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A NABLA FRACTIONAL RIEMANN-LIOUVILLE CALCULUS ON TIME SCALES WITH APPLICATION TO DYNAMIC EQUATIONS

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ABSTRACT. We introduce more general concepts of Riemann-Liouville fractional nabla integral and nabla derivative on time scales. By using the fixed point theorem, sufficient conditions for existence and uniqueness of solution to an initial value problem described by nabla derivatives on time scales. Some properties of the new operator are proved and illustrated with examples.

1. INTRODUCTION

In recent decades, fractal calculus and fractional calculus have been becoming hot topics in both mathematics and engineering for nondifferential solutions [5, 6, 7, 8, 9, 9, 10, 12, 13, 14, 16, 18, 19]. Fractional calculus was introduced and developed by Leibniz, Liouville, Riemann, Letnikov, and Grünwald . This branch was applied in physics, natural and social sciences. Another important area of study is dynamic equations on time scales, which goes back to 1988 and the work of Aulbach and Hilger [17], and has been used with success to unify differential and difference equations [15]. In 2016, Benkhettou et al. [16], introduced a concept of fractional derivative of Riemann-Liouville on time scales. Several authors have obtained important results about different subjects on time scales. See for instance M. Rchid et al [19], A. Abdeljawad et al [18], T. Gülsen et al [14].

This article is organized as follows, we define a Riemann–Liouville fractional nabla derivative on time scales (Definition 3.1) and Riemann–Liouville fractional nabla integrations on time scales (Definition 3.2) help us to study of their important properties. This is the first step in this direction. On the other hand, we present initial value problem and proving some sufficient conditions for uniqueness and existence of solutions of a structural nabla derivatives initial value problem on an arbitrary time scale (see Theorems 3.7), we end by illustrative examples.

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2. Preliminaries

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma,\ \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively. We say that t is right-scattered (resp., left-scattered) if $\sigma(t) > t$ (resp., if $\rho(t) < t$); that t is isolated if it is right-scattered and left-scattered. Also, if $t < \sup \mathbb{T}$ and $t = \sigma(t)$, we say that t is right-dense. If $t > \inf \mathbb{T}$ and $t = \rho(t)$, we say that t is left dense. Points that are right dense and left dense are called dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} \backslash \{M\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. The backward graininess $\nu : \mathbb{T} \to [0, \infty)$ is defined by $\nu(t) := t - \rho(t)$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \backslash \{m\}$, otherwise, $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ is bounded, then $\mathbb{T}_0 \subseteq \mathbb{T}_k$ where $\mathbb{T}_0 = \mathbb{T} \backslash \{\min \mathbb{T}\}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$.

Definition 2.1. [15]The function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense point in $\mathbb{T}$ and has a right-sided limits exist at right-dense points in $\mathbb{T}$, write $f \in C_{ld}(\mathbb{T}, \mathbb{R})$.

Definition 2.2. [15]For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$, the ∇-derivative of f at t, denoted by $f^{\nabla}(t)$, is defined to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $\mathcal{U}$ of t such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right| \leq \varepsilon \left| \rho(t) - s \right|, \quad \text{for all } s \in \mathcal{U}.$$

We say that f is ∇-differentiable if $f^{\nabla}(t)$ exists for every $t \in \mathbb{T}_k$. The function $f^{\nabla} : \mathbb{T} \to \mathbb{R}$ is then called the ∇-derivative of f on $\mathbb{T}_k$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ which are ∇-differentiable and whose ∇-derivative is ld-continuous is denoted by $C^1_{ld}(\mathbb{T}, \mathbb{R})$.

In what follows, with $\mathcal{C}([a, b]_{\mathbb{T}}, \mathbb{R})$ we denote the Banach space of all continuous functions from $[a, b]_{\mathbb{T}}$ into $\mathbb{R}$, where $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$, with the norm

$$\|x\|_{\infty} := \sup\{|x(t)| : t \in [a, b]_{\mathbb{T}}\}.$$

Definition 2.3. [15]Let $[a, b]_{\mathbb{T}}$ denote a closed bounded interval in $\mathbb{T}$. A function $F : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is called a delta antiderivative function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ provided F is continuous on $[a, b]_{\mathbb{T}}$, delta differentiable on $[a, b)$, and $F^{\nabla}(t) = f(t)$, for all $t \in [a, b)$. Then, we define the ∇-integral of f from a to b by:

$$\int_a^b f(t) \nabla t = F(b) - F(a).$$

3. Main Results

We introduce a new notion of fractional nabla derivative on time scales. Before that, we define the fractional nabla integral on a time scale $\mathbb{T}$.

