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IMPROVEMENT OF A KNOWN VARIANT OF THE HARDY-HILBERT INTEGRAL INEQUALITY

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Abstract. Over the past few decades, numerous variants of the Hardy-Hilbert integral inequality have been established. In this article, we present an improved version of a notable variant that combines the power, maximum, and absolute difference functions of the variables. The constant factor has an elegant form and is closely related to the beta function. A complete and detailed proof is provided. We also discuss how our result can be used to determine other variants.

Keywords. Integral inequalities, Hardy-Hilbert-type integral inequalities, beta function
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1. Introduction

Integral inequalities inspired by the Hardy-Hilbert integral framework are valuable results in modern analysis. They provide precise bounds for solving a variety of mathematical problems, including those in operator theory, functional analysis, partial differential equations, and mathematical physics. Further information can be found in [2, 13, 14]. Classical modified Hardy-Hilbert-type integral inequalities involve the maximum function. The most basic result of this kind is from [4], as described below. Let $p > 1$ and $q = p/(p-1)$ be the Hölder conjugate of p. Let $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be two functions, where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$. Then the following inequality holds:

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{\max(x, y)} f(x)g(y)dxdy \leq pq \left[\int_{\mathbb{R}_+} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+} g^q(y)dy\right]^{1/q},$$

with $pq = p^2/(p-1)$, provided that the integrals in the upper bound converge. Other maximum-Hardy-Hilbert-type integral inequalities can be found in [12, 5, 1, 6, 8, 11, 9, 10, 7, 3]. For the purposes of this article, we emphasize [11, Theorem 4], which is stated below. Let $p > 1$, $q = p/(p-1)$ and $\kappa > 0$. Let $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$

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be two functions. Then the following inequality holds:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]} f(x)g(y)dxdy$$
$$\leq 2B(\kappa,\kappa)\left[\int_{\mathbb{R}_+} x^{\kappa} f^{p}(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+} y^{\kappa} g^{q}(y)dy\right]^{1/q}, \tag{1.1}$$

where $B(\kappa,\kappa)=\int_0^1 t^{\kappa-1}(1-t)^{\kappa-1}dt$, i.e., the classical beta function taken at the parameters κ and κ, provided that the integrals in the upper bound converge. This inequality is particularly interesting for the following reasons:

- there is no particular restriction on κ except that $\kappa>0$,
- the functionalities combine the power, maximum and absolute difference functions of the variables, which is quite original in this context,
- the constant factor is simple and closely linked to the beta function.

It has been the object of a two-parameter extension in [3, Theorem 4.1].

In this article, we propose to complete the theory of [11, Theorem 4]. Especially, we determine an auxiliary bivariate function ψ such that, for any $x,y>0$, we have

$$\psi(x,y)\leq 1$$

and

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\psi(x,y)} f(x)g(y)dxdy$$
$$\leq 2B(\kappa,\kappa)\left[\int_{\mathbb{R}_+} x^{\kappa} f^{p}(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+} y^{\kappa} g^{q}(y)dy\right]^{1/q},$$

so

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]} f(x)g(y)dxdy$$
$$\leq \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\psi(x,y)} f(x)g(y)dxdy$$
$$\leq 2B(\kappa,\kappa)\left[\int_{\mathbb{R}_+} x^{\kappa} f^{p}(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+} y^{\kappa} g^{q}(y)dy\right]^{1/q}.$$

We therefore derive an analytical refinement of the inequality in Equation (1.1), while maintaining the same upper bound. An interesting feature is that the function ψ will depend on the power and minimum of the variables, introducing a new structural dimension to the original double integral. For the sake of completeness and clarity, the full proof is presented in detail. We also provide a discussion of possible other variants based on our result.

The article is organized as follows: Section 2 gives the complete statement and proof of our refined Hardy-Hilbert integral inequality. Section 3 provides a conclusion.

2. Improvement

2.1. **Theorem.** Our improved Hardy-Hilbert integral inequality is given below, along with its proof and a discussion.

