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EXISTENCE OF RENORMALIZED SOLUTIONS FOR FOURIER BOUNDARY-VALUE PROBLEMS INVOLVING VARIABLE EXPONENT AND MEASURE DATA

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Abstract. In this manuscript, we establish an existence result in the sense of renormalized solutions for some elliptic anisotropic Leray-Lions problems including Fourier boundary condition, variable exponent, measure data and some nonlinearities that are Carathéodory and satisfy only some growth conditions.

Keywords. Renormalized solution, anisotropic elliptic equations, anisotropic Sobolev spaces, Fourier boundary condition, measure data, variable exponents.
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1. Introduction and assumptions

Recently, the study of problems involving anisotropic variable exponent is a very interesting topic. The intensive studying of such problems is due to their applications in mathematical modeling of important physical and computer science problems including image processing or robotics [2, 15], elastic mechanics [35], electromagnetism [8, 33], porous media [6, 21], nonstandard growth [4, 29, 34] and electrorheological fluids [3, 5, 20, 31, 32]. It should be mentioned that most of the previous works focused on the Dirichlet or Neumann homogeneous boundary conditions (see [11, 12, 26]). In this note, we deal with existence of renormalized solutions for Fourier boundary-value problems of the form

$$\begin{aligned} -\mathcal{A}u + \mathcal{N}(x,u,\nabla u) + |u|^{\pi(x)-2}u = \mathcal{R}(x,u,\nabla u) + \mu - \mathrm{div}\phi(u) \qquad \text{in } \Omega, \\ \mathcal{B}u + \kappa u = h \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω represents a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ and κ denotes a strictly positive real constant. $\mathcal{A}$ is a Leray-Lions type operator acting from $W^{1,\vec{q}(.)}(\Omega)$ into its dual denote $W^{-1,\vec{q}'(.)}(\Omega)$, defined by

$$\mathcal{A}u = \sum_{i=1}^{N} \partial_{x_i} a_i\big(x,u,\nabla u\big) \tag{1.2}$$

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and for $\nu = (\nu^1, \nu^2, \cdots, \nu^N)$ the unit outward normal on $\partial\Omega$, $\mathcal{B}$ is defined by

$$\mathcal{B}u = \sum_{i=1}^{N} a_i\Big(x, u, \nabla u\Big)\nu^i. \tag{1.3}$$

For our study, we assume that for any $i = 1, \ldots, N$, $a_i : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying

$$\exists\ \alpha > 0, \quad a_i(x, s, \xi).\xi_i \geq \alpha|\xi_i|^{q_i(x)}, \tag{1.4}$$

$$\exists\ \beta > 0, \quad |a_i(x, s, \xi)| \leq \beta\big(j_i(x) + |s|^{q_i(x)-1} + |\xi_i|^{q_i(x)-1}\big) \tag{1.5}$$

and

$$\big(a_i(x, s, \xi) - a_i(x, s, \eta)\big)(\xi_i - \eta_i) > 0 \quad \text{for } \xi_i \neq \eta_i \tag{1.6}$$

for almost every $x \in \overline{\Omega}$ and for every $(s, \xi, \eta) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, where j_i is a positive function in $L^{q_i'(.)}(\Omega)$.
The function ϕ is such that

$$\phi = (\phi_1, \cdots, \phi_N) \in C^0(\mathbb{R}, \mathbb{R}^N), \tag{1.7}$$

and the nonlinearities $\mathcal{N}, \mathcal{R} : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are Carathéodory functions verifying the growth conditions

$$|\mathcal{N}(x, s, \xi)| \leq \gamma_0(x) + b(|s|)\sum_{i=1}^{N} |\xi_i|^{q_i(x)}, \tag{1.8}$$

and

$$|\mathcal{R}(x, s, \xi)| \leq \rho(x) + |s|^{\delta_0(x)} + \sum_{i=1}^{N} |\xi_i|^{\delta_i(x)}, \tag{1.9}$$

where $\gamma_0(.),\ \rho(.) : \Omega \longrightarrow \mathbb{R}_+$ are measurable functions in $L^1(\Omega)$ and $b(\cdot) : \mathbb{R} \to \mathbb{R}^+$ is assumed to be a decreasing function that lying in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. $\delta(.) : \Omega \longrightarrow \mathbb{R}^{N+1}$ is a vector function satisfying

$$0 \leq \delta_0(x) < \pi(x) - 1$$

and

$$0 \leq \delta_i(x) < \frac{q_i(x)(\pi(x) - 1)}{\pi(x) + 1} \quad \text{a. e. in } \Omega \text{ for any } i = 0, 1, \ldots, N.$$

Also, μ is a measure datum such as

$$\mu = f - \operatorname{div}(g) \quad \text{in } L^1(\Omega) + W^{-1,\vec{q}'(x)}(\overline{\Omega}) \tag{1.10}$$

and

$$h \in L^1(\partial\Omega). \tag{1.11}$$

Essentially, we restrict our attention to point out existence result to Problem (1.1) under Fourier boundary condition which makes our study more complex and interesting. Our approach is based on an approximation method and the use of some suitable a priori estimates. We deal with the notion of renormalized solutions proposed by Dal Maso et al [16] for the study of general degenerate elliptic equations whose data is a measure, and then extended by Bidaut-Véron [13]. Our main contribution is to combine tools, techniques and ideas developed in [10] and [22] to handle with some difficulties due to the incorporation of the Fourier boundary condition in the definition of the considered notion of solutions. It should be mentioned that the result of our investigation extend to the case of Fourier boundary-value problems including measure data the existence result of Benboubker et al. [10]. More precisely, these authors considered the following strongly nonlinear elliptic problem with Neumann homogeneous boundary conditions

$$\begin{gathered} -\operatorname{div} a(x,u,\nabla u) + g(x,u,\nabla u) + |u|^{q(.)-2}u = f(x,u,\nabla u) \quad \text{in } \Omega, \\ \sum_{i=1}^{N} a_i\big(x,u,\nabla u\big)\cdot \eta_i = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.12}$$

Moreover, it enables us also to generalize in the sense of renormalized solutions the existence result of Houede et al. [22] where the authors proved existence of entropy solutions for the following strongly nonlinear elliptic problem

$$\begin{gathered} -Au + g(x,u,\nabla u) + \delta |u|^{p_0(x)-2}u = \mu - \operatorname{div}\phi(u) \quad \text{in } \ \Omega, \\ Bu + \lambda u = h \quad \text{on} \quad \partial\Omega. \end{gathered} \tag{1.13}$$

The remaining part of this paper is organized as follows: In Section 2, we recall some notations and preliminaries about the anisotropic variable exponent Sobolev spaces. We also present our main result for existence of renormalized solutions to Problem (1.1). In Section 3, we give the main steps of our approximation method to establish the proof of our result.

2. Preliminaries and main results

Let Ω be a bounded and open subset of $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega$ and $\operatorname{meas}(\Omega) > 0$. We recall some known results on variable exponent Lebesgue and Sobolev spaces. See [12, 14, 17, 18, 19, 23, 24, 25] and references therein for more details. Define

$$C_+(\overline{\Omega}) = \left\{ q : \overline{\Omega} \longrightarrow \mathbb{R} \text{ measurable and } 1 < q^- \leq q^+ < N \right\},$$

where

$$q^- = \operatorname{ess\,inf}\{q(x) \mid x \in \overline{\Omega}\} \qquad \text{and} \qquad q^+ = \operatorname{ess\,sup}\{q(x) \mid x \in \overline{\Omega}\}.$$

For any $q \in C_+(\overline{\Omega})$,

$$L^{q(.)}(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{q(x)}dx \ < \ \infty \right\}$$

is the variable exponent Lebesgue space endowed with the so-called Luxemburg norm

$$|u|_{q(.)} := \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\}.$$

The mapping $\rho_{q(.)} : L^{q(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{q(.)}(u) := \int_\Omega |u(x)|^{q(x)}dx$$

is the convex modular of the $L^{q(.)}(\Omega)$ space and satisfies the following inequalities (see [17, 19]): for any $u \in L^{q(.)}(\Omega)$

$$\min\left\{|u|_{q(.)}^{q^-};\ |u|_{q(.)}^{q^+}\right\} \le \rho_{q(.)}(u) \le \max\left\{|u|_{q(.)}^{q^-};\ |u|_{q(.)}^{q^+}\right\} \le |u|_{q(.)}^{q^+} + 1 \tag{2.1}$$

and

$$\min\left\{\rho_{q(.)}^{1/q^+}(u);\ \rho_{q(.)}^{1/q^-}(u)\right\} \le |u|_{q(.)} \le \max\left\{\rho_{q(.)}^{1/q^+}(u);\ \rho_{q(.)}^{1/q^-}(u)\right\} \le \left(\rho_{q(.)}(u)+1\right)^{1/q^-}. \tag{2.2}$$

$\left(L^{q(.)}(\Omega), |\cdot|_{q(.)}\right)$ is a separable reflexive Banach space (see [17, 25]) and we denote by $L^{q'(.)}(\Omega)$ its conjugate space where $q'(x) = \frac{q(x)}{q(x)-1}$, for all $x \in \Omega$. For any $u \in L^{q(.)}(\Omega)$ and $v \in L^{q'(.)}(\Omega)$, the following Hölder type inequality holds true (see [7, 25, 26]):

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{q^-} + \frac{1}{(q')^-}\right)|u|_{q(.)}|v|_{q'(.)} \le 2|u|_{q(.)}|v|_{q'(.)}. \tag{2.3}$$

Let us denote $\vec{q}(.) = \big(q_0(.), q_1(.), \ldots, q_N(.)\big)$ such that for any $i = 0, 1, \ldots, N$, $q_i(.) \in C_+(\bar{\Omega})$ with

$$q_0(x) := \max\{q_i(x)\} \ \text{ for all } x \in \Omega \text{ and } i = 1, \ldots N,$$

and put

$$1 < q_i^- := ess\inf_{x\in\Omega} q_i(x) \le ess\sup_{x\in\Omega} q_i(x) := q_i^+ < \infty,$$

$$q_m(x) := \min\{q_1(x), \ldots, q_N(x)\} \ \text{ for all } x \in \Omega, \quad q_m^- := \min\{q_1^-, \ldots, q_N^-\}.$$

Now, the anisotropic variable exponent Sobolev space is defined as:

$$W^{1,\vec{q}(.)}(\Omega) := \left\{u \in L^{q_0(.)}(\Omega) \ \text{ and } \ \partial_{x_i}u \in L^{q_i(.)}(\Omega),\ \forall\, i = 1, \ldots, N\right\}$$

equipped with the following norm

$$\|u\|_{\vec{q}(.)} = |u|_{q_0(.)} + \sum_{i=1}^N |\partial_{x_i}u|_{q_i(.)}. \tag{2.4}$$

Then $W^{1,\vec{q}(.)}(\Omega)$ is a separable and reflexive Banach space (see [30]) and its dual is denoted by $W^{-1,\vec{q}'(.)}(\Omega)$, where $\vec{q}'(\cdot) = \big(q_0'(\cdot), q_1'(\cdot), \ldots, q_N'(\cdot)\big)$ with

$$1 < q_i'(.) = \frac{q_i(.)}{q_i(.)-1},\ \forall\, i = 0, 1, \ldots, N.$$

The following three lemmas are well known (see [1, 9, 10, 28]):

Lemma 2.1. *The notation $\hookrightarrow\hookrightarrow$ denotes continuous and compact embedding. Then we have the following properties (see* [10]*):*

- (i) *If $q_m^- < N$, then $W^{1,\vec{q}(.)}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ for all $p \in [q_m^-, q_m^*[$ where $q_m^* = Nq_m^-/(N - q_m^-)$.*
- (ii) *If $q_m^- = N$, then $W^{1,\vec{q}(.)}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ for all $p \in [q_m^-, +\infty[$.*
- (iii) *If $q_m^- > N$, then $W^{1,\vec{q}(.)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.*

Lemma 2.2. *(see* [9, 28]*) Let $\vartheta \in L^{q(.)}(\Omega)$ and $\vartheta^n \in L^{q(.)}(\Omega)$ with $\|\vartheta^n\|_{q(.)} \le C$ for $1 < q(.) < \infty$. If $\vartheta^n(x) \longrightarrow \vartheta(x)$ a.e. in Ω, then $\vartheta^n \rightharpoonup \vartheta$ in $L^{q(.)}(\Omega)$.*

Lemma 2.3. *(see [1, 11]) Assume that (1.4)-(1.6) hold, let $(u^n)_n$ be a sequence in $W^{1,\vec{q}(.)}(\Omega)$ such that $u^n \rightharpoonup u$ weakly in $W^{1,\vec{q}(.)}(\Omega)$ and*

$$\lim_{n\to+\infty}\left\{\sum_{i=1}^{N}\int_\Omega \big[a_i(x,u^n,\nabla u^n)-a_i(x,u^n,\nabla u)\big]\big(\partial_{x_i}u^n-\partial_{x_i}u\big)\right.$$
$$\left.+\int_\Omega \big[|u^n|^{q_0(.)-2}u^n-|u|^{q_0(.)-2}u\big](u^n-u)dx\right\}=0. \tag{2.5}$$

then $u^n \to u$ strongly in $W^{1,\vec{q}(.)}(\Omega)$ for a subsequence.

To properly introduce our notion of solution, we put

$$\mathcal{T}^{1,\vec{q}(.)}(\Omega) := \left\{u:\Omega\longrightarrow\mathbb{R},\ \text{measurable and } T_k(u)\in W^{1,\vec{q}(.)}(\Omega),\ \forall k>0\right\},$$

where, T_k is the truncation function at height $k>0$ defined by

$$T_k(s)=\max\{-k;\min\{k;s\}\}.$$

It is obvious that T_k is a uniformly continuous Lipschitz function and

$$T_k(0)=0,\quad \lim_{k\to\infty}T_k(s)=s,\quad |T_k(s)|=\min\{|s|;k\}.$$

Proposition 2.4 (See [10]). *Let $u\in\mathcal{T}^{1,\vec{q}(.)}(\Omega)$, for any $i\in\{1\ldots,N\}$ there exists a unique measurable function $\nu_i:\Omega\longrightarrow\mathbb{R}$ such that*

$$\forall k>0,\quad \partial_{x_i}T_k(u)=\nu_i\chi_{\{|u|<k\}},\quad a.e.\ x\in\Omega.$$

where χ_A is the indicator function of a measurable set A. The functions ν_i are called the weak partial derivatives of u and are still denoted $\partial_{x_i}u$. Moreover, if u belongs to $W^{1,1}(\Omega)$, then ν_i coincides with the standard distributional derivative of u, that is, $\nu_i=\partial_{x_i}u$.

Next, we denote by $\mathcal{T}_{tr}^{1,\vec{q}(.)}(\Omega)$ the set of measurable functions $u\in\mathcal{T}^{1,\vec{q}(.)}(\Omega)$ such that there exists a sequence $(u^n)_n\in W^{1,\vec{q}(.)}(\Omega)$ satisfying $u^n\longrightarrow u$ a.e in Ω, $\partial_{x_i}T_k(u^n)\longrightarrow\partial_{x_i}T_k(u)$ in $L^1(\Omega)$, $\forall\ k>0$ and there exists a measurable function v on $\partial\Omega$ such that $u^n\longrightarrow v$ a.e on $\partial\Omega$.