Definition 3.1 (Riemann–Liouville fractional nabla integral on time scales)**.** Suppose $\mathbb{T}$ is a time scale, $[a,b]$ is an interval of $\mathbb{T}$, and h is an integrable function on $[a,b]$. Let $0<\alpha<1$. Then the (left) fractional nabla integral of order α of h is defined by:

$$ {}_a^{\mathbb{T}}I_{t,\nabla}^{\alpha}h(t)=\int_a^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\nabla s, $$

where Γ is the gamma function.

Definition 3.2 (Riemann–Liouville fractional nabla derivative on time scales)**.** Let $\mathbb{T}$ be a time scale, $t\in\mathbb{T}$, $0<\alpha<1$, and $h:\mathbb{T}\to\mathbb{R}$. The (left) Riemann–Liouville fractional nabla derivative of order α of h is defined by:

$$ {}_a^{\mathbb{T}}D_{t,\nabla}^{\alpha}h(t)=\left(\int_a^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}h(s)\nabla s\right)^{\nabla}=\left({}_a^{\mathbb{T}}I_t^{1-\alpha}h(t)\right)^{\nabla}. $$

Proposition 3.3. *If $\mathbb{T}=\mathbb{R}$, the Riemann–Liouville fractional nabla integral satisfies*

$$ {}_a^{\mathbb{T}}I_{t,\nabla}^{\beta}\circ{}_a^{\mathbb{T}}I_{t,\nabla}^{\alpha}={}_a^{\mathbb{T}}I_{t,\nabla}^{\beta+\alpha},\quad \text{for } \alpha>0 \quad \text{and } \beta>0. \tag{3.1} $$

The following counter example, to prove that the equality (3.1) is not always satisfied on time scales.

Example 3.4 (Counter example)**.** We take $\mathbb{T}=\mathbb{N}$, $a=1$ and $h:\mathbb{T}\to\mathbb{R}$, $h(t)=1$. Let $\alpha>0$ and $b>0$, then by Definition 3.1, we have

$$ {}_1^{\mathbb{N}}I_{t,\nabla}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_1^t(t-s)^{\alpha-1}\nabla s=\frac{1}{\Gamma(\alpha)}\sum\nolimits_{s=1}^{s=t-2}s^{\alpha-1}. $$

By, the last equality, deduce

$$ {}_1^{\mathbb{N}}I_{t,\nabla}^{\alpha+\beta}h(t)=\frac{1}{\Gamma(\alpha+\beta)}\sum\nolimits_{s=1}^{s=t-2}s^{\alpha+\beta-1}. $$

On the other hand, we have

$$ \left({}_1^{\mathbb{T}}I_{t,\nabla}^{\beta}\circ{}_1^{\mathbb{T}}I_{t,\nabla}^{\alpha}\right)h(t)=\frac{1}{\Gamma(\beta)}\sum\nolimits_{s=1}^{s=t-1}(t-s)^{\beta-1}\left({}_1^{\mathbb{N}}I_{t,\nabla}^{\alpha}h(s)\right). $$

Thus,

$$ \left({}_1^{\mathbb{T}}I_{t,\nabla}^{\beta}\circ{}_1^{\mathbb{T}}I_{t,\nabla}^{\alpha}\right)h(3)\neq{}_1^{\mathbb{T}}I_{t,\nabla}^{\alpha+\beta}h(3). $$

Remark 3.5. By counter example 3.4, we conclude that ${}_a^{\mathbb{T}}I_{t,\nabla}^{\beta}\circ{}_a^{\mathbb{T}}I_{t,\nabla}^{\alpha}={}_a^{\mathbb{T}}I_{t,\nabla}^{\beta+\alpha}$, for $\alpha>0$, $\beta>0$ are not always correct on the time scales, if you suggest a counterexample to ${}_a^{\mathbb{T}}D_{t,\nabla}^{\alpha}\circ{}_a^{\mathbb{T}}I_{t,\nabla}^{\alpha}=Id$ and ${}_a^{\mathbb{T}}I_{t,\nabla}^{\alpha}\circ{}_a^{\mathbb{T}}D_{t,\nabla}^{\alpha}=Id$, $\alpha>0$, you leave to provide exact calculations in Example 3.4. Example 3.4 is a for ${}_a^{\mathbb{T}}I_{t,\nabla}^{\beta}\circ{}_a^{\mathbb{T}}I_{t,\nabla}^{\alpha}\neq{}_a^{\mathbb{T}}I_{t,\nabla}^{\beta+\alpha}$ only.

