

RESULTS ON ORDER DECREASING FULL TRANSFORMATION SEMIGROUP VIA TROPICAL GEOMETRY

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ABSTRACT. Let $\mathcal{DY}_\eta$ be the subsemigroup of order decreasing full transformation on $\Omega_\eta = \{1, 2, \dots, \eta\}$. In this paper, the tropical geometry were applied on $\mathcal{DY}_\eta$ for $1 \leq \eta \leq 4$. The elements in $\mathcal{DY}_\eta$ of classical algebra were degenerated into tropical polynomial and some results were obtained through their tropical curve.

1. INTRODUCTION AND PRELIMINARIES

A strange space with mysterious properties hide behind the enigmatic name of tropical geometry. This strange object is a line because of the geometric properties it satisfied. The name "tropical" was coined by French mathematician. Tropical geometry has established itself as an important new field bridging algebraic geometry whose techniques have been used to attack problems, these include enumerative geometry and arithmetic geometry. It builds on the older area of tropical mathematics more commonly known as max-plus algebra which arises in semigroup theory, computer science and optimization.

As in classical algebra, a tropical polynomial expression $\Gamma(x) = \sum_{j=0}^d a_j x^j$ induces a tropical polynomial function denoted by Γ on $\mathbb{T}$;

$$\Gamma : \mathbb{T} \rightarrow \mathbb{T} \tag{1.1}$$

The set of tropical numbers is defined as $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ endowed with the operations called tropical addition and multiplication:

$$"\theta + \vartheta" = \max\{\theta, \vartheta\}; \tag{1.2}$$

$$"\theta \times \vartheta" = \theta + \vartheta \tag{1.3}$$

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with the usual conventions:

$$\forall x \in \mathbb{T}, "x + (-\infty)" = \max(x, -\infty) = x \text{ and } "x \times (-\infty)" = x + (-\infty) = -\infty \quad (1.4)$$

The tropical numbers along with these two operations form a semi-field, that is, they satisfies all the axioms of a field except the existence of an inverse for the law of addition. In tropical operations, " $5x$ " $\neq$ " $x + x + x + x + x$ " but " $5x$ " = " $x + 5$ " and " $0x$ " = " x " but not equal to 0. In this paper, tropical operations will be placed under quotation marks. A tropical polynomial is a convex piecewise affine function and each piece has an integer slope.

A semigroup is an algebraic structure consisting of a non-empty set $\mathcal{K}$ together with an associative binary operation. A transformation of $\mathcal{T}$ is a function from $\mathcal{T}$ to itself. Transformation semigroups is one of the most fundamental mathematical objects that occurs in theoretical computer science where properties of language depend on algebraic properties of various transformation semigroups related to them.

Let $\Omega_\eta = \{1, 2, \dots, \eta\}$ be a finite set ordered in standard way. The map $\psi : \text{Dom}\psi \subseteq \Omega_\eta \rightarrow \text{Im}\psi \subseteq \Omega_\eta$ is called full or total if $\text{Dom}\psi = \Omega_\eta$, partial if $\text{Dom}\psi \subsetneq \Omega_\eta$ or else it is called strictly partial. The set Υ_η of a full transformations on Ω_η forms a semigroup under composition of mappings called the full transformation semigroup.

Tropical geometry has exploded as an area of research in the last decade with many new connections and applications appearing each year. These include enumerate geometry and arithmetic geometry. It builds on the older area of tropical mathematics more commonly known as max-plus algebra which arises in semigroup theory, computer science and optimization. [3] studied some aspects of tropical geometry, [8] looked into introduction to tropical algebraic geometry. Also, [9] worked on tropical intersection product in matroidal fans. Furthermore, [2] analysed obstructions to approximating tropical curves in surfaces via intersection theory. Recently, [11] studied tropicalization of idempotent elements on full transformation semigroup and [1] worked on application of tropical geometry on subsemigroup of order preserving full transformation. For standard concepts and terms in semigroup and tropical geometry theory see [5, 10, 2].

Our work focus on the tropical geometry and semigroup namely the study of the tropical polynomials, their roots and multiplicities in $(\mathcal{D}\Upsilon_\eta)$. The elements in $(\mathcal{D}\Upsilon_\eta)$ were degenerated into tropical polynomial and some results were established with relevant examples to validate it.