Theorem 2.5. *Under the assumptions (1.4)–(1.11), there exists at least one renormalized solution of Problem (1.1), that is, a function $u\in\mathcal{T}_{tr}^{1,\vec{q}(.)}(\Omega)$ such that $\mathcal{N}(x,u,\nabla u)\in L^1(\Omega)$, $\mathcal{R}(x,u,\nabla u)\in L^1(\Omega)$,*

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{i=1}^{N}\int_{\{|u|\le m\}}a_i(x,u,\nabla u)\partial_{x_i}u dx=0, \tag{2.6}$$

and for any smooth function $\gamma(.) \in W^{1,\infty}(\mathbb{R})$ with a compact support, u verifies the following equality

$$\begin{cases} \displaystyle\sum_{i=1}^{N}\int_\Omega a_i\big(x,u,\nabla u\big)\big(\gamma'(u)\varphi\partial_{x_i}u+\gamma(u)\partial_{x_i}\varphi\big)dx+\int_\Omega \mathcal{N}(x,u,\nabla u)\gamma(u)\varphi dx \\ \displaystyle+\int_\Omega |u|^{\pi(x)-2}u\gamma(u)\varphi dx \quad +\sum_{i=1}^{N}\int_{\partial\Omega}\phi_i(u)\gamma(u)\varphi d\sigma+\kappa\int_{\partial\Omega}u\gamma(u)\varphi d\sigma \\ \qquad \displaystyle=\int_\Omega \gamma(u)\varphi d\mu+\int_{\partial\Omega}h\gamma(u)\varphi d\sigma+\int_\Omega \mathcal{R}(x,u,\nabla u)\gamma(u)\varphi dx \\ \qquad\quad \displaystyle+\sum_{i=1}^{N}\int_\Omega \phi_i(u)\partial_{x_i}\big(\gamma'(u)\varphi\partial_{x_i}u+\gamma(u)\partial_{x_i}\varphi\big)dx \end{cases} \tag{2.7}$$

for every $\varphi \in W^{1,\vec{q}(.)}(\Omega)\cap L^\infty(\Omega)$, where:

$$-\int_{\partial\Omega}\mathcal{B}u\gamma(u)\varphi d\sigma=\kappa\int_{\partial\Omega}u\gamma(u)\varphi d\sigma-\int_{\partial\Omega}h\gamma(u)\varphi d\sigma$$

is the explicit expression of the terms arising from the boundary condition, and the formulation of the divergence term is rigorously given by

$$\begin{aligned} &-\int_\Omega div\phi(u)\gamma(u)\varphi dx \\ &\qquad =\sum_{i=1}^{N}\int_\Omega \phi_i(u)\partial_{x_i}\big(\gamma'(u)\varphi\partial_{x_i}u+\gamma(u)\partial_{x_i}\varphi\big)dx-\sum_{i=1}^{N}\int_{\partial\Omega}\phi_i(u)\gamma(u)\varphi d\sigma. \end{aligned}$$

3. Proof of the main result (Theorem 2.5)

Throughout the rest of the work, $C_i,\ i=1,2\cdots$ and C denote positive constants.

Step 1: Approximate problems.

We introduce $E:=W^{1,\vec{q}(.)}(\Omega)\times L^{q_0(.)}(\partial\Omega)$ a reflexive space and $E_0=\{(u,v)\in E: v=\tau(u)\}$, where $\tau(u)$ is the trace of $u\in \mathcal{T}_{tr}^{1,\vec{q}(.)}(\Omega)$ in the usual sense, since $u\in W^{1,\vec{q}(.)}(\Omega)$. In the sequel, we will identify an element (u,v) of E_0 with is representative $u\in W^{1,\vec{q}(.)}(\Omega)$.
Let us put

$$\begin{cases} f^n:=T_n(f), \qquad \mu^n:=f^n-\mathrm{div}g, \qquad \phi_i^n(s):=\phi_i(T_n(s)), \qquad h^n:=T_n(h), \\ \mathcal{N}_n(x,s,\xi):=T_n(\mathcal{N}(x,s,\xi)), \qquad \text{and} \quad \mathcal{R}_n(x,T_n(s),\xi):=T_n(\mathcal{R}(x,s,\xi)). \end{cases} \tag{3.1}$$

It is not hard to show that:

$$f^n\in L^\infty(\Omega), \quad f^n \underset{n\to+\infty}{\longrightarrow} f \text{ in } L^1(\Omega), \quad \text{and} \quad \|f^n\|_{L^1(\Omega)}\le \|f\|_{L^1(\Omega)}. \tag{3.2}$$

For any $n \in \mathbb{N}$, we consider the following Fourier boundary-value problems

$$\begin{cases} u^n \in E_0, \\ -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i\big(x, T_n(u^n), \nabla u^n\big) + \mathcal{N}_n(x, u^n, \nabla u^n) + |T_n(u^n)|^{\pi(x)-2} T_n(u^n) \\ \qquad + \dfrac{1}{n}|u^n|^{q_0(x)-2}u^n = \mathcal{R}_n(x, T_n(u^n), \nabla u^n) + \mu^n - \mathrm{div}\phi^n(u^n), \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, T_n(u^n), \nabla u^n\right) \nu^i + \kappa T_n(u^n) = h^n. \end{cases} \tag{3.3}$$

Let us introduce the operators $J_n : E_0 \longrightarrow E_0'$ defined for all $u, v \in E_0$ as follows:

$$\begin{aligned} \langle J_n u, v\rangle &= \sum_{i=1}^{N} \int_\Omega a_i\big(x, T_n(u), \nabla u\big)\partial_{x_i} v dx + \int_\Omega \mathcal{N}_n(x, u, \nabla u) v dx \\ &+ \int_\Omega |T_n(u)|^{\pi(x)-2} T_n(u) v dx + \kappa \int_{\partial\Omega} T_n(u) v d\sigma + \frac{1}{n}\int_\Omega |u|^{q_0(x)-2} u v dx \\ &- \int_\Omega \mathcal{R}_n(x, T_n(u), \nabla u) v dx + \sum_{i=1}^{N}\int_{\partial\Omega} \phi_i^n(u) v d\sigma - \sum_{i=1}^{N}\int_\Omega \phi_i^n(u)\partial_{x_i} v dx. \end{aligned}$$

Lemma 3.1. *The operator $J_n : E_0 \longrightarrow E_0'$ is both bounded and pseudo-monotone. Moreover, J_n satisfies the following coercive condition:*

$$\frac{\langle J_n v, v\rangle}{\|v\|_{\vec{q}(.)}} \longrightarrow +\infty \quad \text{as } \|v\|_{\vec{q}(.)} \to +\infty \quad \text{for } v \in E_0.$$

Proof of Lemma 3.1:

*** Operator $J_n : E_0 \longrightarrow E_0'$ is bounded and coercive.**

Using (2.1)-(2.2), (1.5) and Hölder type inequality, we have, for all $u, v \in E_0$

$$\begin{aligned} |\langle J_n u, v\rangle| &\leq \beta \sum_{i=1}^{N} \int_\Omega \big(j_i(x) + |T_n(u)|^{q_i(x)-1} + |\partial_{x_i} u|^{q_i(x)-1}\big)|\partial_{x_i} v| dx + n^{\pi^+-1}\int_\Omega |v| dx \\ &+ 2n\int_\Omega |v|dx + \frac{1}{n}|u|_{q_0(.)}^{q_0(x)-1}|v|_{q_0(.)} + \kappa n\int_{\partial\Omega}|v|d\sigma + C(n)\sum_{i=1}^{N}\Big(\int_\Omega |\partial_{x_i} v| dx + \int_{\partial\Omega}|v|d\sigma\Big) \\ &\leq \beta\sum_{i=1}^{N}\big(|j_i(x)|_{q_i'(.)} + |n|_{q_i(.)}^{q_i^+-1} + |\partial_{x_i} u|_{q_i(.)}^{q_i^+-1} + 2\big)|\partial_{x_i} v|_{q_i(.)} + (n^{\pi^+-1} + 2n)|v|_{L^1(\Omega)} \\ &\quad + \frac{1}{n}\|u\|_{\vec{q}(.)}^{q_0(x)-1}\|v\|_{\vec{q}(.)} + \kappa n|v|_{L^1(\partial\Omega)} + C(n)\|v\|_{\vec{q}(.)} \\ &\leq C_1(n)\|v\|_{\vec{q}(.)} \end{aligned}$$

where $C_1(n) = C\big(1 + n^{q^+-1} + \|u\|_{\vec{q}(.)}^{p^+-1}\big) + (n^{\pi^+-1} + 2n + \kappa n) + C(n)$.
Thus, J_n is bounded.

Let $v \in E_0$, using (1.4), we have

$$\begin{aligned}
\langle J_n v, v\rangle &\geq \alpha \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} v|^{q_i(x)} dx + \frac{1}{n}\int_\Omega |v|^{q_0(x)} dx - 2n\int_\Omega |v| dx \\
&\quad -C(n) \sum_{i=1}^{N} \left(\int_\Omega |\partial_{x_i} v| dx + \int_{\partial\Omega} |v| d\sigma \right) \\
&\geq \alpha \sum_{i=1}^{N} \left|\partial_{x_i} v\right|_{q_i(.)}^{\bar{q_m}} - N\alpha + \frac{1}{n}\int_\Omega |v|^{q_0(x)} dx - 2n\int_\Omega |v| dx \\
&\quad -C(n) \sum_{i=1}^{N} \left(\int_\Omega |\partial_{x_i} v| dx + \int_{\partial\Omega} |v| d\sigma \right)
\end{aligned}$$

$$\langle J_n v, v\rangle \geq C_2 \|v\|_{\vec{q}(.)}^{\bar{q_m}} - C_3 - N\alpha - 2n\|v\|_{\vec{q}(.)} - C_4(n)\|v\|_{\vec{q}(.)}.$$

Consequently,

$$\frac{\langle J_n v, v\rangle}{\|v\|_{\vec{q}(.)}} = \frac{C_2 \|v\|_{\vec{q}(.)}^{\bar{q_m}} - C_3 - N\alpha - 2n\|v\|_{\vec{q}(.)} - C_4(n)\|v\|_{\vec{q}(.)}}{\|v\|_{\vec{q}(.)}} \longrightarrow +\infty$$

as $\|v\|_{\vec{q}(.)} \to +\infty$; i.e., J_n is coercive on E_0.

*** Operator $J_n : E_0 \longrightarrow E_0'$ is pseudo-monotone.**

Namely, we show that if $(u^k)_k$ is a sequence in E_0 such that

$$\begin{cases} u^k \rightharpoonup u & \text{in} \quad E_0, \\ J_n u^k \rightharpoonup \chi & \text{in} \quad E_0', \\ \limsup\limits_{k\to\infty} \langle J_n u^k, u^k\rangle \leq \langle \chi, u\rangle, \end{cases} \tag{3.4}$$

then

$$\chi = J_n u \qquad \text{and} \qquad \langle J_n u^k, u^k\rangle \longrightarrow \langle \chi, u\rangle \quad \text{as} \quad k \to +\infty.$$

In view of the compact embedding $W^{1,\vec{q}(.)}(\Omega) \hookrightarrow\hookrightarrow L^{\bar{q_m}(.)}(\Omega)$ and the first line of Assumption (3.4), we deduce that

$$\begin{gathered} (u^k)_k \text{ is bounded in } W^{1,\vec{q}(.)}(\Omega) \\ \text{and } u^k \to u \text{ strongly in } L^{\bar{q_m}(.)}(\Omega) \text{ for a subsequence.} \end{gathered} \tag{3.5}$$

From the boundedness of the subsequence $(u^k)_k$ and the growth Condition (1.5), it is clear that the sequence $\big(a_i(x, T_n(u^k), \nabla u^k)\big)_k$ is bounded in $L^{q_i'(.)}(\Omega)$ for any $i = 1, \dots, N$. Therefore, for $i = 1, \dots, N$ there exists a function $\varphi_i^n \in L^{q_i'(.)}(\Omega)$ such that

$$a_i(x, T_n(u^k), \nabla u^k) \rightharpoonup \varphi_i^n \quad \text{weakly in} \quad L^{q_i'(.)}(\overline{\Omega}) \quad \text{as} \quad k \to \infty. \tag{3.6}$$

Similarly, it is not hard to show that $(\mathcal{N}_n(x,u^k,\nabla u^k))_k$ and $(\mathcal{R}_n(x,T_n(u^k),\nabla u^k))_k$ are bounded in $L^{(\bar{q_m})'(.)}(\Omega)$, then there exists a function $\psi^n,\theta^n \in L^{(\bar{q_m})'(.)}(\Omega)$ such that

$$\begin{aligned} &\mathcal{N}_n(x,u^k,\nabla u^k) \rightharpoonup \psi^n \\ \text{and} \quad &\mathcal{R}_n(x,T_n(u^k),\nabla u^k) \rightharpoonup \theta^n \text{ weakly in } L^{(\bar{q_m})'(.)}(\Omega) \quad \text{as } k\to\infty. \end{aligned} \tag{3.7}$$

Since $\phi_i^n = \phi_i \circ T_n$ is a bounded continuous function and $u^k \to u$ in $L^{q_i(.)}(\Omega)$ it follows that

$$\phi_i^n(u^k) \longrightarrow \phi_i^n(u) \quad \text{in} \quad L^{q_i'(.)}(\overline{\Omega}) \quad \text{as} \quad k\to\infty. \tag{3.8}$$

Moreover, we have

$$\begin{aligned} &|T_n(u^k)|^{\pi(x)-2}T_n(u^k) \to |T_n(u)|^{\pi(x)-2}T_n(u) \\ \text{and} \quad &\frac{1}{n}|u^k|^{q_0(x)-2}u^k \to \frac{1}{n}|u|^{q_0(x)-2}u \text{ strongly in } L^{(\bar{q_m})'(.)}(\Omega). \end{aligned} \tag{3.9}$$

Due to the second line of Assumption (3.4) and using (3.6)−(3.9), we observe that for all $v\in E_0$

$$\begin{aligned} \langle \chi, v\rangle &= \lim_{k\to\infty} \langle J_n u^k, v\rangle \\ &= \lim_{k\to\infty} \Big\{ \sum_{i=1}^N \int_\Omega a_i\big(x,T_n(u^k),\nabla u^k\big)\partial_{x_i} v dx + \int_\Omega \mathcal{N}_n(x,u^k,\nabla u^k) v dx \\ &\qquad - \int_\Omega \mathcal{R}_n(x,T_n(u^k),\nabla u^k) v dx + \int_\Omega |T_n(u^k)|^{\pi(x)-2}T_n(u^k) v dx \\ &\qquad + \kappa \int_{\partial\Omega} T_n(u^k) v d\sigma + \frac{1}{n}\int_\Omega |u^k|^{q_0(x)-2}u^k v dx \\ &\qquad + \sum_{i=1}^N \int_{\partial\Omega} \phi_i^n(u^k) v d\sigma - \sum_{i=1}^N \int_\Omega \phi_i^n(u^k)\partial_{x_i} v dx \Big\} \\ &= \sum_{i=1}^N \int_\Omega \varphi_i^n \partial_{x_i} v dx + \int_\Omega \psi^n v dx - \int_\Omega \theta^n v dx \\ &\quad + \int_\Omega |T_n(u)|^{\pi(x)-2}T_n(u) v dx + \frac{1}{n}\int_\Omega |u|^{q_0(x)-2}u v dx \\ &\quad + \kappa \int_{\partial\Omega} T_n(u) v d\sigma + \sum_{i=1}^N \int_{\partial\Omega} \phi_i^n(u) v d\sigma - \sum_{i=1}^N \int_\Omega \phi_i^n(u)\partial_{x_i} v dx. \end{aligned} \tag{3.10}$$

Therefore, the last line of Assumption (3.4) leads to

$$\begin{aligned}
&\limsup_{k\to+\infty}\langle J_n(u^k),u^k\rangle \\
&= \limsup_{k\to+\infty}\Big\{\sum_{i=1}^N\int_\Omega a_i\big(x,T_n(u^k),\nabla u^k\big)\partial_{x_i}u^k dx+\int_\Omega \mathcal{N}_n(x,u^k,\nabla u^k)u^k dx \\
&-\int_\Omega \mathcal{R}_n(x,T_n(u^k),\nabla u^k)u^k dx+\int_\Omega |T_n(u^k)|^{\pi(x)}dx+\frac{1}{n}\int_\Omega |u^k|^{q_0(x)}dx \\
&+\kappa\int_{\partial\Omega}T_n(u^k)u^k d\sigma+\sum_{i=1}^N\int_{\partial\Omega}\phi_i^n(u^k)u^k d\sigma-\sum_{i=1}^N\int_\Omega \phi_i^n(u^k)\partial_{x_i}u^k dx\Big\} \\
&\le \sum_{i=1}^N\int_\Omega \varphi_i^n\partial_{x_i}u dx+\int_\Omega \psi^n u dx-\int_\Omega \theta^n u dx+\int_\Omega |T_n(u)|^{\pi(x)}dx+\frac{1}{n}\int_\Omega |u|^{q_0(x)}dx \\
&+\kappa\int_{\partial\Omega}T_n(u)u d\sigma+\sum_{i=1}^N\int_{\partial\Omega}\phi_i^n(u)u d\sigma-\sum_{i=1}^N\int_\Omega \phi_i^n(u)\partial_{x_i}u dx. \qquad (3.11)
\end{aligned}$$

So, using the convergences (3.7)-(3.9), we can say that

$$\limsup_{k\to+\infty}\sum_{i=1}^N\int_\Omega a_i\big(x,T_n(u^k),\nabla u^k\big)\partial_{x_i}u^k dx\le\sum_{i=1}^N\int_\Omega \varphi_i^n\partial_{x_i}u dx. \qquad (3.12)$$

On the other hand, thanks to the Assumption (1.6), we see that

$$\sum_{i=1}^N\int_\Omega\Big[a_i\big(x,T_n(u^k),\nabla u^k\big)-a_i\big(x,T_n(u^k),\nabla u\big)\Big]\big(\partial_{x_i}u^k-\partial_{x_i}u\big)dx\ge 0. \qquad (3.13)$$