For simplification, we note $\mathcal{J}_{a,b}=[a,\rho(b)]_{\mathbb{T}}$. We consider the following initial value problem:

$$ \begin{cases}\beta x^{\nabla}(t)+{}_a^{\mathbb{T}}D_{t,\nabla}^{\alpha}(x(t))=f(t,x(t)),\text{ for } t\in\mathcal{J}_{a,b},\ 0<\alpha<1,\\ x(a)=0,\end{cases} \tag{3.2} $$

where ${}_{a}^{\mathbb{T}}D_{t,\nabla}^{\alpha}$ is the Riemann–Liouville fractional nabla derivative operator of order α defined on $\mathbb{T}$. The problem (3.2) will be studied under the following assumptions $f \in \mathcal{C}(\mathcal{J}_{a,b} \times \mathbb{R}, \mathbb{R})$ and $\beta \neq 0$. Our main results give necessary and sufficient conditions for the existence and uniqueness of solution to the problem (3.2).

Lemma 3.6. *Let $\alpha \in (0,1)$ and $f : \mathcal{J}_{a,b} \times \mathbb{R} \to \mathbb{R}$. Function $x \in \mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$ is a solution of the problem (3.2) if and only if it is a solution of the following integral equation:*

$$\beta x(t) = \int_a^t f(s, x(s)) \nabla s - \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} x(s) \nabla s.$$

Proof. By Definition 3.1, we have

$$\begin{aligned} \beta x^{\nabla}(t) &= f(t, x(t)) - \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} x(s) \nabla s \right)^{\nabla} \\ &= f(t, x(t)) - \left({}_{a}^{\mathbb{T}}I_{t,\nabla}^{1-\alpha} x(t) \right)^{\nabla} \\ &= f(t, x(t)) - {}_{a}^{\mathbb{T}}D_{t,\nabla}^{\alpha}(x(t)). \end{aligned}$$

The proof is complete. □

Our main result is based on the Banach fixed point theorem [2].

Theorem 3.7. *Let $\alpha \in (0,1)$ and $f \in \mathcal{C}(\mathcal{J}_{a,b} \times \mathbb{R}, \mathbb{R})$, there exists a positive and continuous function $r : \mathcal{J}_{a,b} \to \mathbb{R}$, such that*

$$|f(t,x) - f(t,y)| \leq r(t)|x-y|, \quad \textit{for all } (x,y,t) \in \mathbb{R}^2 \times \mathcal{J}_{a,b}. \tag{3.3}$$

If

$$\int_a^b r(s) \nabla s + \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} < |\beta|, \tag{3.4}$$

then the problem (3.2) has a unique solution on $\mathcal{J}_{a,b}$.

Proof. We transform the problem (3.2) into a fixed point problem. Consider the operator $T : \mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R}) \to \mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$ defined by:

$$\mathcal{T}x(t) = \frac{1}{\beta} \int_a^t \left[f(s, x(s)) - \frac{1}{\Gamma(1-\alpha)} (t-s)^{-\alpha} p(s) x(s) \right] \nabla s. \tag{3.5}$$

We need to prove that $\mathcal{T}$ has a fixed point, which is a unique solution of (3.2) on $\mathcal{J}_{a,b}$. For that, we show that $\mathcal{T}$ is a contraction. Let $x, y \in \mathcal{C}(\mathcal{J}_{a,b}, \mathbb{R})$. For $t \in \mathcal{J}_{a,b}$, we have

$$\begin{aligned} |\mathcal{T}x(t) - \mathcal{T}y(t)| &\leq \frac{1}{|\beta|} \int_a^t |f(s, x(s)) - f(s, y(s))| \nabla s \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} |x(s) - y(s)| \nabla s \\ &\leq \left(\frac{1}{|\beta|} \int_a^b r(s) \nabla s + \frac{(b-a)^{1-\alpha}}{|\beta| \Gamma(2-\alpha)} \right) \|x-y\|. \end{aligned}$$

By (3.4), $\mathcal{T}$ is a contraction and thus, by the contraction mapping theorem, we deduce that $\mathcal{T}$ has a unique fixed point. This fixed point is the unique solution of (3.2). $\square$

4. Example

In this section, we give an example to illustrate our main result.