Definition 1.1. [2] A tropical number τ is a root of a Polynomial $\Gamma(x)$ in one variable in which the points x_0 on the graph $\Gamma(x)$ has a corner at x_0 for $-\infty \leq \tau \leq \infty$.

Definition 1.2. [2] The Multiplicity of a tropical root τ denoted by $\rho(\tau)$ defined as

$$\rho(\tau) = |\varrho_j - \varrho_{j-1}|$$

where ϱ_j are the slopes of the lines in the tropical graph $\Gamma(x)$ intersecting above τ .

Proposition 1.3. [2] *The tropical semi-field is algebraically closed. That is, every tropical polynomial of degree $\delta > 0$ has exactly δ roots when counted with multiplicities.*

Definition 1.4. [10] Let $S = \mathcal{DY}_\eta$ and $\psi \in \mathcal{DY}_\eta$. Then, the height of $\psi = |Im\psi|$

Definition 1.5. [10] $\mathcal{DY}_\eta = \{\psi \in \mathcal{DY}_\eta : (\forall \vartheta \in Dom\psi) \vartheta\psi \leq \vartheta\}$, is the sub-semigroup of $\mathcal{Y}_\eta$ containing all *order-decreasing full transformation semigroup* on Ω_η .

2. MAIN RESULTS

Theorem 2.1. *Let $S = \mathcal{DY}_\eta$, $|\tau|$ be the order of the roots of the tropical polynomial in $\mathcal{DY}_4$. Then, for all elements in $\mathcal{DY}_4$ satisfy $|\tau| < 4$ has unique multiplicity of height two.*

Proof. Let $\Phi, \Psi, \mu \in \mathcal{DY}_4$ such that $\Phi = \begin{pmatrix} \varepsilon_p & \varepsilon_{p+1} & \varepsilon_{p+2} & \varepsilon_{p+3} \\ \zeta_i & \zeta_{i+1} & \zeta_{i+2} & \zeta_{i+3} \end{pmatrix}$, $\Psi = \begin{pmatrix} \omega_p & \omega_{p+1} & \omega_{p+2} & \omega_{p+3} \\ \alpha_j & \alpha_{j+1} & \alpha_{j+2} & \alpha_{j+3} \end{pmatrix}$, and $\mu = \begin{pmatrix} \lambda_p & \lambda_{p+1} & \lambda_{p+2} & \lambda_{p+3} \\ \beta_k & \beta_{k+1} & \beta_{k+2} & \beta_{k+3} \end{pmatrix}$ with the polynomial defines as follows;

$\Phi(x) = \varepsilon_p x^\eta + \varepsilon_{p+i} x^{\eta-1} + \varepsilon_{p+2} x^{\eta-2} + \varepsilon_{p+3} x^{\eta-3} + \zeta_i x^{\eta-1} + \zeta_{i+1} x^{\eta-2} + \zeta_{i+2} x^{\eta-3} + \zeta_{i+3}$,
 $\Psi(x) = \omega_p x^\eta + \omega_{p+i} x^{\eta-1} + \omega_{p+2} x^{\eta-2} + \omega_{p+3} x^{\eta-3} + \alpha_j x^{\eta-1} + \alpha_{j+1} x^{\eta-2} + \alpha_{j+2} x^{\eta-3} + \alpha_{j+3}$
 $\mu(x) = \lambda_p x^\eta + \lambda_{p+i} x^{\eta-1} + \lambda_{p+2} x^{\eta-2} + \lambda_{p+3} x^{\eta-3} + \beta_k x^{\eta-1} + \beta_{k+1} x^{\eta-2} + \beta_{k+2} x^{\eta-3} + \beta_{k+3}$
for $p = i = j = k = 1$, $\varepsilon_p = \omega_p = \lambda_p \in dom((\Phi, \psi, \mu))$ and $\zeta_i \neq \alpha_j \neq \beta_k \in Im(\Phi, \psi, \mu)$, $\forall \eta = 4$
having the tropical polynomials;