Then

$$\begin{aligned}
\sum_{i=1}^N\int_\Omega a_i\big(x,T_n(u^k),\nabla u^k\big)\partial_{x_i}u^k dx \ \ge\ & -\sum_{i=1}^N\int_\Omega a_i\big(x,T_n(u^k),\nabla u\big)\partial_{x_i}u dx \\
&+\sum_{i=1}^N\int_\Omega a_i\big(x,T_n(u^k),\nabla u^k\big)\partial_{x_i}u dx \\
&+\sum_{i=1}^N\int_\Omega a_i\big(x,T_n(u^k),\nabla u\big)\partial_{x_i}u^k dx.
\end{aligned}$$

Taking into account (3.6), we get

$$\liminf_{k\to+\infty}\sum_{i=1}^N\int_\Omega a_i\big(x,T_n(u^k),\nabla u^k\big)\partial_{x_i}u^k dx\ge\sum_{i=1}^N\int_\Omega \varphi_i^n\partial_{x_i}u dx. \qquad (3.14)$$

Consequently, we establish that

$$\lim_{k\to+\infty}\sum_{i=1}^N\int_\Omega a_i\big(x,T_n(u^k),\nabla u^k\big)\partial_{x_i}u^k dx=\sum_{i=1}^N\int_\Omega \varphi_i^n\partial_{x_i}u dx. \qquad (3.15)$$

Combining (3.10) and (3.15), we conclude that

$$\langle J_n u^k, u^k\rangle \longrightarrow \langle \chi, u\rangle \quad \text{as } k\to\infty.$$

Also, in view of (3.15), we get the relation

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega \big[a_i\big(x,T_n(u^k),\nabla u^k\big)-a_i\big(x,T_n(u^k),\nabla u\big)\big]\big(\partial_{x_i}u^k-\partial_{x_i}u\big)dx=0. \quad (3.16)$$

Next, since $u^k\to u$ strongly in $L^{q_m^-(.)}(\Omega)$, we end up with

$$\lim_{k\to+\infty}\Big\{\sum_{i=1}^{N}\int_\Omega \big[a_i\big(x,T_n(u^k),\nabla u^k\big)-a_i\big(x,T_n(u^k),\nabla u\big)\big]\big(\partial_{x_i}u^k-\partial_{x_i}u\big)$$
$$+\int_\Omega \big(|u^k|^{q_0(.)-2}u^k-|u|^{q_0(.)-2}u\big)(u^k-u)dx\Big\}=0.$$

Finally, Lemma 2.3 implies the following convergences along a subsequence:

$$u^k\to u \quad \text{in } \ W^{1,\overrightarrow{q}(.)}(\Omega) \qquad \text{and} \qquad \partial_{x_i}u^k\to\partial_{x_i}u \quad \text{a.e. in} \quad \Omega, \qquad (3.17)$$

then

$$\varphi_i^n=a_i\big(x,T_n(u),\nabla u\big) \quad \text{in} \quad L^{q_i'(.)}(\Omega) \quad \text{for } i=1,\dots,N,$$

and

$$\psi^n=\mathcal{N}_n\big(x,u,\nabla u\big) \quad \text{and} \quad \theta^n=\mathcal{R}_n\big(x,T_n(u),\nabla u\big) \quad \text{in} \quad L^{(q_m^-)'(.)}(\Omega).$$

Evidently, in view of (3.9) the equality $\chi=J_n u$ is satisfied, which completes the proof of the Lemma 3.1. □

Thanks to Lemma 3.1 (see [10], [27] and [28]), there exists at least one weak solution $u^n\in E_0$ for Fourier boundary-value problems (3.3). Which means that, for any $v\in E_0$, the following integral identity holds true:

$$\underbrace{\sum_{i=1}^{N}\int_\Omega a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}v dx}_{\mathcal{I}_1^n}+\underbrace{\int_\Omega \mathcal{N}_n(x,u^n,\nabla u^n)v dx}_{\mathcal{I}_2^n}+\underbrace{\int_\Omega |T_n(u^n)|^{\pi(x)-2}T_n(u^n)v dx}_{\mathcal{I}_3^n}$$
$$+\underbrace{\frac{1}{n}\int_\Omega |u^n|^{q_0(x)-2}u^n v dx}_{\mathcal{I}_4^n}+\underbrace{\kappa\int_{\partial\Omega}T_n(u^n)v d\sigma}_{\mathcal{I}_5^n}+\underbrace{\sum_{i=1}^{N}\int_{\partial\Omega}\phi_i^n(u^n)v d\sigma}_{\mathcal{I}_6^n} \qquad (3.18)$$
$$=\underbrace{\int_\Omega v d\mu^n}_{\mathcal{I}_7^n}+\underbrace{\int_{\partial\Omega}h^n v d\sigma}_{\mathcal{I}_8^n}+\underbrace{\int_\Omega \mathcal{R}_n(x,T_n(u^n),\nabla u^n)v dx}_{\mathcal{I}_9^n}+\underbrace{\sum_{i=1}^{N}\int_\Omega \phi_i^n(u^n)\partial_{x_i}v dx}_{\mathcal{I}_{10}^n}.$$

Step 2: Weak convergence of truncations.

Let us define

$$B(\tau)=\frac{2}{\alpha}\int_0^\tau b(|\theta|)d\theta \quad \text{and} \quad \chi(\tau)=\left(2-\frac{1}{(1+|\tau|)^{\sigma-1}}\right)$$

where

$$1<\sigma<\left(q_i^-/\delta_i^+-1\right)(\pi(x)-1).$$

Evidently,

$$1<\chi(\tau)<2 \quad \text{and} \quad 0\leq B(\infty):=\frac{2}{\alpha}\int_0^{+\infty} b(|\theta|)d\theta=C<\infty. \tag{3.19}$$

Replacing v by $T_k(u^n)\chi(u^n)e^{B(|u^n|)}\in E_0$ in (3.18), we derive useful estimates by analyzing the different terms:

• **Estimate of $\mathcal{I}_1^n$**: This test function allows to rewrite our first term in (3.18) as follow:

$$\begin{aligned}\mathcal{I}_1^n=&\sum_{i=1}^N\int_\Omega a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}T_k(u^n)\chi(u^n)e^{B(|u^n|)}dx\\&+(\sigma-1)\sum_{i=1}^N\int_\Omega \frac{a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}u^n}{(1+|u^n|)^\sigma}|T_k(u^n)|e^{B(|u^n|)}dx\\&+\frac{2}{\alpha}\sum_{i=1}^N\int_\Omega a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}u^n b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx\end{aligned}$$

and, using Hypothesis (1.4), we get

$$\begin{aligned}\mathcal{I}_1^n\geq&\ \alpha\sum_{i=1}^N\int_\Omega|\partial_{x_i}T_k(u^n)|^{q_i(x)}\chi(u^n)e^{B(|u^n|)}dx\\&+\alpha(\sigma-1)\sum_{i=1}^N\int_\Omega\frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_k(u^n)|e^{B(|u^n|)}dx\\&+2\sum_{i=1}^N\int_\Omega|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx.\end{aligned}\tag{3.20}$$

• **Estimate of $\mathcal{I}_2^n$**: Using Assumption (1.8), we deduce from the second term of (3.18) that:

$$\begin{aligned}|\mathcal{I}_2^n|\leq&\int_\Omega|\gamma_0(x)||T_k(u^n)||\chi(u^n)|e^{B(|u^n|)}dx\\&+\sum_{i=1}^N\int_\Omega|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx\\\leq&\ 2ke^{B(\infty)}\|\gamma_0\|_{L^1(\Omega)}+\sum_{i=1}^N\int_\Omega|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx.\end{aligned}\tag{3.21}$$

• **Estimate of $\mathcal{I}_7^n$:** From the definition of μ^n given in (3.1), we rewrite the seventh term of (3.18) as:

$$\begin{aligned}\mathcal{I}_7^n = \int_\Omega f^n T_k(u^n)\chi(u^n)e^{B(|u^n|)}dx - \sum_{i=1}^N \int_{\partial\Omega} g_i T_k(u^n)\chi(u^n)e^{B(|u^n|)}dx \\ + \sum_{i=1}^N \int_\Omega g_i \partial_{x_i} T_k(u^n)\chi(u^n)e^{B(|u^n|)}dx \\ + \frac{2}{\alpha}\sum_{i=1}^N \int_\Omega g_i \partial_{x_i} u^n b(|u^n|)|T_k(u^n)|\varphi(u^n)e^{B(|u^n|)}dx \\ +(\sigma-1)\sum_{i=1}^N \int_\Omega \frac{g_i\partial_{x_i}u^n}{(1+|u^n|)^\sigma}|T_k(u^n)|e^{B(|u^n|)}dx\end{aligned}$$

then, by applying Young's Inequality, we have

$$\begin{aligned}|\mathcal{I}_7^n| &\leq \int_\Omega |f^n||T_k(u^n)||\chi(u^n)|e^{B(|u^n|)}dx + \sum_{i=1}^N \int_{\partial\Omega} |g_i||T_k(u^n)||\chi(u^n)|e^{B(|u^n|)}dx \\ &+\frac{\alpha}{4}\sum_{i=1}^N \int_\Omega |\partial_{x_i}T_k(u^n)|^{q_i(x)}\chi(u^n)e^{B(|u^n|)}dx + C(\alpha)\sum_{i=1}^N \int_\Omega |g_i|^{q_i'(x)}\chi(u^n)e^{B(|u^n|)}dx \\ &+\frac{\alpha}{4}\sum_{i=1}^N \int_\Omega |\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\ &+C(\alpha)\sum_{i=1}^N \int_\Omega |g_i|^{q_i'(x)}b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\ &+\frac{\alpha}{3}(\sigma-1)\sum_{i=1}^N \int_\Omega \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_k(u^n)|e^{B(|u^n|)}dx \\ &+C(\alpha)(\sigma-1)\sum_{i=1}^N \int_\Omega \frac{|g_i|^{q_i'(x)}}{(1+|u^n|)^\sigma}|T_k(u^n)|e^{B(|u^n|)}dx \\ &\leq 2ke^{B(\infty)}\big(\|f\|_{L^1(\Omega)} + \sum_{i=1}^N \int_{\partial\Omega}|g_i|dx\big) + C(\alpha)e^{B(\infty)}\big(2+k(\sigma-1) \\ &+2k\|b(|.|)\|_{L^1(\mathbb{R})}\big)\sum_{i=1}^N \int_\Omega |g_i|^{q_i'(x)}dx + \frac{\alpha}{4}\sum_{i=1}^N \int_\Omega |\partial_{x_i}T_k(u^n)|^{q_i(x)}\chi(u^n)e^{B(|u^n|)}dx \\ &+\frac{1}{2}\sum_{i=1}^N \int_\Omega |\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\ &+\frac{\alpha}{3}(\sigma-1)\sum_{i=1}^N \int_\Omega \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_k(u^n)|e^{B(|u^n|)}dx.\end{aligned} \tag{3.22}$$

• **Estimate of** $\mathcal{I}_9^n$: Combining Hypothesis (1.9) and Young's Inequality, the ninth term of (3.18) gives

$$
\begin{aligned}
|\mathcal{I}_9^n| \leq & \int_\Omega \rho(x) T_k(u^n)\chi(u^n) e^{B(|u^n|)}dx + \int_\Omega |T_n(u^n)|^{\delta_0(x)} |T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
& + \int_\Omega \sum_{i=1}^N |\partial_{x_i} u^n|^{\delta_i(x)} |T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
\leq & \; 2ke^{B(\infty)}\|\rho\|_{L^1(\Omega)} + \frac{1}{2}\int_\Omega |T_n(u^n)|^{\pi(x)-1}|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
& + C_0 \int_\Omega |T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
& + \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u^n|^{\delta_i(x)}}{(1+|u^n|)^{\frac{\sigma\delta_i(x)}{q_i(x)}}} (1+|u^n|)^{\frac{\sigma\delta_i(x)}{q_i(x)}} |T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
\leq & \; 2ke^{B(\infty)}\|\rho\|_{L^1(\Omega)} + \frac{1}{2}\int_\Omega |T_n(u^n)|^{\pi(x)-1}|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
& + C_0 \int_\Omega |T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
& + \frac{\alpha}{3}(\sigma-1) \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u^n|^{q_i(x)}}{(1+|u^n|)^\sigma} |T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
& + C' \int_\Omega \sum_{i=1}^N (1+|u^n|)^{\frac{\sigma\delta_i(x)}{q_i(x)-\delta_i(x)}} |T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
\leq & \; 2ke^{B(\infty)}\|\rho\|_{L^1(\Omega)} + \frac{1}{2}\int_\Omega |T_n(u^n)|^{\pi(x)-1}|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
& + (C_0 + C_0') \int_\Omega |T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
& \frac{\alpha}{3}(\sigma-1) \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u^n|^{q_i(x)}}{(1+|u^n|)^\sigma} |T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
& + \frac{1}{2}\int_\Omega |T_n(u^n)|^{\pi(x)-1}|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx.
\end{aligned}
\tag{3.23}
$$

• **Estimate of** $\mathcal{I}_{10}^n$: The last integral in the right hand side of (3.18) becomes

$$
\begin{aligned}
\mathcal{I}_{10}^n = & \sum_{i=1}^N \int_\Omega \phi_i^n(u^n)\partial_{x_i} T_k(u^n)\chi(u^n)e^{B(|u^n|)}dx \\
& + \frac{2}{\alpha}\sum_{i=1}^N \int_\Omega \phi_i^n(u^n)\partial_{x_i} u^n b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
& + (\sigma-1)\sum_{i=1}^N \int_\Omega \frac{\phi_i^n(u^n)\partial_{x_i} u^n}{(1+|u^n|)^\sigma} |T_k(u^n)|e^{B(|u^n|)}dx
\end{aligned}
$$

and, exploiting Young's Inequality, we have

$$\begin{aligned}
|\mathcal{I}_{10}^n| \leq{}& \frac{\alpha}{4}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}T_k(u^n)|^{q_i(x)}\chi(u^n)e^{B(|u^n|)}dx\\
&+C(\alpha)\sum_{i=1}^{N}\int_\Omega |\phi_i^n(u^n)|^{q_i'(x)}\chi(u^n)e^{B(|u^n|)}dx\\
&+\frac{\alpha}{4}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx\\
&+C(\alpha)\sum_{i=1}^{N}\int_\Omega |\phi_i^n(u^n)|^{q_i'(x)}b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx\\
&+\frac{\alpha}{3}(\sigma-1)\sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_k(u^n)|e^{B(|u^n|)}dx\\
&+C(\alpha)(\sigma-1)\sum_{i=1}^{N}\int_\Omega \frac{|\phi_i^n(u^n)|^{q_i'(x)}}{(1+|u^n|)^\sigma}|T_k(u^n)|e^{B(|u^n|)}dx\\
\leq{}& C(\alpha)e^{B(\infty)}\big(2+k(\sigma-1)+2k\|b(|.|)\|_{L^1(\mathbb{R})}\big)\sum_{i=1}^{N}\int_\Omega |\phi_i^n(u^n)|^{q_i'(x)}dx\\
&+\frac{\alpha}{4}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}T_k(u^n)|^{q_i(x)}\chi(u^n)e^{B(|u^n|)}dx\\
&+\frac{1}{2}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|T_k(u^n)|\chi(u^n)e^{B(|u^n|)}dx\\
&+\frac{\alpha}{3}(\sigma-1)\sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_k(u^n)|e^{B(|u^n|)}dx.
\end{aligned}\tag{3.24}$$

Next, combining (3.18) and estimates (3.20)-(3.24), we end up at

$$\frac{\alpha}{2}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}T_k(u^n)|^{q_i(x)}dx+\frac{1}{n}\int_\Omega |T_k(u^n)|^{q_0(x)}dx \leq kC_5+C_6(\alpha,\sigma),$$

which implies that

$$\min\left\{\frac{\alpha}{2},\frac{1}{n}\right\}\Big[\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}T_k(u^n)|^{q_i(x)}dx+\int_\Omega |T_k(u^n)|^{q_0(x)}dx\Big] \leq kC_7.$$

Namely

$$\sum_{i=0}^{N}\int_\Omega |\partial_{x_i}T_k(u^n)|^{q_i(x)}dx \leq C_8k \quad \text{for all } k>0.$$

Using Condition (2.2), it follows that

$$\sum_{i=0}^{N} |\partial_{x_i} T_k(u^n)|_{q_i(.)}^{q_m^-} \leq \sum_{i=0}^{N} \int_\Omega |\partial_{x_i} T_k(u^n)|^{q_i(.)} dx + N \leq C_8 k + N \quad \text{for all } k > 0.$$

In other hand, according to the convexity of the application $t \in \mathbb{R}^+ \longrightarrow t^{q_m^-}$, $q_m^- > 1$, we have:

$$\frac{1}{N^{q_m^- - 1}} \left(\sum_{i=0}^{N} |\partial_{x_i} T_k(u^n)|_{q_i(.)} \right)^{q_m^-} \leq \sum_{i=0}^{N} |\partial_{x_i} T_k(u^n)|_{q_i(.)}^{q_m^-} \leq C_8 k + N \quad \text{for all } k > 0,$$

then,

$$\left(\sum_{i=0}^{N} |\partial_{x_i} T_k(u^n)|_{q_i(.)} \right)^{q_m^-} \leq (C_8 k + N) N^{q_m^- - 1}.$$

So, we get

$$\sum_{i=0}^{N} |\partial_{x_i} T_k(u^n)|_{q_i(.)} \leq C_9 (k+1)^{1/q_m^-}, \quad \text{with} \ \ C_9 = \left[\max(C_8 N^{q_m^- - 1}, N^{q_m^-}) \right]^{1/q_m^-},$$

which ensures that

$$\|T_k(u^n)\|_{\vec{q}(.)} = |T_k(u^n)|_{q_0(.)} + \sum_{i=1}^{N} \partial_{x_i} T_k(u^n)|_{q_i(.)} \leq C_9 (k+1)^{1/q_m^-} \quad \forall k > 0, \tag{3.25}$$

where C_9 is positive constant that does not depend on k and n.

