near-perfect hop matching*, or *near-complete hop matching*, is a hop matching such that exactly one vertex of the graph is not an end vertex of an element of the hop matching.

Theorem 3.10. *If G is a graph of order n with $\delta_h \geq 1$ and $\mathrm{Dist}(G;2)$ has no isolated vertex, then*

$$\gamma_{2sr}(G) \leq \frac{2\ln(\delta_h+1)+\delta_h+3}{\delta_h+1}n - 2\nu_h(G)$$

and

$$\gamma_{t2sr}(G) \leq \frac{2\ln\delta_h+\delta_h+2}{\delta_h}n - 2\nu_h(G)$$

Proof. Let D be a minimum hop dominating set of G. Then $|D| = \gamma_h(G) =: \gamma_h$. Let H be a maximum hop matching of G. Then $\nu_h := \nu_h(G) = |H|$. Let $H = \{P_2^1, P_2^2, \ldots, P_2^{\nu_h}\}$. If all of the end vertices of the elements in H are not in D, we may replace an element of H with a path of size two such that one of the end vertex is in D. Hence, we may assume that the first k paths in H have at least one end vertex contained in D. It follows that $\nu_h - k$ paths in H have both end vertices contained in $V(G) \setminus D$. Since $k \leq |D| = \gamma_h(G)$, $\nu_h - \gamma_h \leq \nu_h - k$. Since $\mathrm{Dist}(G;2)$ has no isolated vertex, by [13, Theorem 2, p.2314], $\nu_h \geq \gamma_h$ so that at least $\nu_h - \gamma_h$ paths in H have both end vertices contained in $V(G) \setminus D$. Let $D' = \{u \in V(G) \setminus D : u$ is not an end vertex of the $\nu_h - k$ paths in $H\}$. Let $S := D \cup D'$. Then S is a 2-step restrained dominating set of G. Now,

$$\begin{aligned} \gamma_{2sr}(G) &\leq |S| \\ &= |D \cup D'| \\ &= n - 2[\nu_h - k] \\ &\leq n - 2[\nu_h - \gamma_h] \\ &= n + 2\gamma_h - 2\nu_h. \end{aligned}$$

In view of [12, Theorem 12, p.926], $\gamma_h \leq \dfrac{\ln(\delta_h + 1) + 1}{\delta_h + 1} n$. Hence,

$$\gamma_{2sr}(G) \leq \frac{2\ln(\delta_h + 1) + \delta_h + 3}{\delta_h + 1} n - 2\nu_h.$$

This proves the first part of the theorem.

Next, let $D^{(t)}$ be a minimum 2-step dominating set of G. Then $|D^{(t)}| = \gamma_{2\mathrm{step}}(G) := \gamma_{2\mathrm{step}}$. Using the same technique employed above, we can construct a total 2-step restrained dominating set $S_t := D^{(t)} \cup D'$ of G so that

$$\gamma_{t2sr}(G) \leq n + 2\gamma_{2\mathrm{step}} - 2\nu_h.$$

By Theorem 3.3, $\gamma_{2\mathrm{step}} \leq \dfrac{\ln \delta_h + 1}{\delta_h} n$. Hence,

$$\gamma_{t2sr}(G) \leq \frac{2\ln \delta_h + \delta_h + 2}{\delta_h} n - 2\nu_h.$$

□

Corollary 3.11. *If a graph G of order n has a perfect hop matching, then*

$$\gamma_{2sr}(G) \leq \frac{\ln(\delta_h + 1) + 1}{\delta_h + 1} 2n$$

and

$$\gamma_{t2sr}(G) \leq \frac{\ln \delta_h + 1}{\delta_h} 2n.$$

Corollary 3.12. *If a graph G of order n has a near-perfect hop matching, then*

$$\gamma_{2sr}(G) \leq \frac{\ln(\delta_h + 1) + 1}{\delta_h + 1} 2n + 1$$

and

$$\gamma_{t2sr}(G) \leq \frac{\ln \delta_h + 1}{\delta_h} 2n + 1$$

Next, we improve the assumption in Corollary 3.11. Let $f_{2sr} : [0,1]^n \to \mathbb{R}$ be the function defined by

$$\begin{aligned} f_{2sr}(\boldsymbol{p}) &= \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} (1-p_i) \prod_{j \in N_2(i)} (1-p_j) \\ &\quad + \sum_{i=1}^{n} (1-p_i) \left(1 - (1-p_i) \prod_{j \in N_2(i)} (1-p_j) \right) \\ &\quad \times \prod_{j \in N_2(i)} \left[p_j + (1-p_j) \prod_{k \in N_2(j)} (1-p_k) \right]. \end{aligned}$$

The proof of the following result is similar with Theorem 3.7 with $Y = \{i \notin X \cup Z : N(i) \subseteq X \cup Z\}$ replaced with $Y = \{i \notin X \cup Z : N_2(i) \subseteq X \cup Z\}$.

Theorem 3.13. *If G is a graph of order n, then*

$$\gamma_{2sr}(G) = \min_{\boldsymbol{p} \in [0,1]^n} f_{2sr}(\boldsymbol{p})$$

Theorem 3.14. *If G is a graph of order n with $\delta_h \geq 1$, then*

$$\gamma_{2sr}(G) \leq \frac{\ln(\delta_h + 1) + 1}{\delta_h + 1} 2n.$$

Similar with section 3.2, the upper bound in Theorem 3.8 can be improved but we need to put additional assumption.

Theorem 3.15. *If G be a graph of order n with $\delta_h \geq 1$ and $n < \dfrac{\delta_h^2}{\ln \delta_h + 1}$, then*

$$\gamma_{2sr}(G) \leq \frac{\ln(\delta_h + 1) + 1}{\delta_h + 1} n$$

and

$$\gamma_{t2sr}(G) \leq \frac{\ln \delta_h + 1}{\delta_h} n.$$

4. Conclusion and Recommendations

This paper has introduced the concepts of restrained hop, total restrined hop, 2-step restrained, and total 2-step restrained dominating sets and constructed their corresponding IP formulations. Sharp upper bounds have also been provided using probabilistic methods. A worthwhile direction for further study is to consider the complexity of these domination parameters.

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