$\mathbb{T}(\Phi(x)) = "\varepsilon_p x^\eta + \varepsilon_{p+1} x^{\eta-1} + \varepsilon_{p+2} x^{\eta-2} + \varepsilon_{p+3} x^{\eta-3} + \zeta_i x^{\eta-1} + \zeta_{i+1} x^{\eta-2} + \zeta_{i+2} x^{\eta-3} + \zeta_{i+3}"$,
 $\mathbb{T}(\Psi(x)) = "\omega_p x^\eta + \omega_{p+i} x^{\eta-1} + \omega_{p+2} x^{\eta-2} + \omega_{p+3} x^{\eta-3} + \alpha_j x^{\eta-1} + \alpha_{j+1} x^{\eta-2} + \alpha_{j+2} x^{\eta-3} + \alpha_{j+3}"$
 $\mathbb{T}(\mu(x)) = "\lambda_p x^\eta + \lambda_{p+i} x^{\eta-1} + \lambda_{p+2} x^{\eta-2} + \lambda_{p+3} x^{\eta-3} + \beta_k x^{\eta-1} + \beta_{k+1} x^{\eta-2} + \beta_{k+2} x^{\eta-3} + \beta_{k+3}"$
which gives;

$$\max\{\eta x + \varepsilon_p, (\eta - 1)x + \varepsilon_{p+1}, (\eta - 2)x + \varepsilon_{p+2}, (\eta - 3)x + \varepsilon_{p+3}, (\eta - 1)x + \zeta_i, \\ (\eta - 2)x + \zeta_{i+1}, (\eta - 3)x + \zeta_{i+2}, \zeta_{i+3}\}$$

$$\max\{\eta x + \omega_p, (\eta - 1)x + \omega_{p+1}, (\eta - 2)x + \omega_{p+2}, (\eta - 3)x + \omega_{p+3}, (\eta - 1)x + \alpha_i, \\ (\eta - 2)x + \alpha_{i+1}, (\eta - 3)x + \alpha_{i+2}, \alpha_{i+3}\}$$

$$\max\{\eta x + \lambda_p, (\eta - 1)x + \lambda_{p+1}, (\eta - 2)x + \lambda_{p+2}, (\eta - 3)x + \lambda_{p+3}, (\eta - 1)x + \beta_i, \\ (\eta - 2)x + \beta_{i+1}, (\eta - 3)x + \beta_{i+2}, \beta_{i+3}\}$$

Then, from the curve we observed that, for all $\Phi(x)$, $\Psi(x)$, and $\mu(x)$ having the order of the tropical roots less than η has a unique multiplicity of height two. i.e $|\rho(\Phi(x))| = |\rho(\Psi(x))| = |\rho(\mu(x))| = 2$ for $|\tau_\Phi(x)|, |\tau_\Psi(x)|, |\tau_\mu(x)| < \eta$ $\square$

Example 2.2. Consider $\Phi, \Psi, \mu \in \mathcal{DY}_4$ such that $\Phi = (1)(4321]$, $\Psi = (1)(2)(3)(41]$ and $\mu = (1)(4)(321]$ with polynomial functions;

$$\Phi(x) = x^4 + 3x^3 + 4x^2 + 6x + 3$$

$$\Psi(x) = x^4 + 3x^3 + 5x^2 + 7x + 1$$

$$\mu(x) = x^4 + 3x^3 + 4x^2 + 6x + 4$$

Then, we have the following tropical polynomials for $\Phi(x)$, $\Psi(x)$ and $\mu(x)$ respectively ;