So, by (3.25), $(T_k(u^n))_n$ is bounded in $W^{1,\vec{q}(.)}(\Omega)$, then there exists a subsequence still denoted $(T_k(u^n))_n$ and $\eta_k \in W^{1,\vec{q}(.)}(\Omega)$ such that

$$\begin{cases} T_k(u^n) \rightharpoonup \eta_k & \text{weakly in } W^{1,\vec{q}(.)}(\Omega) \quad \text{as } n \to \infty, \forall k > 0, \\ T_k(u^n) \to \eta_k & \text{strongly in } L^{q_m^-(.)}(\Omega) \quad \text{and a.e. in } \Omega \ \forall k > 0. \end{cases} \tag{3.26}$$

Now, we will show that $(u^n)_n$ is a Cauchy sequence in measure in Ω. Indeed, let $k > 0$ be large enough, combining Hölder's type inequality and (3.25), one has

$$\begin{aligned} k \ \text{meas}(\{|u^n| > k\}) &= \int_{\{|u^n|>k\}} |T_k(u^n)| \, dx \leq \int_\Omega |T_k(u^n)| \, dx \\ &\leq \big(1/q_0^- + 1/(q_0')^-\big) |1|_{q_0'(.)} |T_k(u^n)|_{q_0(.)} \\ &\leq \big(1/q_0^- + 1/(q_0')^-\big) (meas(\Omega) + 1)^{1/(q_0')^-} ||T_k(u^n)||_{\vec{q}(.)} \\ &\leq C_{10} (k+1)^{1/q_m^-}, \end{aligned}$$

where

$$C_{10} = \big(1/q_0^- + 1/(q_0')^-\big)(meas(\Omega) + 1)^{1/(q_0')^-} C.$$

This yields

$$\text{meas}(\{|u^n| > k\}) \leq C_{10} \big(1/k^{-1+q_m^-} + 1/k^{q_m^-}\big)^{1/q_m^-} \longrightarrow 0 \quad \text{as } k \to \infty. \tag{3.27}$$

Since, for every $\nu > 0$

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \nu\} &\le \text{meas}\{|u_n| > k\} \\ &+ \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \nu\}, \end{aligned} \tag{3.28}$$

for all $\varepsilon > 0$, using (3.27), we may choose $k = k(\varepsilon) > 0$ large enough such that

$$\text{meas}\{|u_n| > k\} \le \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \le \frac{\varepsilon}{3}. \tag{3.29}$$

Therefore, from (3.26), we have $T_k(u^n) \to \eta_k$ in $L^{\bar{q_m}(\cdot)}(\Omega)$ and a.e. in Ω. Consequently, we can assume that $(T_k(u^n))_n$ is a Cauchy sequence in measure in Ω. So, for all $k > 0$ and $\nu, \varepsilon > 0$, there exists $n_0 = n_0(k, \nu, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \nu\} \le \frac{\varepsilon}{3} \quad \text{for all } m, n \ge n_0(k, \nu, \varepsilon). \tag{3.30}$$

Finally, by combining (3.29) and (3.30) we obtain that

$$\forall \nu, \varepsilon > 0, \quad \exists n_0 = n_0(\nu, \varepsilon) \text{ such that } \quad \text{meas}\{|u_n - u_m| > \nu\} \le \varepsilon \quad \forall m, n \ge n_0.$$

It follows that $(u^n)_n$ is a Cauchy sequence in measure in Ω, then converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{cases} T_k(u^n) \rightharpoonup T_k(u) & \text{weakly in } W^{1,\vec{q}(\cdot)}(\Omega) \quad \text{as } n \to \infty, \forall k > 0, \\ T_k(u^n) \to T_k(u) & \text{strongly in } L^{\bar{q_m}(\cdot)}(\Omega) \quad \text{and a.e. in } \Omega \ \forall k > 0. \end{cases} \tag{3.31}$$

Invoking Lebesgue's dominated convergence Theorem, we get

$$T_k(u^n) \to T_k(u) \quad \text{strongly in } L^{q_i(\cdot)}(\Omega) \quad \text{and a.e. in } \Omega \text{ for } i = 1, \dots, N. \tag{3.32}$$

Moreover, from (3.27), we obtain that $\dfrac{T_k(u^n)}{k} \to 0$ strongly in $L^1(\Omega)$ and weak* in $L^\infty(\Omega)$.

Step 3: A priori estimates.

In this step, taking $v = \frac{1}{m} T_m(u^n)\chi(u^n)e^{B(|u^n|)}$ as test function in the weak Formulation (3.18) we establish some a priori estimates for the subsequence $(u^n)_{n\in\mathbb{N}}$. As in Step 2, we start by treating certain terms separately:

- **Estimate of $\mathcal{I}_1^n$**: Putting this test function in (3.18), the first term becomes

$$\begin{aligned} \mathcal{I}_1^n &= \frac{1}{m} \sum_{i=1}^N \int_\Omega a_i\big(x, T_n(u^n), \nabla u^n\big) \partial_{x_i} T_m(u^n) \chi(u^n) e^{B(|u^n|)} dx \\ &+ \frac{(\sigma - 1)}{m} \sum_{i=1}^N \int_\Omega \frac{a_i\big(x, T_n(u^n), \nabla u^n\big) \partial_{x_i} u^n}{(1 + |u^n|)^\sigma} |T_m(u^n)| e^{B(|u^n|)} dx \\ &+ \frac{2}{\alpha m} \sum_{i=1}^N \int_\Omega a_i\big(x, T_n(u^n), \nabla u^n\big) \partial_{x_i} u^n |T_m(u^n)| \chi(u^n) b(|u^n|) e^{B(|u^n|)} dx, \end{aligned}$$

then, using Condition (1.4), we obtain

$$\begin{aligned}\mathcal{I}_1^n \geq & \frac{1}{m}\sum_{i=1}^{N}\int_\Omega a_i\big(x, T_n(u^n), \nabla u^n\big)\partial_{x_i}T_m(u^n)\chi(u^n)e^{B(|u^n|)}dx \\ & +\frac{\alpha(\sigma-1)}{m}\sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_m(u^n)|e^{B(|u^n|)}dx \\ & +\frac{2}{m}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u^n|^{q_i(x)}|T_m(u^n)|\chi(u^n)b(|u^n|)e^{B(|u^n|)}dx.\end{aligned} \tag{3.33}$$

• **Estimate of $\mathcal{I}_2^n$**: From Hypothesis (1.8), the second term in (3.18) gives

$$\begin{aligned}|\mathcal{I}_2^n| \leq & \frac{1}{m}\int_\Omega \gamma_0(x)|T_m(u^n)||\chi(u^n)|e^{B(|u^n|)}dx \\ & +\frac{1}{m}\int_\Omega\sum_{i=1}^{N}|\partial_{x_i}u^n|^{q_i(x)}|T_m(u^n)|\chi(u^n)b(|u^n|)e^{B(|u^n|)}dx \\ \leq & 2e^{B(\infty)}\int_\Omega |\gamma_0(x)|\frac{|T_m(u^n)|}{m}dx \\ & +\frac{1}{m}\int_\Omega\sum_{i=1}^{N}|\partial_{x_i}u^n|^{q_i(x)}|T_m(u^n)|\chi(u^n)b(|u^n|)e^{B(|u^n|)}dx.\end{aligned} \tag{3.34}$$

• **Estimate of $\mathcal{I}_7^n$**: In view of the definition of μ^n given in (3.1), we rewrite that

$$\begin{aligned}\mathcal{I}_7^n = & \frac{1}{m}\int_\Omega f^n T_m(u^n)\chi(u^n)e^{B(|u^n|)}dx \\ & -\frac{1}{m}\sum_{i=1}^{N}\int_{\partial\Omega} g_i T_m(u^n)\chi(u^n)e^{B(|u^n|)}dx \\ & +\frac{1}{m}\sum_{i=1}^{N}\int_\Omega g_i\partial_{x_i}T_m(u^n)\chi(u^n)e^{B(|u^n|)}dx \\ & +\frac{2}{\alpha m}\sum_{i=1}^{N}\int_\Omega g_i\partial_{x_i}u^n b(|u^n|)|T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\ & +\frac{(\sigma-1)}{m}\sum_{i=1}^{N}\int_\Omega \frac{g_i\partial_{x_i}u^n}{(1+|u^n|)^\sigma}|T_m(u^n)|e^{B(|u^n|)}dx\end{aligned}$$

then, by applying Young's inequality, we have

$$\begin{aligned}
|\mathcal{I}_7^n| \leq 2e^{B(\infty)} &\int_\Omega |f^n|\frac{|T_m(u^n)|}{m}dx + C(\alpha)(\sigma-1)\sum_{i=1}^N\int_\Omega |g_i|^{q_i'(x)}\frac{|T_m(u^n)|}{m}e^{B(|u^n|)}dx \\
&+C(\alpha)\sum_{i=1}^N\int_\Omega |g_i|^{q_i'(x)}b(|u^n|)\frac{|T_m(u^n)|}{m}\chi(u^n)e^{B(|u^n|)}dx \\
&+\sum_{i=1}^N\int_{\partial\Omega}|g_i|\frac{|T_m(u^n)|}{m}\chi(u^n)e^{B(|u^n|)}d\sigma + \frac{C(\alpha)}{m}\sum_{i=1}^N\int_\Omega |g_i|^{q_i'(x)}\chi(u^n)e^{B(|u^n|)}dx \\
&+\frac{\alpha(\sigma-1)}{m}\sum_{i=1}^N\int_\Omega \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_m(u^n)|e^{B(|u^n|)}dx \\
&+\frac{2}{m}\sum_{i=1}^N\int_\Omega |\partial_{x_i}u^n|^{q_i(x)}|T_m(u^n)|\chi(u^n)b(|u^n|)e^{B(|u^n|)}dx \\
&+\frac{\alpha}{4m}\sum_{i=1}^N\int_\Omega |\partial_{x_i}T_m(u^n)|^{q_i(x)}\chi(u^n)e^{B(|u^n|)}dx.
\end{aligned} \tag{3.35}$$

• **Estimate of $\mathcal{I}_9^n$**: Exploiting Hypothesis (1.9) and Young's inequality, it coming that

$$\begin{aligned}
|\mathcal{I}_9^n| &\leq \frac{1}{m}\int_\Omega \rho(x)T_m(u^n)\chi(u^n)e^{B(|u^n|)}dx + \frac{1}{m}\int_\Omega |T_n(u^n)|^{\delta_0(x)}T_m(u^n)\chi(u^n)e^{B(|u^n|)}dx \\
&\quad+\frac{1}{m}\int_\Omega\sum_{i=1}^N |\partial_{x_i}u^n|^{\delta_i(x)}T_m(u^n)\chi(u^n)e^{B(|u^n|)}dx \\
&\leq \frac{1}{4m}\int_\Omega |T_n(u^n)|^{\pi(x)-1}|T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
&\quad+\frac{C_0}{m}\int_\Omega |T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx + 2e^{B(\infty)}\int_\Omega |\rho(x)|\frac{|T_m(u^n)|}{m}dx \\
&\quad+\frac{1}{m}\int_\Omega\sum_{i=1}^N \frac{|\partial_{x_i}u^n|^{\delta_i(x)}}{(1+|u^n|)^{\frac{\sigma\delta_i(x)}{q_i(x)}}}(1+|u^n|)^{\frac{\sigma\delta_i(x)}{q_i(x)}}|T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
&\leq \frac{1}{4m}\int_\Omega |T_n(u^n)|^{\pi(x)-1}|T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
&\quad+\frac{C_0}{m}\int_\Omega |T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx + 2e^{B(\infty)}\int_\Omega |\rho(x)|\frac{|T_m(u^n)|}{m}dx \\
&\quad+\frac{\alpha(\sigma-1)}{4m}\int_\Omega\sum_{i=1}^N \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
&\quad+\frac{C_1}{m}\int_\Omega\sum_{i=1}^N (1+|u^n|)^{\frac{\sigma\delta_i(x)}{q_i(x)-\delta_i(x)}}|T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx,
\end{aligned}$$

ultimately, we have

$$
\begin{aligned}
|\mathcal{I}_9^n| \leq \frac{1}{4m}\int_\Omega |T_n(u^n)|^{\pi(x)-1}|T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
+\frac{C_0}{m}\int_\Omega |T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx + 2e^{B(\infty)}\int_\Omega |\rho(x)|\frac{|T_m(u^n)|}{m}dx \\
+\frac{\alpha(\sigma-1)}{4m}\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
+\frac{1}{4m}\int_\Omega |T_n(u^n)|^{\pi(x)-1}|T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx \\
+\frac{C_2}{m}\int_\Omega |T_m(u^n)|\chi(u^n)e^{B(|u^n|)}dx.
\end{aligned}
\tag{3.36}
$$

• **Estimate of $\mathcal{I}_{10}^n$**: The last integral in the right hand side of (3.18) becomes

$$
\begin{aligned}
\mathcal{I}_{10}^n = \frac{1}{m}\sum_{i=1}^N\int_\Omega \phi_i^n(u^n)\partial_{x_i}T_m(u^n)\chi(u^n)e^{B(|u^n|)}dx \\
+\frac{(\sigma-1)}{m}\sum_{i=1}^N\int_\Omega \frac{\phi_i^n(u^n)\partial_{x_i}u^n}{(1+|u^n|)^\sigma}|T_m(u^n)|e^{B(|u^n|)}dx \\
+\frac{2}{\alpha m}\sum_{i=1}^N\int_\Omega \phi_i^n(u^n)\partial_{x_i}u^n|T_m(u^n)|\chi(u^n)b(|u^n|)e^{B(|u^n|)}dx
\end{aligned}
$$

then, using Young Inequality, we get

$$
\begin{aligned}
|\mathcal{I}_{10}^n| \leq C(\alpha)\sum_{i=1}^N\int_\Omega |\phi_i^n|^{q_i'(x)}b(|u^n|)\frac{|T_m(u^n)|}{m}\chi(u^n)e^{B(|u^n|)}dx \\
+C(\alpha)(\sigma-1)\sum_{i=1}^N\int_\Omega |\phi_i^n|^{q_i'(x)}\frac{|T_m(u^n)|}{m}e^{B(|u^n|)}dx \\
+\frac{C(\alpha)}{m}\sum_{i=1}^N\int_\Omega |\phi_i^n|^{q_i'(x)}\chi(u^n)e^{B(|u^n|)}dx \\
+\frac{\alpha(\sigma-1)}{m}\sum_{i=1}^N\int_\Omega \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_m(u^n)|e^{B(|u^n|)}dx \\
+\frac{2}{m}\sum_{i=1}^N\int_\Omega |\partial_{x_i}u^n|^{q_i(x)}|T_m(u^n)|\chi(u^n)b(|u^n|)e^{B(|u^n|)}dx \\
+\frac{\alpha}{4m}\sum_{i=1}^N\int_\Omega |\partial_{x_i}T_m(u^n)|^{q_i(x)}\chi(u^n)e^{B(|u^n|)}dx.
\end{aligned}
\tag{3.37}
$$