Theorem 2.1. *Let $p > 1$, $q = p/(p-1)$ and $\kappa > 0$. Let $f, g : \mathbb{R}'_+ \to \mathbb{R}_+$ be two functions. Then we have*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\psi(x,y)} f(x)g(y)dxdy$$
$$\leq 2B(\kappa,\kappa)\left[\int_{\mathbb{R}_+} x^{\kappa} f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+} y^{\kappa} g^q(y)dy\right]^{1/q},$$

where

$$\psi(x,y) = \left\{\min\left[\left(\frac{y}{x}\right)^{\kappa+1},1\right]\right\}^{1/p}\left\{\min\left[\left(\frac{x}{y}\right)^{\kappa+1},1\right]\right\}^{1/q}$$
$$= \begin{cases} \left(\dfrac{y}{x}\right)^{(\kappa+1)/p} & \text{if } y \leq x, \\ \left(\dfrac{x}{y}\right)^{(\kappa+1)/q} & \text{if } y > x, \end{cases} \tag{2.1}$$

provided that the integrals in the upper bound converge.

Proof. Decomposing the integrand using the definition of ψ in Equation (2.1) and $1/p + 1/q = 1$, and applying the Hölder integral inequality, we have

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\psi(x,y)} f(x)g(y)dxdy$$
$$= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{(1-\kappa)/p}\left\{\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(y/x)^{\kappa+1},1\right]\right\}^{1/p}} f(x)$$
$$\times \frac{1}{|x-y|^{(1-\kappa)/q}\left\{\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(x/y)^{\kappa+1},1\right]\right\}^{1/q}} g(y)dxdy$$
$$\leq \mathfrak{A}^{1/p}\mathfrak{B}^{1/q}, \tag{2.2}$$

where

$$\mathfrak{A} = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(y/x)^{\kappa+1},1\right]} f^p(x)dxdy$$

and

$$\mathfrak{B} = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(x/y)^{\kappa+1},1\right]} g^q(y)dxdy.$$

Let us express $\mathfrak{A}$ and $\mathfrak{B}$ in turn.

For $\mathfrak{A}$, the Fubini-Tonelli integral theorem implies that

$$\mathfrak{A}=\int_{\mathbb{R}_+}\left\{\int_{\mathbb{R}_+}\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(y/x)^{\kappa+1},1\right]}dy\right\}f^p(x)dx. \tag{2.3}$$

For the central integral term, using the definitions of $\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]$ and $\min\left[(y/x)^{\kappa+1},1\right]$ combined with the Chasles integral relation, simplifying, and making the changes of variables $u=y/x$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+}\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(y/x)^{\kappa+1},1\right]}dy\\
&=\int_0^x\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(y/x)^{\kappa+1},1\right]}dy\\
&+\int_x^{+\infty}\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(y/x)^{\kappa+1},1\right]}dy\\
&=\int_0^x\frac{1}{(x-y)^{1-\kappa}(x/y)^{2\kappa}(y/x)^{\kappa+1}}dy+\int_x^{+\infty}\frac{1}{(y-x)^{1-\kappa}(y/x)^{2\kappa}\times 1}dy\\
&=x^{\kappa}\left[\int_0^x\frac{(y/x)^{\kappa-1}}{(1-y/x)^{1-\kappa}}\frac{1}{x}dy+\int_x^{+\infty}\frac{(y/x)^{-2\kappa}}{(y/x-1)^{1-\kappa}}\frac{1}{x}dy\right]\\
&=x^{\kappa}\left[\int_0^1\frac{u^{\kappa-1}}{(1-u)^{1-\kappa}}du+\int_1^{+\infty}\frac{u^{-2\kappa}}{(u-1)^{1-\kappa}}du\right].
\end{aligned} \tag{2.4}$$

Let us now focus on the second integral. Making the change of variables $u=1/v$, we obtain

$$\int_1^{+\infty}\frac{u^{-2\kappa}}{(u-1)^{1-\kappa}}du=\int_1^0\frac{(1/v)^{-2\kappa}}{(1/v-1)^{1-\kappa}}\left(-\frac{1}{v^2}dv\right)=\int_0^1\frac{v^{\kappa-1}}{(1-v)^{1-\kappa}}dv. \tag{2.5}$$

Joining Equations (2.4) and (2.5), uniformizing the notation and recognizing the beta function, we get

$$\begin{aligned}
&\int_{\mathbb{R}_+}\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(y/x)^{\kappa+1},1\right]}dy\\
&=2x^{\kappa}\int_0^1\frac{u^{\kappa-1}}{(1-u)^{1-\kappa}}du=2x^{\kappa}\int_0^1u^{\kappa-1}(1-u)^{\kappa-1}du\\
&=2x^{\kappa}B(\kappa,\kappa).
\end{aligned} \tag{2.6}$$