Example 4.1. Let $\mathbb{T}=\mathbb{Z}$, $a=1$ and $b=m$, where $m\in\mathbb{N}$. We consider the following initial value problem:

$$\begin{cases} \beta x^{\nabla}(t) + {}_1^{\mathbb{Z}} D_{t,\nabla}^{\frac{1}{2}} x(t) = x(t), \text{ for } t\in\mathcal{J}_{1,m-1}, \\ x(1)=0. \end{cases} \tag{4.1}$$

Here, $\beta>0$, $\alpha=\frac{1}{2}$, and $f(t,x)=x$, for $t\in[1,m-1]_{\mathbb{Z}}$ and $x\in\mathbb{R}$.Wwe find that the problem (4.1) is a private case of the problem (3.2) in $\mathbb{T}=\mathbb{Z}$. Then (3.3) holds. If $2\leq m<\frac{\pi|\beta|+2\pi+2}{\pi+2}$, then (3.4) holds, Thus, the conditions of Theorem 3.7 are satisfied, and we conclude that there is a function $x\in\mathcal{C}(\mathcal{J}_{1,m-1},\mathbb{R})$ the unique solution of (4.1).

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References

1. A. Cabada, D. Vivero, *Expression of the Lebesgue $\Delta-$integral on time scales as a usual Lebesgue integral; application to the calculus of $\Delta-$ antiderivatives,* Mathematical and Computer Modelling, **43**, 2006, 194-207.
2. A. Granas, J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.
3. A. Benaissa Cherif, F. Z. Ladrani, *Hardy's-sobolev's-type inequalities on time scale via alpha conformable fractional integral*, J. Fra. Cal. Appl, **8**, (2017), 156-165.
4. A. Benaissa Cherif, A. Hammoudi, F. Z. Ladrani, *Density problems in $L_{\Delta}^{p}(\mathbb{T},\mathbb{R})$ space,* Electronic Journal of Mathematical Analysis and Applications, **1**, (2013), 178–187.
5. A. Benaissa Cherif, F. Z. Ladrani, *Density problems in sobolev's spaces on time scales*, kragujevac journal of mathematics,**45**(2021), 215–223.
6. A. Benaissa Cherif, F. Z. Ladrani,*A symptotic behavior of solution for a fractional Riemann-Liouville differential equations on time scales,* Malaya. J. Mat, **5**, (2017), 561–568.
7. A.Benaissa Cherif, F. Z. Ladrani, New properties of the time-scale fractional operators with application to dynamic equations, Mathematica Moravica, **25**(2021), 123–136.
8. B. Bendouma, A. Benaissa Cherif, A. Hammoud, *Systems of first-order nabla dynamic equations on time scales*, Malaya Journal of Matematik,**6** 2018, 757-765.
9. B. Bendouma, A. Hammoudi, *Nonlinear Functional Boundary Value Problems for Conformable Fractional Dynamic Equations on Time Scales*, Mediterr. J. Math, **16**(2019).
10. D. Mozyrska, D. F.M.Torres, M. Wyrwas, *Solutions of systems with the Caputo–Fabrizio fractional delta derivative on time scales*, Nonlinear Analysis: Hybrid Systems, **32**(2019), 168-176.
11. E. G. Bajlekova, *Fractional evolution equations in Banach spaces*, Eindhoven University of Technology, Eindhoven, 2001.
12. F. Z. Ladrani, A. Benaissa Cherif, *Oscillation Tests for Conformable Fractional Differential Equations with Damping*, Punjab Univ. j. math, **52** (2020), 73-82.

13. G. M. Bahaa, Delfim F. M. Torres, *Time-Fractional Optimal Control of Initial Value Problems on Time Scales*, Nonlinear Analysis and Boundary Value Problems, **292**,(2019), 229–242.
14. T. Gülse, E. Yilmaz, and H. Kemaloğlu, *Conformable fractional Sturm–Liouville equation and some existence results on time scales*, Turk. J. Math, **42**, (2018), 1348-1360.
15. M. Bohner, A.C.Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkäuser Boston, Inc., Boston, MA, 2001.
16. N.Benkhettou, A. Hammoudi, D. F.M. Torres, *Existence and uniqueness of solution for a fractional Riemann–Liouville initial value problem on time scales*, Journal of King Saud University-Science, **28** (2016), 87–92.
17. S. Hilger, *Ein Ma ßkettenakalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. Thesis, Universität Würzburg, 1988 (in German).
18. T. Abdeljawad, D. F. M. Torres, *Symmetric duality for left and right Riemann–Liouville and Caputo fractional differences*, Arab. J. Math. Sci, **23** (2017), 157–172.
19. M. R. Sidi Ammi, D. F. M. Torres, *Existence and uniqueness results for a fractional Riemann-Liouville nonlocal thermistor problem on arbitrary time scales*, J. King. Saud. Univ. Sci., **30** (2018), 381–385.

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