Next, combining estimates (3.33), (3.34) (3.35), (3.36) and (3.37), it follows that

$$\begin{aligned}
&\frac{1}{2m}\sum_{i=1}^{N}\int_\Omega a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}T_m(u^n)dx\\
&+\frac{\alpha(\sigma-1)}{4m}\sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_m(u^n)|dx\\
&+\frac{1}{2m}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u^n|^{q_i(x)}|T_m(u^n)|b(|u^n|)dx\\
&+\frac{1}{2m}\int_\Omega |T_n(u^n)|^{\pi(x)-2}T_n(u^n)T_m(u^n)dx+\frac{1}{nm}\int_\Omega |u^n|^{q_0(x)-2}u^nT_m(u^n)dx\\
&\leq 2e^{B(\infty)}\int_\Omega \big(|\gamma_0(x)|+|\rho(x)|+|f^n|+C_0+C_2\big)\frac{|T_m(u^n)|}{m}dx\\
&+C(\alpha)e^{B(\infty)}(\sigma-1)\sum_{i=1}^{N}\int_\Omega\big(|g_i|^{p_i'(x)}+|\phi_i^n(u^n)|^{p_i'(x)}\big)\frac{|T_m(u^n)|}{m}dx\\
&+C(\alpha)2e^{B(\infty)}\sum_{i=1}^{N}\int_\Omega\big(|g_i|^{p_i'(x)}+|\phi_i^n(u^n)|^{p_i'(x)}\big)b(|u^n|)\frac{|T_m(u^n)|}{m}dx\\
&+2e^{B(\infty)}\int_{\partial\Omega}|h^n|\frac{|T_m(u^n)|}{m}d\sigma+2e^{B(\infty)}\sum_{i=1}^{N}\int_{\partial\Omega}\big(|\phi_i^n(u^n)|+|g_i|\big)\frac{|T_m(u^n)|}{m}d\sigma\\
&+2e^{B(\infty)}\frac{C(\alpha)}{m}\sum_{i=1}^{N}\int_\Omega\big(|g_i|^{p_i'(x)}+|\phi_i^n(u^n)|^{p_i'(x)}\big)dx.
\end{aligned}\tag{3.38}$$

From (3.27), we have: $\text{meas}(\{|u^n|>m\})\longrightarrow 0$ as m tends to infinity, so $\frac{|T_m(u^n)|}{m}\rightharpoonup 0$ weak* in $L^\infty(\Omega)$. Then, using the Lebesgue's dominated convergence theorem, all the integral terms on the right-hand side of the inequality (3.38) converge to zero. Thus, by sending m tends to infinity in (3.38), we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{m}\sum_{i=1}^{N}\int_{\{|u^n|\leq m\}}a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}u^ndx=0.\tag{3.39}$$

Moreover, we have

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{|u^n|>m\}}\frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}dx=0,\tag{3.40}$$

then

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{|u^n|>m\}}|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)dx=0.\tag{3.41}$$

Step 4: The equi-integrability of $(|T_n(u^n)|^{\pi(x)-2}T_n(u^n))_n$ **and** $(\frac{1}{n}|u^n|^{q_0(.)-2}u^n)_n$.

In order to pass to the limit in approximate Problem (3.18), let us prove that

$$\begin{aligned} |T_n(u^n)|^{\pi(x)-2}T_n(u^n) &\longrightarrow |u|^{\pi(x)-2}u \\ \text{and} \quad \frac{1}{n}|u^n|^{q_0(.)-2}u^n &\longrightarrow 0 \quad \text{strongly in } L^1(\Omega). \end{aligned} \tag{3.42}$$

From the convergence (3.31), we have

$$|T_n(u^n)|^{\pi(x)-2}T_n(u^n) \longrightarrow |u|^{\pi(x)-2}u \quad \text{and} \quad |u^n|^{q_0(.)-2}u^n \longrightarrow |u|^{q_0(.)-2}u \quad a.e \text{ in } \Omega.$$

So, considering Vitali's Theorem, it remains to prove that $(|u^n|^{\pi(x)-2}u^n)_n$ and $(\frac{1}{n}|u^n|^{q_0(.)-2}u^n)_n$ are uniformly equi-integrable. Indeed, on the one hand, thanks to (3.38) and (3.39)-(3.41), we can deduce

$$\int_{\{|u^n|>m\}} |T_n(u^n)|^{\pi(x)-1}dx + \frac{1}{n}\int_{\{|u^n|>m\}} |u^n|^{q_0(x)-1}dx \longrightarrow 0 \quad \text{as } m \to \infty, \tag{3.43}$$

hence, it follows that for any $\varepsilon > 0$, there exists a sufficiently large $m(\varepsilon) > 0$ such that

$$\int_{\{|u^n|>m(\varepsilon)\}} |T_n(u^n)|^{\pi(x)-1}dx + \frac{1}{n}\int_{\{|u^n|>m(\varepsilon)\}} |u^n|^{q_0(x)-1}dx \leq \frac{\varepsilon}{2}. \tag{3.44}$$

On the other hand, for any measurable subset $E \subset \Omega$, there holds

$$\begin{aligned} &\int_E |T_n(u^n)|^{\pi(x)-1}dx + \frac{1}{n}\int_E |u^n|^{q_0(x)-1}dx \\ &\leq \int_E |T_{m(\varepsilon)}(u^n)|^{\pi(x)-1}dx + \frac{1}{n}\int_E |T_{m(\varepsilon)}(u^n)|^{q_0(x)-1}dx \\ &+ \int_{\{|u^n|>m(\varepsilon)\}} |T_n(u^n)|^{\pi(x)-1}dx + \frac{1}{n}\int_{\{|u^n|>m(\varepsilon)\}} |u^n|^{q_0(x)-1}dx. \end{aligned} \tag{3.45}$$

Evidently, there exists $\mu(\varepsilon) > 0$, such that for all $E \subset \Omega$ with $\text{meas}(E) \leq \mu(\varepsilon)$, we have

$$\int_E |T_{m(\varepsilon)}(u^n)|^{\pi(x)-1}dx + \frac{1}{n}\int_E |T_{m(\varepsilon)}(u^n)|^{q_0(x)-1}dx \leq \frac{\varepsilon}{2}. \tag{3.46}$$

Finally, combining (3.44) and (3.46), we can assume that: For any $\varepsilon > 0$, there exists $\mu(\varepsilon) > 0$ such that

$$\int_E |T_n(u^n)|^{\pi(x)-1}dx + \frac{1}{n}\int_E |u^n|^{q_0(x)-1}dx \leq \varepsilon, \tag{3.47}$$

for any $E \subset \Omega$, such that $\text{meas}(E) \leq \mu(\varepsilon)$.
This implies that the sequence $(|T_n(u^n)|^{\pi(x)-2}T_n(u^n))_n$ and $(\frac{1}{n}|u^n|^{q_0(.)-2}u^n)_n$ are equi-integrable, which conclude the proof of the convergence (3.42).

Step 5: Strong convergence of truncations.

In this step, in order to verify part of the hypotheses of Theorem 2.5, we establish the following strong convergences

$$T_k(u^n) \longrightarrow T_k(u) \text{ strongly in } W^{1,\vec{q}(.)}(\Omega)$$
$$\text{and} \quad \partial_{x_i} u^n \longrightarrow \partial_{x_i} u \quad \text{a.e in } \Omega \text{ for } i = 1, \dots, N,$$

and prove that the condition (2.6) is satisfied.
Let us put

$$\theta_m(\nu) = 1 - \frac{|T_{2m}(\nu) - T_m(\nu)|}{m}$$

and

$$\Psi(s) = s\exp(\frac{\vartheta^2 s^2}{2}) \quad \text{with } \vartheta = 2(\sigma - 1) + \frac{6}{\alpha}\|b(.)\|_{L^\infty(\mathbb{R})}.$$

Evidently,

$$\Psi'(s) - \vartheta|\Psi(s)| \geq \frac{1}{2} \quad \forall s \in \mathbb{R}. \tag{3.48}$$

Let $m \geq k \geq 1$ and $z_n = T_k(u^n) - T_k(u)$. It is clear that for $|u^n| > k$, we have

$$z_n u^n \geq 0 \quad \text{and} \quad \partial_{x_i} z_n = -\partial_{x_i} T_k(u). \tag{3.49}$$

From (3.32), we get the weak convergence $z_n \rightharpoonup 0$ weakly in $L^{q_i(.)}(\Omega)$, thus, we have the weak convergence

$$\partial_{x_i} z_n \rightharpoonup 0 \quad \text{weakly in } L^{q_i(.)}(\Omega). \tag{3.50}$$

On again, in view of (3.32), we get

$$\Psi(z_n) \longrightarrow 0 \text{ a.e. in } \Omega, \text{ as } n \to \infty, \tag{3.51}$$

and

$$|\Psi(z_n)| \leq \Psi(2k), \qquad 1 \leq \Psi'(z_n) \leq \Psi'(2k), \quad n \in \mathbb{N}. \tag{3.52}$$

Using (3.51)-(3.52), we deduce the convergence

$$|\Psi(z_n)| \rightharpoonup 0 \text{ weak* in } L^\infty(\Omega). \tag{3.53}$$

Replacing v by $\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}$ in (3.18) and thanks to (3.49), we obtain

$$\begin{aligned}
\mathcal{I}_1^n = \sum_{i=1}^N \int_\Omega a_i\big(x, T_n(u^n), \nabla u^n\big)\partial_{x_i} z_n \Psi'\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx \\
-\frac{1}{m}\sum_{i=1}^N \int_{\{m<|u^n|\leq 2m\}} a_i\big(x, T_n(u^n), \nabla u^n\big)\partial_{x_i} u^n |\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx \\
+(\sigma - 1)\sum_{i=1}^N \int_\Omega \frac{a_i\big(x, T_n(u^n), \nabla u^n\big)\partial_{x_i} u^n}{(1 + |u^n|)^\sigma}\text{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx \\
+\frac{2}{\alpha}\sum_{i=1}^N \int_\Omega a_i\big(x, T_n(u^n), \nabla u^n\big)\partial_{x_i} u^n b(|u^n|)\text{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx,
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{I}_2^n+\mathcal{I}_3^n+\mathcal{I}_4^n+\mathcal{I}_5^n+\mathcal{I}_6^n\\
&=\int_\Omega \mathcal{N}_n(x,u^n,\nabla u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+\int_\Omega |T_n(u^n)|^{\pi(x)-2}T_n(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+\frac{1}{n}\int_\Omega |u^n|^{q_0(x)-2}u^n\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+\kappa\int_{\partial\Omega} T_n(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}d\sigma\\
&+\sum_{i=1}^N\int_{\partial\Omega}\phi_i^n(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}d\sigma,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_7^n=&\int_\Omega f^n\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx-\sum_{i=1}^N\int_{\partial\Omega} g_i\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}d\sigma\\
&+\sum_{i=1}^N\int_\Omega g_i\partial_{x_i}z_n\Psi'\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&-\frac{1}{m}\sum_{i=1}^N\int_{\{m<|u^n|\le 2m\}} g_i\partial_{x_i}u^n|\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx\\
&+(\sigma-1)\sum_{i=1}^N\int_\Omega \frac{g_i\partial_{x_i}u^n}{(1+|u^n|)^\sigma}\operatorname{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx\\
&+\frac{2}{\alpha}\sum_{i=1}^N\int_\Omega g_i\partial_{x_i}u^n b(|u^n|)\operatorname{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_8^n+\mathcal{I}_9^n+\mathcal{I}_{10}^n=&\int_{\partial\Omega} h^n\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}d\sigma\\
&+\int_\Omega \mathcal{R}_n(x,T_n(u^n),\nabla u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+\sum_{i=1}^N\int_\Omega \phi_i^n(u^n)\partial_{x_i}z_n\Psi'\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&-\frac{1}{m}\sum_{i=1}^N\int_{\{m<|u^n|\le 2m\}}\phi_i^n(u^n)\partial_{x_i}u^n|\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx\\
&+(\sigma-1)\sum_{i=1}^N\int_\Omega\frac{\phi_i^n(u^n)\partial_{x_i}u^n}{(1+|u^n|)^\sigma}\operatorname{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx\\
&+\frac{2}{\alpha}\sum_{i=1}^N\int_\Omega\phi_i^n(u^n)\partial_{x_i}u^n b(|u^n|)\operatorname{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx.
\end{aligned}
$$

Next, according to (3.52) and (1.8)-(1.9), we can deduce the following inequality

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}z_n\Psi'\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+(\sigma-1)\sum_{i=1}^{N}\int_\Omega \frac{a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}u^n}{(1+|u^n|)^\sigma}\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx\\
&+\frac{2}{\alpha}\sum_{i=1}^{N}\int_\Omega a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}u^n b(|u^n|)\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&\leq 2e^{B(\infty)}\int_\Omega\big(\gamma_0(x)+\rho(x)+|f^n|\big)|\Psi\big(z_n\big)|dx+2e^{B(\infty)}\int_\Omega |T_n(u^n)|^{\delta_0(x)}|\Psi\big(z_n\big)|dx\\
&+2e^{B(\infty)}\int_\Omega |T_n(u^n)|^{\pi(x)-1}|\Psi\big(z_n\big)|dx+\frac{2e^{B(\infty)}}{n}\int_\Omega |u^n|^{q_0(x)-1}|\Psi\big(z_n\big)|dx\\
&+\frac{2\Psi(2k)e^{B(\infty)}}{m}\sum_{i=1}^{N}\int_{\{m<|u^n|\leq 2m\}} a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}u^n dx\\
&+2e^{B(\infty)}\int_{\partial\Omega}|h^n||\Psi\big(z_n\big)|d\sigma+2e^{B(\infty)}\kappa\int_{\partial\Omega}|T_n(u^n)||\Psi\big(z_n\big)|d\sigma\\
&+2e^{B(\infty)}\sum_{i=1}^{N}\int_{\partial\Omega}|\phi_i^n(u^n)||\Psi\big(z_n\big)|d\sigma+2e^{B(\infty)}\sum_{i=1}^{N}\int_{\partial\Omega}|g_i||\Psi\big(z_n\big)|d\sigma\\
&+\frac{2e^{B(\infty)}\Psi(2k)}{m}\sum_{i=1}^{N}\int_{\{m<|u^n|\leq 2m\}}|g_i||\partial_{x_i}u^n|dx\\
&+\frac{2e^{B(\infty)}\Psi(2k)}{m}\sum_{i=1}^{N}\int_{\{m<|u^n|\leq 2m\}}|\phi_i^n(u^n)||\partial_{x_i}u^n|dx\\
&+2e^{B(\infty)}\Psi'(2k)\sum_{i=1}^{N}\int_\Omega|\phi_i^n(u^n)||\partial_{x_i}z_n|dx+2e^{B(\infty)}\Psi'(2k)\sum_{i=1}^{N}\int_\Omega|g_i||\partial_{x_i}z_n|dx\\
&+\int_\Omega\sum_{i=1}^{N}|\partial_{x_i}u^n|^{\delta_i(x)}|\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+\int_\Omega\sum_{i=1}^{N}|\partial_{x_i}u^n|^{q_i(x)}|b(|u^n|)\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+(\sigma-1)\sum_{i=1}^{N}\int_\Omega\frac{g_i\partial_{x_i}u^n}{(1+|u^n|)^\sigma}\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx\\
&+\frac{2}{\alpha}\sum_{i=1}^{N}\int_\Omega g_i\partial_{x_i}u^n b(|u^n|)\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+(\sigma-1)\sum_{i=1}^{N}\int_\Omega\frac{\phi_i^n(u^n)\partial_{x_i}u^n}{(1+|u^n|)^\sigma}\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx\\
&+\frac{2}{\alpha}\sum_{i=1}^{N}\int_\Omega \phi_i^n(u^n)\partial_{x_i}u^n b(|u^n|)\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx.
\end{aligned}
\tag{3.54}
$$

Let us denote by $\varepsilon_i(n)$, $n, i \in \mathbb{N}$, some functions which converge to 0 as n tends to infinity, similarly for $\varepsilon_i(m)$ and $\varepsilon_i(n,m)$. Since $\gamma_0(x)$, $\rho(x)$ and f^n belong to $L^1(\Omega)$, applying the convergence (3.53) we deduce that

$$\varepsilon_0(n) = 2e^{B(\infty)} \int_\Omega \big(\gamma_0(x) + \rho(x) + |f^n|\big)|\Psi\big(z_n\big)|dx \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.55}$$

In the same way, since $\delta_0(x) \leq \pi(x) - 1$ a.e. in Ω and in view of convergence (3.42), we claim that

$$\varepsilon_1(n) = 2e^{B(\infty)} \int_\Omega |T_n(u^n)|^{\delta_0(x)}|\Psi\big(z_n\big)|dx \longrightarrow 0 \quad \text{as } n \to \infty \tag{3.56}$$

and

$$\varepsilon_2(n) = 2e^{B(\infty)} \int_\Omega |T_n(u^n)|^{\pi(x)-1}|\Psi\big(z_n\big)|dx \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.57}$$