It follows from Equations (2.3) and (2.6) that

$$\mathfrak{A}=\int_{\mathbb{R}_+}2x^{\kappa}B(\kappa,\kappa)f^p(x)dx=2B(\kappa,\kappa)\int_{\mathbb{R}_+}x^{\kappa}f^p(x)dx. \tag{2.7}$$

For $\mathfrak{B}$, we proceed in a similar way, but with some important differences. The Fubini-Tonelli integral theorem implies that

$$\mathfrak{B}=\int_{\mathbb{R}_+}\left\{\int_{\mathbb{R}_+}\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(x/y)^{\kappa+1},1\right]}dx\right\}g^q(y)dy. \tag{2.8}$$

For the central integral term, using the definitions of $\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]$ and $\min\left[(y/x)^{\kappa+1},1\right]$ combined with the Chasles integral relation, simplifying, and making the changes of variables $w=x/y$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+}\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(x/y)^{\kappa+1},1\right]}dx\\
&=\int_0^y\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(x/y)^{\kappa+1},1\right]}dx\\
&+\int_y^{+\infty}\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(x/y)^{\kappa+1},1\right]}dx\\
&=\int_0^y\frac{1}{(y-x)^{1-\kappa}(y/x)^{2\kappa}(x/y)^{\kappa+1}}dx+\int_y^{+\infty}\frac{1}{(x-y)^{1-\kappa}(x/y)^{2\kappa}\times 1}dx\\
&=y^{\kappa}\left[\int_0^y\frac{(x/y)^{\kappa-1}}{(1-x/y)^{1-\kappa}}\frac{1}{y}dx+\int_y^{+\infty}\frac{(x/y)^{-2\kappa}}{(x/y-1)^{1-\kappa}}\frac{1}{y}dx\right]\\
&=y^{\kappa}\left[\int_0^1\frac{w^{\kappa-1}}{(1-w)^{1-\kappa}}dw+\int_1^{+\infty}\frac{w^{-2\kappa}}{(w-1)^{1-\kappa}}dw\right]. \qquad (2.9)
\end{aligned}$$

Let us now work on the second integral. Making the change of variables $z=1/w$, we get

$$\int_1^{+\infty}\frac{w^{-2\kappa}}{(w-1)^{1-\kappa}}dw=\int_1^0\frac{(1/z)^{-2\kappa}}{(1/z-1)^{1-\kappa}}\left(-\frac{1}{z^2}dz\right)=\int_0^1\frac{z^{\kappa-1}}{(1-z)^{1-\kappa}}dz. \qquad (2.10)$$

Joining Equations (2.9) and (2.10), uniformizing the notation and recognizing the beta function, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+}\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\min\left[(x/y)^{\kappa+1},1\right]}dx\\
&=2y^{\kappa}\int_0^1\frac{w^{\kappa-1}}{(1-w)^{1-\kappa}}dw=2y^{\kappa}\int_0^1 w^{\kappa-1}(1-w)^{\kappa-1}dw\\
&=2y^{\kappa}B(\kappa,\kappa). \qquad (2.11)
\end{aligned}$$

It follows from Equations (2.8) and (2.11) that

$$\mathfrak{B}=\int_{\mathbb{R}_+}2y^{\kappa}B(\kappa,\kappa)g^q(y)dy=2B(\kappa,\kappa)\int_{\mathbb{R}_+}y^{\kappa}g^q(y)dy. \qquad (2.12)$$

Combining Equations (2.2), (2.7) and (2.12) and using $1/p+1/q=1$, we get

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\psi(x,y)}f(x)g(y)dxdy\\
&\leq\left[2B(\kappa,\kappa)\int_{\mathbb{R}_+}x^{\kappa}f^p(x)dx\right]^{1/p}\left[2B(\kappa,\kappa)\int_{\mathbb{R}_+}y^{\kappa}g^q(y)dy\right]^{1/q}\\
&=2B(\kappa,\kappa)\left[\int_{\mathbb{R}_+}x^{\kappa}f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+}y^{\kappa}g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This concludes the proof of Theorem 2.1. □

To our knowledge, the inequality in Theorem 2.1 is a new variant of the Hardy-Hilbert integral inequality. The simplicity of the constant factor and the weighted integral norms of f and g in the upper bound are among its key features. Compared with the proof of [11, Theorem 4], we see that the introduced function ψ plays a decisive role in allowing exact expressions for $\mathfrak{A}$ and $\mathfrak{B}$ instead of upper bounds. The theory is thus significantly improved.