Analogously, thanks to (3.53), we get the following estimates

$$\varepsilon_3(n) = \frac{2e^{B(\infty)}}{n} \int_\Omega |u^n|^{q_0(x)-1}\Psi\big(z_n\big)dx \longrightarrow 0 \quad \text{as } n \to \infty, \tag{3.58}$$

$$\varepsilon_4(n) = 2e^{B(\infty)}\kappa \int_{\partial\Omega} |T_n(u^n)||\Psi\big(z_n\big)|d\sigma \longrightarrow 0 \quad \text{as } n \to \infty, \tag{3.59}$$

$$\varepsilon_5(n) = 2e^{B(\infty)} \int_{\partial\Omega} |h^n||\Psi\big(z_n\big)|d\sigma \longrightarrow 0 \quad \text{as } n \to \infty, \tag{3.60}$$

$$\varepsilon_6(n) = 2e^{B(\infty)} \sum_{i=1}^N \int_{\partial\Omega} |\phi_i^n(u^n)||\Psi\big(z_n\big)|d\sigma \longrightarrow 0 \quad \text{as } n \to \infty, \tag{3.61}$$

and

$$\varepsilon_7(n) = 2e^{B(\infty)} \sum_{i=1}^N \int_{\partial\Omega} |g_i||\Psi\big(z_n\big)|d\sigma \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.62}$$

From estimate (3.39), we obtain evidently,

$$\begin{cases} \varepsilon_8(m) = \dfrac{2\Psi(2k)e^{B(\infty)}}{m} \\ \qquad \times \displaystyle\sum_{i=1}^N \int_{\{m<|u^n|\leq 2m\}} a_i\big(x, T_n(u^n), \nabla u^n\big)\partial_{x_i}u^n dx \longrightarrow 0 \quad \text{as } m \to \infty, \end{cases} \tag{3.63}$$

$$\varepsilon_9(m) = \frac{2e^{B(\infty)\Psi(2k)}}{m} \sum_{i=1}^N \int_{\{m<|u^n|\leq 2m\}} g_i\partial_{x_i}u^n dx \longrightarrow 0 \text{ as } m \to \infty \tag{3.64}$$

and

$$\varepsilon_{10}(m) = \frac{2e^{B(\infty)\Psi(2k)}}{m} \sum_{i=1}^N \int_{\{m<|u^n|\leq 2m\}} |\phi_i^n(u^n)|\partial_{x_i}u^n dx \longrightarrow 0 \text{ as } m \to \infty. \tag{3.65}$$

Moreover, from the convergence (3.50), we get the estimates

$$\varepsilon_{11}(n) = 2e^{B(\infty)}\Psi'(2k)\sum_{i=1}^{N}\int_{\Omega}|\phi_i^n(u^n)||\partial_{x_i}z_n|dx \longrightarrow 0 \quad \text{as } n\to\infty \tag{3.66}$$

and

$$\varepsilon_{12}(n) = 2e^{B(\infty)}\Psi'(2k)\sum_{i=1}^{N}\int_{\Omega}|g_i||\partial_{x_i}z_n|dx \longrightarrow 0 \quad \text{as } n\to\infty. \tag{3.67}$$

Combining (3.54) and estimates (3.55)−(3.67), we obtain

$$\begin{aligned}
&\underbrace{\sum_{i=1}^{N}\int_{\Omega}a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}z_n\Psi'\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx}_{\mathcal{S}_1(n,m)}\\
&\underbrace{+(\sigma-1)\sum_{i=1}^{N}\int_{\Omega}\frac{a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}u^n}{(1+|u^n|)^{\sigma}}\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx}_{\mathcal{S}_2(n,m)}\\
&\underbrace{+\frac{2}{\alpha}\sum_{i=1}^{N}\int_{\Omega}a_i\big(x,T_n(u^n),\nabla u^n\big)\partial_{x_i}u^n b(|u^n|)\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx}_{\mathcal{S}_3(n,m)}\\
&\leq \underbrace{\int_{\Omega}\sum_{i=1}^{N}|\partial_{x_i}u^n|^{\delta_i(x)}|\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx}_{\mathcal{S}_4(n,m)}\\
&\underbrace{+\int_{\Omega}\sum_{i=1}^{N}|\partial_{x_i}u^n|^{q_i(x)}|b(|u^n|)\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx}_{\mathcal{S}_5(n,m)}\\
&\underbrace{+(\sigma-1)\sum_{i=1}^{N}\int_{\Omega}\frac{g_i\partial_{x_i}u^n}{(1+|u^n|)^{\sigma}}\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)e^{B(|u_n|)}dx}_{\mathcal{S}_6(n,m)}\\
&\underbrace{+\frac{2}{\alpha}\sum_{i=1}^{N}\int_{\Omega}g_i\partial_{x_i}u^n b(|u^n|)\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx}_{\mathcal{S}_7(n,m)}\\
&\underbrace{+(\sigma-1)\sum_{i=1}^{N}\int_{\Omega}\frac{\phi_i^n(u^n)\partial_{x_i}u^n}{(1+|u^n|)^{\sigma}}\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx}_{\mathcal{S}_8(n,m)}\\
&\underbrace{+\frac{2}{\alpha}\sum_{i=1}^{N}\int_{\Omega}\phi_i^n(u^n)\partial_{x_i}u^n b(|u^n|)\mathrm{sign}(u^n)\Psi\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx}_{\mathcal{S}_9(n,m)}+\varepsilon_{13}(n,m).
\end{aligned}$$

Next, applying (3.49), we get

$$\begin{aligned}
&\mathcal{S}_1(n,m)+\mathcal{S}_2(n,m)+\mathcal{S}_3(n,m)\\
&=\sum_{i=1}^N\int_\Omega a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}z_n\Psi'\big(z_n\big)\chi(u^n)e^{B(|u^n|)}dx\\
&-\sum_{i=1}^N\int_{\{k<|u^n|\le 2m\}} a_i\big(x,T_{2m}(u^n),\nabla T_{2m}(u^n)\big)\partial_{x_i}T_k(u)\Psi'\big(z_n\big)\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&-(\sigma-1)\sum_{i=1}^N\int_\Omega \frac{a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}T_k(u^n)}{(1+|u^n|)^\sigma}|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\
&+\alpha(\sigma-1)\sum_{i=1}^N\int_{\{k<|u^n|\le 2m\}}\frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|\Psi\big(z_n\big)|\theta_m(u^n)e^{B(|u^n|)}dx\\
&-\frac{2}{\alpha}\sum_{i=1}^N\int_\Omega a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}T_k(u^n)b(|u^n|)|\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx\\
&+2\sum_{i=1}^N\int_{\{k<|u^n|\le 2m\}}|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx,
\end{aligned}\tag{3.68}$$

$$\begin{aligned}
&\mathcal{S}_4(n,m)+\mathcal{S}_5(n,m)\\
&=\int_\Omega\sum_{i=1}^N|\partial_{x_i}u^n|^{\delta_i(x)}|\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+\int_\Omega\sum_{i=1}^N|\partial_{x_i}u^n|^{q_i(x)}|b(|u^n|)\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{S}_6(n,m)+\mathcal{S}_7(n,m)\\
&=-(\sigma-1)\sum_{i=1}^N\int_\Omega\frac{g_i\partial_{x_i}T_k(u^n)}{(1+|u^n|)^\sigma}|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\
&+(\sigma-1)\sum_{i=1}^N\int_{\{k<|u^n|\le 2m\}}\frac{g_i\partial_{x_i}u^n}{(1+|u^n|)^\sigma}|\Psi\big(z_n\big)|\theta_m(u^n)e^{B(|u^n|)}dx\\
&-\frac{2}{\alpha}\sum_{i=1}^N\int_\Omega g_i\partial_{x_i}T_k(u^n)b(|u^n|)|\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx\\
&+\frac{2}{\alpha}\sum_{i=1}^N\int_{\{k<|u^n|\le 2m\}}g_i\partial_{x_i}u^nb(|u^n|)|\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{S}_8(n,m)+\mathcal{S}_9(n,m)\\
&=-(\sigma-1)\sum_{i=1}^{N}\int_\Omega \frac{\phi_i^n(u^n)\partial_{x_i}T_k(u^n)}{(1+|u^n|)^\sigma}|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\
&+(\sigma-1)\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} \frac{\phi_i^n(u^n)\partial_{x_i}u^n}{(1+|u^n|)^\sigma}|\Psi\big(z_n\big)|\theta_m(u^n)e^{B(|u^n|)}dx\\
&-\frac{2}{\alpha}\sum_{i=1}^{N}\int_\Omega \phi_i^n(u^n)\partial_{x_i}T_k(u^n)b(|u^n|)|\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx\\
&+\frac{2}{\alpha}\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} \phi_i^n(u^n)\partial_{x_i}u^n b(|u^n|)|\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx.
\end{aligned}$$

In view of Young's Inequality, it follows that

$$\begin{aligned}
\mathcal{S}_6(n,m)+\mathcal{S}_7(n,m)\le{}& \frac{\alpha(\sigma-1)}{3}\sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}T_k(u^n)|^{q_i(x)}}{(1+|u^n|)^\sigma}|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\
&+\underbrace{(\sigma-1)C(\alpha)e^{B(\infty)}\sum_{i=1}^{N}\int_\Omega |g_i|^{q_i'(x)}|\Psi\big(z_n\big)|dx}_{\varepsilon_{14}(n)}\\
&+\frac{\alpha(\sigma-1)}{3}\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|\Psi\big(z_n\big)|\theta_m(u^n)e^{B(|u^n|)}dx\\
&+\underbrace{(\sigma-1)C(\alpha)e^{B(\infty)}\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} |g_i|^{q_i'(x)}|\Psi\big(z_n\big)|\theta_m(u^n)dx}_{\varepsilon_{15}(n)}\\
&+\frac{1}{3}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}T_k(u^n)|^{q_i(x)}b(|u^n|)|\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx\\
&+\underbrace{2C(\alpha)e^{B(\infty)}\|b(.)\|_{L^\infty(\mathbb{R})}\sum_{i=1}^{N}\int_\Omega |g_i|^{q_i'(x)}|\Psi\big(z_n\big)|dx}_{\varepsilon_{16}(n)}\\
&+\frac{1}{2}\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} |\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+\underbrace{2C(\alpha)e^{B(\infty)}\|b(.)\|_{L^\infty}\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} |g_i|^{q_i'(x)}|\Psi\big(z_n\big)|\theta_m(u^n)dx}_{\varepsilon_{17}(n)}
\end{aligned}\tag{3.69}$$

and

$$\begin{aligned}
&\mathcal{S}_8(n,m)+\mathcal{S}_9(n,m)\\
&\leq \frac{\alpha(\sigma-1)}{3}\sum_{i=1}^{N}\int_\Omega \frac{|\partial_{x_i}T_k(u^n)|^{q_i(x)}}{(1+|u^n|)^\sigma}|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\
&\underbrace{+(\sigma-1)C(\alpha)e^{B(\infty)}\sum_{i=1}^{N}\int_\Omega |\phi_i^n(u^n)|^{q_i'(x)}|\Psi\big(z_n\big)|dx}_{\varepsilon_{18}(n)}\\
&+\frac{\alpha(\sigma-1)}{3}\sum_{i=1}^{N}\int_{\{k<|u^n|\leq 2m\}} \frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|\Psi\big(z_n\big)|\theta_m(u^n)e^{B(|u^n|)}dx\\
&\underbrace{+(\sigma-1)C(\alpha)e^{B(\infty)}\sum_{i=1}^{N}\int_{\{k<|u^n|\leq 2m\}} |\phi_i^n(u^n)|^{q_i'(x)}|\Psi\big(z_n\big)|\theta_m(u^n)dx}_{\varepsilon_{19}(n)}\\
&+\frac{1}{3}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}T_k(u^n)|^{q_i(x)}b(|u^n|)|\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx\\
&\underbrace{+2C(\alpha)e^{B(\infty)}\|b(.)\|_{L^\infty}\sum_{i=1}^{N}\int_\Omega |\phi_i^n(u^n)|^{q_i'(x)}|\Psi\big(z_n\big)|dx}_{\varepsilon_{20}(n)}\\
&+\frac{1}{2}\sum_{i=1}^{N}\int_{\{k<|u^n|\leq 2m\}} |\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&\underbrace{+2C(\alpha)e^{B(\infty)}\|b(.)\|_{L^\infty}\sum_{i=1}^{N}\int_{\{k<|u^n|\leq 2m\}} |\phi_i^n(u^n)|^{q_i'(x)}|\Psi\big(z_n\big)|\theta_m(u^n)dx}_{\varepsilon_{21}(n)}+\varepsilon_{13}(n,m).
\end{aligned} \tag{3.70}$$

Exploiting the convergence (3.53), we deduce the following estimates

$$\varepsilon_{14}(n)=C(\alpha)(\sigma-1)e^{B(\infty)}\sum_{i=1}^{N}\int_\Omega |g_i|^{q_i'(x)}|\Psi\big(z_n\big)|dx\longrightarrow 0 \quad \text{as } n\to\infty, \tag{3.71}$$

$$\begin{cases}\varepsilon_{15}(n)=(\sigma-1)C(\alpha)e^{B(\infty)}\\ \qquad\times\displaystyle\sum_{i=1}^{N}\int_{\{k<|u^n|\leq 2m\}} |g_i|^{q_i'(x)}|\Psi\big(z_n\big)|\theta_m(u^n)dx\longrightarrow 0 \quad \text{as } n\to\infty,\end{cases} \tag{3.72}$$

$$\begin{cases}\varepsilon_{16}(n)=2C(\alpha)e^{B(\infty)}\|b(.)\|_{L^\infty(\mathbb{R})}\\ \qquad\qquad\times\displaystyle\sum_{i=1}^{N}\int_\Omega |g_i|^{q_i'(x)}|\Psi\big(z_n\big)|dx\longrightarrow 0 \quad \text{as } n\to\infty,\end{cases} \tag{3.73}$$

$$\begin{cases} \varepsilon_{17}(n) = 2C(\alpha)e^{B(\infty)}\|b(.)\|_{L^\infty} \\ \qquad \times \sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} |g_i|^{q_i'(x)}|\Psi\big(z_n\big)|\theta_m(u^n)dx \longrightarrow 0 \quad \text{as } n\to\infty, \end{cases} \tag{3.74}$$

$$\begin{cases} \varepsilon_{18}(n) = (\sigma-1)C(\alpha)e^{B(\infty)} \\ \qquad \times \sum_{i=1}^{N}\int_{\Omega} |\phi_i^n(u^n)|^{q_i'(x)}|\Psi\big(z_n\big)|dx \longrightarrow 0 \text{ as } n\to\infty, \end{cases} \tag{3.75}$$

$$\begin{cases} \varepsilon_{19}(n) = (\sigma-1)C(\alpha)e^{B(\infty)} \\ \qquad \times \sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} |\phi_i^n(u^n)|^{q_i'(x)}|\Psi\big(z_n\big)|\theta_m(u^n)dx \longrightarrow 0 \quad \text{as } n\to\infty, \end{cases} \tag{3.76}$$

$$\begin{cases} \varepsilon_{20}(n) = 2C(\alpha)e^{B(\infty)}\|b(.)\|_{L^\infty} \\ \qquad \times \sum_{i=1}^{N}\int_{\Omega} |\phi_i^n(u^n)|^{q_i'(x)}|\Psi\big(z_n\big)|dx \longrightarrow 0 \quad \text{as } n\to\infty \end{cases} \tag{3.77}$$

and

$$\begin{cases} \varepsilon_{21}(n) = 2C(\alpha)e^{B(\infty)}\|b(.)\|_{L^\infty} \\ \qquad \times \sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} |\phi_i^n(u^n)|^{q_i'(x)}|\Psi\big(z_n\big)|\theta_m(u^n)dx \longrightarrow 0 \text{ as } n\to\infty. \end{cases} \tag{3.78}$$

Then, combining (3.68),(3.69),(3.70) and (3.71)-(3.76), we derive

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}z_n\Psi'\big(z_n\big)\chi(u^n)e^{B(|u^n|)}dx\\
&-2\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} a_i\big(x,T_{2m}(u^n),\nabla T_{2m}(u^n)\big)\partial_{x_i}T_k(u)\Psi'\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx\\
&-(\sigma-1)\sum_{i=1}^{N}\int_{\Omega}\frac{a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}T_k(u^n)}{(1+|u^n|)^{\sigma}}|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\
&+\frac{\alpha(\sigma-1)}{3}\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}}\frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^{\sigma}}|\Psi\big(z_n\big)|\theta_m(u^n)e^{B(|u^n|)}dx\\
&-\frac{2}{\alpha}\sum_{i=1}^{N}\int_{\Omega} a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}T_k(u^n)b(|u^n|)|\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx\\
&+\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}}|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&\le\int_{\Omega}\sum_{i=1}^{N}|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)|\Psi\big(z_n\big)|\theta_m(u^n)\chi(u^n)e^{B(|u^n|)}dx\\
&+\frac{\alpha(\sigma-1)}{3}\int_{\Omega}\sum_{i=1}^{N}\frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^{\sigma}}|\Psi\big(z_n\big)|\theta_m(u^n)e^{B(|u^n|)}dx\\
&+\frac{1}{2}\int_{\Omega}|T_n(u^n)|^{\pi(x)-1}|\Psi\big(z_n\big)|\theta_m(u^n)e^{B(|u^n|)}dx+C_2\int_{\Omega}|\Psi\big(z_n\big)|\theta_m(u^n)e^{B(|u^n|)}dx\\
&+\frac{2\alpha(\sigma-1)}{3}\sum_{i=1}^{N}\int_{\Omega}\frac{|\partial_{x_i}T_k(u^n)|^{q_i(x)}}{(1+|u^n|)^{\sigma}}|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\
&+\frac{2}{3}\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}T_k(u^n)|^{q_i(x)}b(|u^n|)|\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx+\varepsilon_{22}(n,m).
\end{aligned}
$$