Some secondary comments are now formulated. For the special case $p = 2$, the inequality in Theorem 2.1 reduces to

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa},(y/x)^{2\kappa}\right]\psi(x,y)} f(x)g(y)dxdy$$
$$\leq 2B(\kappa,\kappa)\sqrt{\int_{\mathbb{R}_+} x^{\kappa} f^2(x)dx}\sqrt{\int_{\mathbb{R}_+} y^{\kappa} g^2(y)dy},$$

where

$$\psi(x,y) = \left[\frac{\min(x,y)}{\max(x,y)}\right]^{(\kappa+1)/2}.$$

Let us consider two values for κ in this case.

Special case 1: If we set $\kappa = 1/2$, since $B(1/2,1/2) = \pi$, then we obtain

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{\sqrt{|x-y|}\max\left(x/y,y/x\right)\psi(x,y)} f(x)g(y)dxdy$$
$$\leq 2\pi\sqrt{\int_{\mathbb{R}_+} \sqrt{x} f^2(x)dx}\sqrt{\int_{\mathbb{R}_+} \sqrt{y} g^2(y)dy},$$

where

$$\psi(x,y) = \left[\frac{\min(x,y)}{\max(x,y)}\right]^{3/4}.$$

This variant is innovative in form, has a manageable function ψ, involves a simple constant factor and is still new.

Special case 2: If we set $\kappa = 1$, since $B(1,1) = 1$ and $\psi(x,y) = \min(x,y)/\max(x,y)$, then we arrive at

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \min\left(\frac{y}{x},\frac{x}{y}\right) f(x)g(y)dxdy$$
$$= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\max(x,y)}{\max\left[(x/y)^2,(y/x)^2\right]\min(x,y)} f(x)g(y)dxdy$$
$$\leq 2\sqrt{\int_{\mathbb{R}_+} x f^2(x)dx}\sqrt{\int_{\mathbb{R}_+} y g^2(y)dy}.$$

This inequality can be established in a more straightforward manner by employing classical analysis arguments.

2.2. **Discussion.** Let us now discuss the precision of the inequality in Theorem 2.1 and how it can be used to produce new results. Based on Equation (2.1), it is immediate that, for any $x, y > 0$,

$$\psi(x,y) = \left\{\min\left[\left(\frac{y}{x}\right)^{\kappa+1}, 1\right]\right\}^{1/p} \left\{\min\left[\left(\frac{x}{y}\right)^{\kappa+1}, 1\right]\right\}^{1/q} \leq 1.$$

It follows from this and Theorem 2.1 that

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa}, (y/x)^{2\kappa}\right]} f(x)g(y)dxdy \\
&\leq \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa}, (y/x)^{2\kappa}\right]\psi(x,y)} f(x)g(y)dxdy \\
&\leq 2B(\kappa,\kappa)\left[\int_{\mathbb{R}_+} x^{\kappa} f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+} y^{\kappa} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

Therefore, Theorem 2.1 implies [11, Theorem 4], as recalled in Equation (1.1). It is an improvement in this sense.

Other upper bounds of the function ψ can lead to new integral inequalities. An example is given below. For any $x, y > 0$, we have

$$\begin{aligned}
\psi(x,y) &= \left\{\min\left[\left(\frac{y}{x}\right)^{\kappa+1}, 1\right]\right\}^{1/p} \left\{\min\left[\left(\frac{x}{y}\right)^{\kappa+1}, 1\right]\right\}^{1/q} \\
&\leq \left(\frac{y}{x}\right)^{(\kappa+1)/p}.
\end{aligned}$$

This and Theorem 2.1 give

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa}, (y/x)^{2\kappa}\right](y/x)^{(\kappa+1)/p}} f(x)g(y)dxdy \\
&\leq \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{|x-y|^{1-\kappa}\max\left[(x/y)^{2\kappa}, (y/x)^{2\kappa}\right]\psi(x,y)} f(x)g(y)dxdy \\
&\leq 2B(\kappa,\kappa)\left[\int_{\mathbb{R}_+} x^{\kappa} f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+} y^{\kappa} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

To the best of our knowledge, this is a variant of the Hardy-Hilbert integral inequality that has not yet been published. Other inequalities can be obtained in the same way.

3. Conclusion

In summary, this article improves a known variant of the Hardy-Hilbert integral inequality by emphasizing its interaction with powers, maximum, and absolute differences of the variables. The elegant form of the constant factor, expressed via the beta function, captures the sharpness and depth of the result. Beyond its immediate scope, this approach provides a foundation for deriving additional variants and generalizations, indicating promising avenues for future research in this field.

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