Now, using (1.4), it follows that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}z_n\Psi'\big(z_n\big)\chi(u^n)e^{B(|u^n|)}dx\\
&-2\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}} a_i\big(x,T_{2m}(u^n),\nabla T_{2m}(u^n)\big)\partial_{x_i}T_k(u)\Psi'\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx\\
&-2(\sigma-1)\sum_{i=1}^{N}\int_{\Omega}\frac{a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}T_k(u^n)}{(1+|u^n|)^{\sigma}}|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\
&-\frac{3}{\alpha}\sum_{i=1}^{N}\int_{\Omega} a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}T_k(u^n)b(|u^n|)|\Psi\big(z_n\big)|\chi(u^n)e^{B(|u^n|)}dx\le\varepsilon_{23}(n,m).
\end{aligned}
\tag{3.79}
$$

Since $\big(a_i(x,T_{2m}(u^n),\nabla T_{2m}(u^n))\big)_k$ is bounded in $L^{q_i'(.)}(\Omega)$, then there exists a function $\varphi_i^m\in L^{q_i'(.)}(\Omega)$ such that

$$a_i(x,T_{2m}(u^n),\nabla T_{2m}(u^n))\rightharpoonup\varphi_i^m\quad\text{weakly in}\quad L^{q_i'(.)}(\overline{\Omega})\quad\text{as}\quad n\to\infty$$

for any $i=1,\dots,N$. From where, concerning the second term of the left-hand side of (3.78), we have

$$\begin{aligned}\varepsilon_{24}(n)&=\sum_{i=1}^{N}\Big|\int_{\{k<|u^n|\le 2m\}}a_i\big(x,T_{2m}(u^n),\nabla T_{2m}(u^n)\big)\partial_{x_i}T_k(u)\Psi'\big(z_n\big)\theta_m(u^n)e^{B(|u^n|)}dx\Big|\\&\le e^{B(\infty)}\Psi'(2k)\sum_{i=1}^{N}\int_{\{k<|u^n|\le 2m\}}\big|a_i\big(x,T_{2m}(u^n),\nabla T_{2m}(u^n)\big)\big|\big|\partial_{x_i}T_k(u)\big|dx\\&\longrightarrow e^{B(\infty)}\Psi'(2k)\sum_{i=1}^{N}\int_{\{k<|u|\le 2m\}}\big|\varphi_i^m\big|\big|\partial_{x_i}T_k(u)\big|dx=0\quad\text{as } n\to\infty.\end{aligned}$$

We conclude that

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}z_n\Psi'\big(z_n\big)\chi(u^n)e^{B(|u^n|)}dx\\&-\big(2(\sigma-1)+\frac{6\|b(.)\|_{L^\infty(\mathrm{R})}}{\alpha}\big)\sum_{i=1}^{N}\int_\Omega a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}T_k(u^n)|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\&\le\varepsilon_{25}(n,m).\end{aligned}$$

Having in mind that $\vartheta=2(\sigma-1)+\dfrac{6\|b(.)\|_{L^\infty(\mathrm{R})}}{\alpha}$, we deduce

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega\big(a_i(x,T_k(u^n),\nabla T_k(u^n))-a_i(x,T_k(u^n),\nabla T_k(u))\big)\partial_{x_i}z_n\Psi'\big(z_n\big)\chi(u^n)e^{B(|u^n|)}dx\\&+\sum_{i=1}^{N}\int_\Omega a_i\big(x,T_k(u^n),\nabla T_k(u)\big)\partial_{x_i}z_n\Psi'\big(z_n\big)\chi(u^n)e^{B(|u^n|)}dx\\&-\vartheta\sum_{i=1}^{N}\int_\Omega\big(a_i(x,T_k(u^n),\nabla T_k(u^n))-a_i(x,T_k(u^n),\nabla T_k(u))\big)\partial_{x_i}z_n|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\&-\vartheta\sum_{i=1}^{N}\int_\Omega a_i\big(x,T_k(u^n),\nabla T_k(u)\big)\partial_{x_i}z_n|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\\&-\vartheta\sum_{i=1}^{N}\int_\Omega a_i\big(x,T_k(u^n),\nabla T_k(u^n)\big)\partial_{x_i}T_k(u)|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\le\varepsilon_{26}(n,m).\end{aligned}\tag{3.80}$$

In view of the convergence (3.32), we have $T_k(u^n)\longrightarrow T_k(u)$ in $L^{q_i(.)}(\Omega)$, then

$$a_i\big(x,T_k(u^n),\nabla T_k(u)\big)\longrightarrow a_i\big(x,T_k(u),\nabla T_k(u)\big)\quad\text{strongly in}\quad L^{q_i'(.)}(\Omega),$$

and using the convergence (3.53), for the second term of the left-hand side of (3.80), we obtain

$$\begin{aligned} \varepsilon_{27}(n) &= \sum_{i=1}^{N}\int_{\Omega} a_i\big(x, T_k(u^n), \nabla T_k(u)\big)\partial_{x_i} z_n \Psi'\big(z_n\big)e^{B(|u^n|)}dx \\ &\leq \Psi'(2k)e^{B(\infty)}\sum_{i=1}^{N}\int_{\Omega} |a_i\big(x, T_k(u^n), \nabla T_k(u)\big)||\partial_{x_i} z_n|dx \longrightarrow 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{3.81}$$

Similarly, for the fourth term of the left-hand side of (3.80), we have

$$\begin{aligned} \varepsilon_{28}(n) &= \sum_{i=1}^{N}\int_{\Omega} a_i\big(x, T_k(u^n), \nabla T_k(u)\big)\partial_{x_i} z_n \Psi\big(z_n\big)e^{B(|u^n|)}dx \\ &\leq \Psi(2k)e^{B(\infty)}\sum_{i=1}^{N}\int_{\Omega} |a_i\big(x, T_k(u^n), \nabla T_k(u)\big)||\partial_{x_i} z_n|dx \longrightarrow 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{3.82}$$

Since $(|a_i(x, T_k(u^n), \partial_{x_i} T_k(u^n))|)n$ is bounded in $L^{q_i'(.)}(\Omega)$, then there exists $\nu_i \in L^{q_i'(.)}(\Omega)$ such that $|a_i(x, T_k(u^n), \partial_{x_i} T_k(u^n))| \rightharpoonup \nu_i$ weakly in $L^{q_i'(.)}(\Omega)$, and since $\partial_{x_i} T_k(u)|\Psi(z_n)|$ tends strongly to 0 in $L^{q_i(.)}$ for any $i = 1, ..., N$, concerning the last term of the left-hand side of (3.80), it follows that

$$\begin{aligned} \varepsilon_{29}(n) &= \left|\sum_{i=1}^{N}\int_{\Omega} a_i\big(x, T_k(u^n), \nabla T_k(u^n)\big)\partial_{x_i} T_k(u)|\Psi\big(z_n\big)|e^{B(|u^n|)}dx\right| \\ &\leq e^{B(\infty)}\sum_{i=1}^{N}\int_{\Omega} |a_i\big(x, T_k(u^n), \nabla T_k(u^n)\big)||\partial_{x_i} T_k(u)||\Psi\big(z_n\big)|dx \longrightarrow 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{3.83}$$

Combining (3.80) and estimates (3.81)-(3.83), it follows that

$$\begin{aligned} 0 &\leq \frac{1}{2}\sum_{i=1}^{N}\int_{\Omega} \big(a_i(x, T_k(u^n), \nabla T_k(u^n)) - a_i(x, T_k(u^n), \nabla T_k(u))\big)\partial_{x_i} z_n dx \\ &\leq \sum_{i=1}^{N}\int_{\Omega} \big(a_i(x, T_k(u^n), \nabla T_k(u^n)) - a_i(x, T_k(u^n), \nabla T_k(u))\big)\partial_{x_i} z_n \left(\Psi'(z_n) - \vartheta|\Psi(z_n|\right) dx \\ &\leq \varepsilon_{30}(n, m) \longrightarrow 0 \quad \text{as } n, m\to\infty. \end{aligned} \tag{3.84}$$

Invoking Lebesgue dominated convergence theorem, we have $T_k(u^n) \longrightarrow T_k(u)$ strongly in $L^{q_0}(\Omega)$. Thus, by letting n then m tend to infinity, we deduce that

$$\begin{aligned} &\sum_{i=1}^{N}\int_{\Omega} \big(a_i(x, T_k(u^n), \nabla T_k(u^n)) - a_i(x, T_k(u^n), \nabla T_k(u))\big)\big(\partial_{x_i} T_k(u^n) - \partial_{x_i} T_k(u)\big)dx \\ &+ \int_{\Omega} \left[|T_k(u^n)|^{q_0(.)-2}T_k(u^n) - |T_k(u)|^{q_0(.)-2}T_k(u)\right]\big(T_k(u^n) - T_k(u)\big)dx \longrightarrow 0 \text{ as } n\to\infty. \end{aligned} \tag{3.85}$$

Thanks to Lemma (2.3), we conclude that

$$\begin{cases} T_k(u^n) \longrightarrow T_k(u) \quad \text{strongly in } W^{1,\vec{q}(\cdot)}(\Omega), \\ \partial_{x_i} u^n \longrightarrow \partial_{x_i} u \quad \text{a.e in } \Omega \text{ for } i = 1, \dots, N. \end{cases} \tag{3.86}$$

Moreover, we have $a_i(x, T_n(u^n), \nabla u^n)\partial u^n$ converges to $a_i(x, u, \nabla u)\partial u$ almost everywhere in Ω, in view of Fatou's lemma and (3.39), we conclude that

$$\begin{aligned} &\lim_{m\to+\infty} \frac{1}{m} \sum_{i=1}^{N} \int_{\{|u|\le m\}} a_i\big(x, u, \nabla u\big)\partial_{x_i} u dx \\ &\le \lim_{m\to+\infty} \liminf_{n\to\infty} \frac{1}{m} \sum_{i=1}^{N} \int_{\{|u^n|\le m\}} a_i\big(x, T_n(u^n), \nabla u^n\big)\partial_{x_i} u^n dx \\ &\le \lim_{m\to+\infty} \limsup_{n\to\infty} \frac{1}{m} \sum_{i=1}^{N} \int_{\{|u^n|\le m\}} a_i\big(x, T_n(u^n), \nabla u^n\big)\partial_{x_i} u^n dx = 0 \end{aligned} \tag{3.87}$$

which prove (2.6).

Step 6: The equi-integrability of $(\mathcal{N}_n(x, u^n, \nabla u^n))_n$ and $(\mathcal{R}_n(x, u^n, \nabla u^n))_n$.

In this step, in order to pass to the limit in approximate Problem (3.18) and also satisfy part of the conditions of the Theorem 2.5, we establish the following convergences

$$\begin{aligned} &\mathcal{N}_n(x, u^n, \nabla u^n) \longrightarrow \mathcal{N}(x, u, \nabla u) \\ \text{and} \quad &\mathcal{R}_n(x, u^n, \nabla u^n) \longrightarrow \mathcal{R}(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \end{aligned} \tag{3.88}$$

From the second line of (3.86), we have

$$\begin{aligned} &\mathcal{N}_n(x, u^n, \nabla u^n) \longrightarrow \mathcal{N}(x, u, \nabla u) \\ \text{and } &\mathcal{R}_n(x, u^n, \nabla u^n) \longrightarrow \mathcal{R}(x, u, \nabla u) \ \ a.e \text{ in } \Omega. \end{aligned}$$

Then, according to Vitali's Theorem, it remains to prove that $\mathcal{N}_n(x, u^n, \nabla u^n)$ and $\mathcal{R}_n(x, u^n, \nabla u^n)$ are uniformly equi-integrable. Indeed, replacing v by $(T_{m+1}(u^n) - T_m(u^n))\chi(u^n)e^{B(|u^n|)}$ in (3.18), we will derive below some useful estimates:
Obviously $(T_{m+1}(u^n) - T_m(u^n))$ has the same sign as u^n, so, using assumption (1.4) and (3.19), evidently,

$$\begin{aligned} \mathcal{I}_1^n \ge\ & \alpha \sum_{i=1}^{N} \int_{\{m\le|u^n|\le m+1\}} |\partial u^n|^{q_i(x)} e^{B(|u^n|)} dx \\ &+ \alpha(\sigma-1) \sum_{i=1}^{N} \int_{\{m\le|u^n|\}} \frac{|\partial u^n|^{q_i(x)}}{(1+|u^n|)^{\sigma}} |T_{m+1}(u^n) - T_m(u^n)| e^{B(|u^n|)} dx \\ &+ 4 \sum_{i=1}^{N} \int_{\{m\le|u^n|\}} |\partial u^n|^{q_i(x)} |T_{m+1}(u^n) - T_m(u^n)| b(|u^n|) e^{B(|u^n|)} dx. \end{aligned} \tag{3.89}$$

Due to assumption (1.8) and (3.19), we obtain

$$
\begin{aligned}
&|\mathcal{I}_2^n| \\
&\quad \leq 2\int_{\{m\leq|u^n|\}}\gamma_0(x)|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx \\
&+2\sum_{i=1}^{N}\int_{\{m\leq|u^n|\}}|\partial_{x_i}u^n|^{q_i(x)}|T_{m+1}(u^n)-T_m(u^n)|b(|u^n|)e^{B(|u^n|)}dx.
\end{aligned}
\tag{3.90}
$$

Next, using the definition of μ^n given in (3.1), Young's inequality and (3.19), it follows that

$$
\begin{aligned}
&|\mathcal{I}_7^n| \\
&\quad \leq 2\int_{\{m\leq|u^n|\}}|f^n||T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx \\
&\quad\quad +C(\alpha)\sum_{i=1}^{N}\int_{\{m\leq|u^n|\leq m+1\}}|g_i|^{q_i'(x)}e^{B(|u^n|)}dx \\
&+(\sigma-1)C(\alpha)\sum_{i=1}^{N}\int_{\{m\leq|u^n|\}}|g_i|^{q_i'(x)}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx \\
&+C(\alpha)\sum_{i=1}^{N}\int_{\{m\leq|u^n|\}}|g_i|^{q_i'(x)}|T_{m+1}(u^n)-T_m(u^n)|b(|u^n|)e^{B(|u^n|)}dx \\
&\quad +2\sum_{i=1}^{N}\int_{\partial\Omega\cap\{m\leq|u^n|\}}|g_i||T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}d\sigma \\
&\quad\quad +\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{m\leq|u^n|\leq m+1\}}|\partial u^n|^{q_i(x)}e^{B(|u^n|)}dx \\
&+\frac{\alpha(\sigma-1)}{4}\sum_{i=1}^{N}\int_{\{m\leq|u^n|\}}\frac{|\partial u^n|^{q_i(x)}}{(1+|u^n|)^{\sigma}}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx \\
&+\frac{1}{2}\sum_{i=1}^{N}\int_{\{m\leq|u^n|\}}|\partial u^n|^{q_i(x)}|T_{m+1}(u^n)-T_m(u^n)|b(|u^n|)e^{B(|u^n|)}dx.
\end{aligned}
\tag{3.91}
$$

Thanks to hypothesis (1.9), Young's inequality and (3.19), it is easy to check that

$$\begin{aligned}
|\mathcal{I}_9^n| &\leq 2\int_{\{m\leq|u^n|\}}\rho(x)|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+2\int_{\{m\leq|u^n|\}}|T_n(u^n)|^{\delta_0(x)}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+2\sum_{i=1}^{N}\int_{\{m\leq|u^n|\}}|\partial_{x_i}u^n|^{\delta_i(x)}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&\leq 2\int_{\{m\leq|u^n|\}}\rho(x)|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+\frac{1}{4}\int_{\{m\leq|u^n|\}}|T_n(u^n)|^{\pi(x)-1}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+C\int_{\{m\leq|u^n|\}}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+\frac{\alpha(\sigma-1)}{4}\int_{\{m\leq|u^n|\}}\sum_{i=1}^{N}\frac{|\partial_{x_i}u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+\frac{1}{4}\int_{\{m\leq|u^n|\}}|T_n(u^n)|^{\pi(x)-1}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx.
\end{aligned}\tag{3.92}$$

From the last integral in the right hand side of (3.18), by using Young's inequality, we get

$$\begin{aligned}
|\mathcal{I}_{10}^n| &\leq C(\alpha)\sum_{i=1}^{N}\int_{\{m\leq|u^n|\leq m+1\}}|\phi_i^n(u^n)|^{q_i'(x)}e^{B(|u^n|)}dx\\
&+(\sigma-1)C(\alpha)\sum_{i=1}^{N}\int_{\{m\leq|u^n|\}}|\phi_i^n(u^n)|^{q_i'(x)}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+C(\alpha)\sum_{i=1}^{N}\int_{\{m\leq|u^n|\}}|\phi_i^n(u^n)|^{q_i'(x)}|T_{m+1}(u^n)-T_m(u^n)|b(|u^n|)e^{B(|u^n|)}dx\\
&+\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{m\leq|u^n|\leq m+1\}}|\partial u^n|^{q_i(x)}e^{B(|u^n|)}dx\\
&+\frac{\alpha(\sigma-1)}{4}\sum_{i=1}^{N}\int_{\{m\leq|u^n|\}}\frac{|\partial u^n|^{q_i(x)}}{(1+|u^n|)^\sigma}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+\frac{1}{2}\sum_{i=1}^{N}\int_{\{m\leq|u^n|\}}|\partial u^n|^{q_i(x)}|T_{m+1}(u^n)-T_m(u^n)|b(|u^n|)e^{B(|u^n|)}dx.
\end{aligned}\tag{3.93}$$

Combining (3.18) with estimates (3.89)-(3.93), yields the following inequality

$$
\begin{aligned}
&\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{m\le|u^n|\le m+1\}}|\partial u^n|^{q_i(x)}e^{B(|u^n|)}dx\\
&+\frac{\alpha(\sigma-1)}{4}\sum_{i=1}^{N}\int_{\{m\le|u^n|\}}\frac{|\partial u^n|^{q_i(x)}}{(1+|u^n|)^{\sigma}}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+\sum_{i=1}^{N}\int_{\{m\le|u^n|\}}|\partial u^n|^{q_i(x)}|T_{m+1}(u^n)-T_m(u^n)|b(|u^n|)e^{B(|u^n|)}dx\\
&+\frac{1}{2}\int_{\{m\le|u^n|\}}|T_n(u^n)|^{\pi(x)-1}|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&\le\int_{\{m\le|u^n|\}}(|f^n|+\gamma_0(x)+\rho(x)+C)|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+2\sum_{i=1}^{N}\int_{\partial\Omega\cap\{m\le|u^n|\}}\big(|\phi_i^n(u^n)|+|g_i|+|h^n|\big)|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}d\sigma\\
&+C(\alpha)\sum_{i=1}^{N}\int_{\{m\le|u^n|\le m+1\}}\big(|g_i|^{q_i'(x)}+|\phi_i^n(u^n)|^{q_i'(x)}\big)e^{B(|u^n|)}dx\\
&+(\sigma-1)C(\alpha)\sum_{i=1}^{N}\int_{\{m\le|u^n|\}}\big(|g_i|^{q_i'(x)}+|\phi_i^n(u^n)|^{q_i'(x)}\big)|T_{m+1}(u^n)-T_m(u^n)|e^{B(|u^n|)}dx\\
&+C(\alpha)\sum_{i=1}^{N}\int_{\{m\le|u^n|\}}\big(|g_i|^{q_i'(x)}+|\phi_i^n(u^n)|^{q_i'(x)}\big)|T_{m+1}(u^n)-T_m(u^n)|b(|u^n|)\chi(u^n)e^{B(|u^n|)}dx.
\end{aligned}
\tag{3.94}
$$

Applying (3.27), we deduce that

$$
\begin{aligned}
&\frac{\alpha(\sigma-1)e^{B(\infty)}}{4}\sum_{i=1}^{N}\int_{\{m+1\le|u^n|\}}\frac{|\partial u^n|^{q_i(x)}}{(1+|u^n|)^{\sigma}}dx\\
&+e^{B(\infty)}\sum_{i=1}^{N}\int_{\{m+1\le|u^n|\}}|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)dx\\
&+\frac{e^{B(\infty)}}{2}\int_{\{m+1\le|u^n|\}}|T_n(u^n)|^{\pi(x)-1}dx\le\eta(m)\longrightarrow 0\quad\text{as } m\to\infty
\end{aligned}
\tag{3.95}
$$

where $\eta(m)$ denote integrals in the right-hand side of the inequality (3.94).
In view of (3.92) and (3.95), we obtain that for any $\varepsilon>0$ there exists $m(\varepsilon)>0$ such that

$$
\int_{\{m(\varepsilon)\le|u^n|\}}\left(|T_n(u^n)|^{\delta_0(x)}+\sum_{i=1}^{N}|\partial_{x_i}u^n|^{\delta_i(x)}\right)dx\le\frac{\varepsilon}{4}
\tag{3.96}
$$

and

$$\sum_{i=1}^{N}\int_{\{m(\varepsilon)\le|u^n|\}}|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)dx\le\frac{\varepsilon}{4}. \tag{3.97}$$

On the other hand, for any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}
&\int_E\Big(|T_n(u^n)|^{\delta_0(x)}+\sum_{i=1}^{N}|\partial_{x_i}u^n|^{\delta_i(x)}\Big)dx+\sum_{i=1}^{N}\int_E|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)dx\\
&\le\int_E\Big(|T_{m(\varepsilon)}(u^n)|^{\delta_0(x)}+\sum_{i=1}^{N}|\partial_{x_i}T_{m(\varepsilon)}(u^n)|^{\delta_i(x)}\Big)dx\\
&\quad+\sum_{i=1}^{N}\int_E|\partial_{x_i}T_{m(\varepsilon)}(u^n)|^{q_i(x)}b(|T_{m(\varepsilon)}(u^n)|)dx\\
&\quad+\int_{\{m(\varepsilon)\le|u^n|\}}\Big(|T_n(u^n)|^{\delta_0(x)}+\sum_{i=1}^{N}|\partial_{x_i}u^n|^{\delta_i(x)}\Big)dx\\
&\quad+\sum_{i=1}^{N}\int_{\{m(\varepsilon)\le|u^n|\}}|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)dx.
\end{aligned}$$

Exploiting (3.86), there exists $\alpha(\varepsilon)>0$ such that for all $E\subset\Omega$ with $\mathrm{meas}(E)\le\alpha(\varepsilon)$, we have the inequality

$$\begin{aligned}
&\int_E\Big(|T_{m(\varepsilon)}(u^n)|^{\delta_0(x)}+\sum_{i=1}^{N}|\partial_{x_i}T_{m(\varepsilon)}(u^n)|^{\delta_i(x)}\Big)dx\\
&\quad+\sum_{i=1}^{N}\int_E|\partial_{x_i}T_{m(\varepsilon)}(u^n)|^{q_i(x)}b(|T_{m(\varepsilon)}(u^n)|)dx\le\frac{\varepsilon}{2}.
\end{aligned}\tag{3.98}$$

Finally, taking into account (3.96)-(3.98), we obtain that

$$\begin{aligned}
&\int_E\Big(|T_n(u^n)|^{\delta_0(x)}+\sum_{i=1}^{N}|\partial_{x_i}u^n|^{\delta_i(x)}\Big)dx\\
&+\sum_{i=1}^{N}\int_E|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|)dx\le\varepsilon,\quad\forall E\subset\Omega\text{ such that }\mathrm{meas}(E)\le\alpha(\varepsilon).
\end{aligned}\tag{3.99}$$

This implies that the sequences $\big(|T_n(u^n)|^{\delta_0(x)}\big)_n$, $\big(\sum_{i=1}^{N}|\partial_{x_i}u^n|^{\delta_i(x)}\big)_n$ and $\big(\sum_{i=1}^{N}|\partial_{x_i}u^n|^{q_i(x)}b(|u^n|\big)_n$ are uniformly equi-integrable. So, in view of (1.8) and (1.9), we deduce the equi-integrability of $\big(\mathcal{N}_n(x,u^n,\nabla u^n)\big)$ and $\big(\mathcal{R}_n(x,u^n,\nabla u^n)\big)$. Consequently, using Vitali's theorem, we conclude the convergence (3.88).

Step 7: Passing to the limit.

Let $\gamma \in W^{1,\infty}(\mathrm{R})$ be such that $\sup(\gamma) \subset [-M, M]$ for $M \geq 0$. For any function $\varphi \in W^{1,\vec{q}(.)}(\Omega) \cap L^\infty(\Omega)$ and $(u^n)_n$ a sequence of weak solutions of the problem (3.18), we have $\gamma(u^n)\varphi \in W^{1,\vec{q}(.)}(\Omega) \cap L^\infty(\Omega)$. We will first show that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i\big(x, T_n(u^n), \nabla u^n\big)(\partial_{x_i}u^n\gamma'(u^n)\varphi + \gamma(u^n)\partial_{x_i}\varphi)dx \\ = \sum_{i=1}^{N}\int_\Omega a_i\big(x, u, \nabla u\big)(\partial_{x_i}u\gamma'(u)\varphi + \gamma(u)\partial_{x_i}\varphi)dx.$$

Indeed, for $n \geq M$, we have

$$\sum_{i=1}^{N}\int_\Omega a_i\big(x, T_n(u^n), \nabla u^n\big)(\partial_{x_i}u^n\gamma'(u^n)\varphi + \gamma(u^n)\partial_{x_i}\varphi)dx \\ = \sum_{i=1}^{N}\int_\Omega a_i\big(x, T_M(u^n), \nabla T_M(u^n)\big)(\partial_{x_i}T_M(u^n)\gamma'(u^n)\varphi + \gamma(T_M(u^n))\partial_{x_i}\varphi)dx. \tag{3.100}$$

Using Convergences (3.86) and Hypothesis (1.5), we deduce the boundedness of $\big(a_i\big(x, T_M(u^n), \nabla T_M(u^n)\big)\big)_n$ in $L^{q_i'(.)}(\Omega)$, and since $a_i\big(x, T_M(u^n), \nabla T_M(u^n)\big)$ tends to $a_i\big(x, T_M(u), \nabla T_M(u)\big)$ a.e. in Ω, we obtain the weak convergence

$$a_i\big(x, T_M(u^n), \nabla T_M(u^n)\big) \rightharpoonup a_i\big(x, T_M(u), \nabla T_M(u)\big) \quad \text{in } L^{q_i'(.)}(\Omega). \tag{3.101}$$

Moreover, using the Convergences (3.86), it follow that

$$\partial_{x_i}T_M(u^n)\gamma'(u^n)\varphi + \gamma(T_M(u^n))\partial_{x_i}\varphi \\ \longrightarrow \partial_{x_i}T_M(u)\gamma'(u)\varphi + \gamma(T_M(u))\partial_{x_i}\varphi \quad \text{strongly in } L^{q_i(.)}(\Omega). \tag{3.102}$$

Combining (3.101) and (3.102), we get

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i\big(x, T_n(u^n), \nabla u^n\big)(\partial_{x_i}u^n\gamma'(u^n)\varphi + \gamma(u^n)\partial_{x_i}\varphi)dx \\
&\quad = \lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i\big(x, T_M(u^n), \nabla T_M(u^n)\big)(\partial_{x_i}T_M(u^n)\gamma'(u^n)\varphi + \gamma(T_M(u^n))\partial_{x_i}\varphi)dx \\
&\quad = \sum_{i=1}^{N}\int_\Omega a_i\big(x, T_M(u), \nabla T_M(u)\big)(\partial_{x_i}T_M(u)\gamma'(u)\varphi + \gamma(T_M(u))\partial_{x_i}\varphi)dx \\
&\quad = \sum_{i=1}^{N}\int_\Omega a_i\big(x, u, \nabla u\big)(\partial_{x_i}u\gamma'(u)\varphi + \gamma(u)\partial_{x_i}\varphi)dx.
\end{aligned} \tag{3.103}$$

Furthermore, we have $\gamma(T_M(u^n)) \rightharpoonup \gamma(T_M(u))$ weak* in $L^\infty(\Omega)$. So, in view of (3.88), we deduce that

$$\int_\Omega \mathcal{N}_n(x,u^n,\nabla u^n)\gamma(T_M(u^n))\varphi dx$$
$$\longrightarrow \int_\Omega \mathcal{N}(x,u,\nabla u)\gamma(T_M(u))\varphi dx = \int_\Omega \mathcal{N}(x,u,\nabla u)\gamma(u)\varphi dx$$

and

$$\int_\Omega \mathcal{R}_n(x,u^n,\nabla u^n)\gamma(T_M(u^n))\varphi dx$$
$$\longrightarrow \int_\Omega \mathcal{R}(x,u,\nabla u)\gamma(T_M(u))\varphi dx = \int_\Omega \mathcal{R}(x,u,\nabla u)\gamma(u)\varphi dx.$$

Also, thanks to (3.42), we have

$$\int_\Omega |T_n(u^n)|^{\pi(x)-2}T_m(u^n)\gamma(T_M(u^n))\varphi dx$$
$$\longrightarrow \int_\Omega |u|^{\pi(x)-2}u\gamma(T_M(u))\varphi dx = \int_\Omega |u|^{\pi(x)-2}u\gamma(u)\varphi dx$$

and

$$\lim_{n\to\infty}\frac{1}{n}\int_\Omega |u^n|^{q_0(.)-2}u^n\gamma(u^n)\varphi dx = 0.$$

Net, using Hypotheses (1.7), (1.11) and the Lebesgue dominated convergence Theorem to get

$$\int_\Omega f^n\gamma(T_M(u^n))\varphi dx \longrightarrow \int_\Omega f\gamma(T_M(u))\varphi dx = \int_\Omega f\gamma(u)\varphi dx,$$
$$\lim_{n\longrightarrow\infty}\int_{\partial\Omega} T_n(u^n)\gamma(T_M(u^n))\varphi d\sigma = \int_{\partial\Omega} u\gamma(T_M(u))\varphi d\sigma = \int_{\partial\Omega} u\gamma(u)\varphi d\sigma,$$
$$\lim_{n\longrightarrow\infty}\int_{\partial\Omega} h^n\gamma(T_M(u^n))\varphi d\sigma = \int_{\partial\Omega} h\gamma(T_M(u))\varphi d\sigma = \int_{\partial\Omega} h\gamma(u)\varphi d\sigma,$$
$$\lim_{n\longrightarrow\infty}\int_{\partial\Omega} \phi_i^n(u^n)\gamma(T_M(u^n))\varphi d\sigma = \int_{\partial\Omega} \phi_i(u)\gamma(T_M(u))\varphi d\sigma = \int_{\partial\Omega} \phi_i(u)\gamma(u)\varphi d\sigma$$

and

$$\lim_{n\longrightarrow\infty}\int_{\partial\Omega} F_i\gamma(T_M(u^n))\varphi d\sigma = \int_{\partial\Omega} F_i\gamma(T_M(u))\varphi d\sigma = \int_{\partial\Omega} F_i\gamma(u)\varphi d\sigma.$$

Moreover, in view of (3.86), it follows that

$$\phi_i^n(T_M(u^n)) \longrightarrow \phi_i(T_M(u)) = \phi_i(u) \text{ strongly in } L^{q_i'(.)}(\Omega),\ \forall\ i = 1,\cdots,N$$

and

$$\partial_{x_i}\gamma(T_M(u^n))\varphi \longrightarrow \partial_{x_i}\gamma(T_M(u))\varphi = \partial_{x_i}\gamma(u)\varphi \text{ strongly in } L^{q_i(.)}(\Omega),$$

so, using Lebesgue's dominated convergence Theorem, we deduce that

$$\int_\Omega \phi_i^n(u^n)\partial_{x_i}\gamma(T_M(u^n))\varphi dx$$
$$\longrightarrow \int_\Omega \phi_i(u)\partial_{x_i}\gamma(T_M(u))\varphi dx = \int_\Omega \phi_i(u)\partial_{x_i}\gamma(u)\varphi dx \quad \text{as } n \to \infty.$$

Similarly, we have

$$\int_\Omega F_i\partial_{x_i}\gamma(T_M(u^n))\varphi dx \longrightarrow \int_\Omega F_i\partial_{x_i}\gamma(T_M(u))\varphi dx = \int_\Omega F_i\partial_{x_i}\gamma(u)\varphi dx \;\; \text{as } n \to \infty.$$

Now, taking $\gamma(u^n)\varphi$ as a test function in approximate Problem (3.18), we have everything at our disposal to pass to the limit and get the desired relation (2.7). This completes the proof of Theorem 2.5.

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