$$"x^4 + 3x^3 + 4x^2 + 6x + 3" = \max(4x, 3x + 3, 2x + 4, 6 + x, 3)$$

,

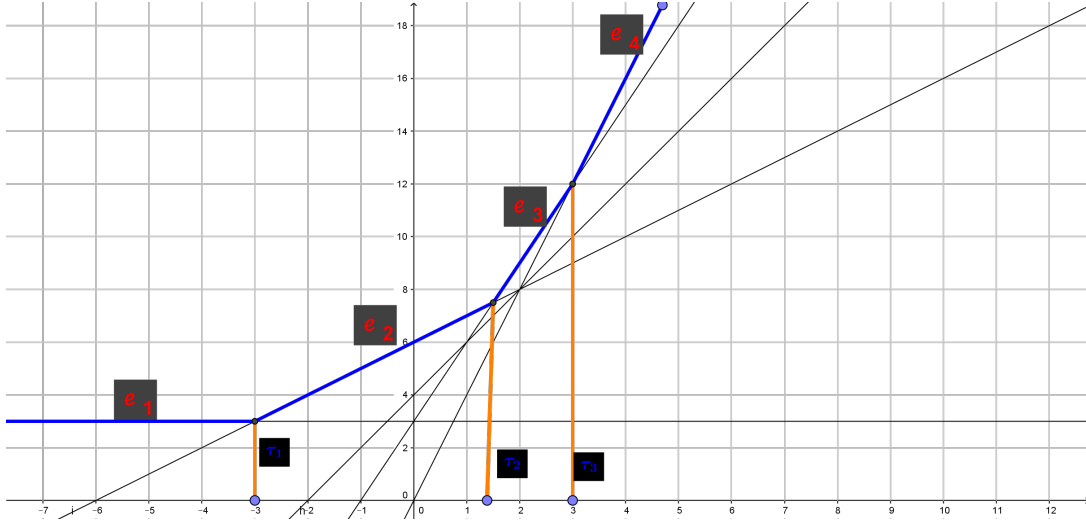
$$"x^4 + 3x^3 + 5x^2 + 7x + 1" = \max(4x, 3x + 3, 2x + 5, 7 + x, 1)$$

,

and

$$"x^4 + 3x^3 + 4x^2 + 6x + 4" = \max(4x, 3x + 3, 2x + 4, 6 + x, 4)$$

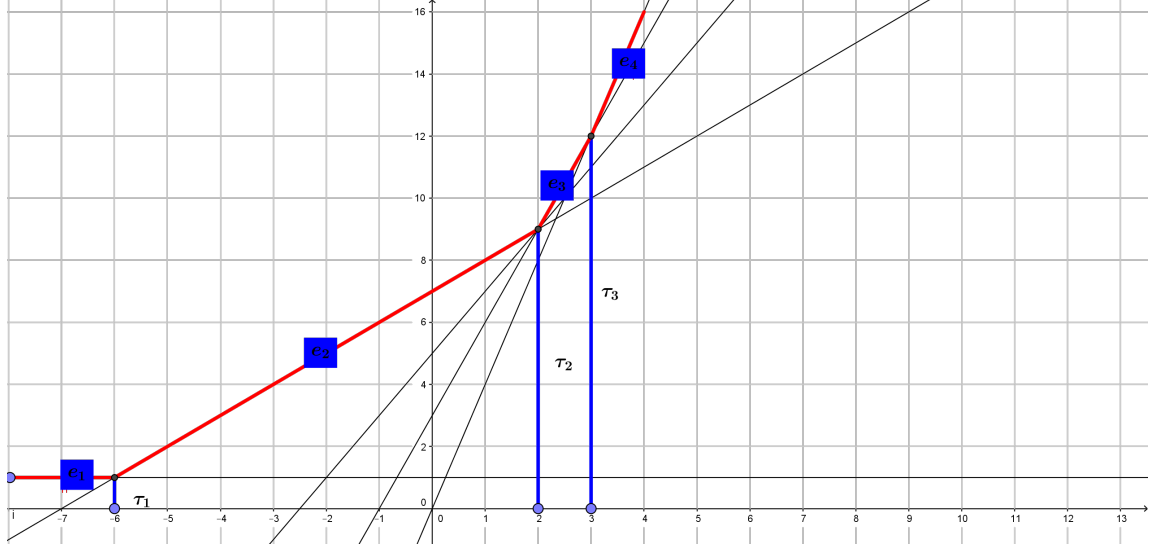
FIGURE 1. Tropical curve for $\Phi(x)$



From the tropical curve $\mathbb{T}_{\Phi(x)}$ above, the tropical roots were obtained as; $\tau_1 = -3, \tau_2 = 1.5$ and $\tau_3 = 3$, that is the $|\tau| = 3$ with slopes $\varrho_1 = 0, \varrho_2 = 1, \varrho_3 = 3$ and $\varrho_4 = 4$ having multiplicities of $\rho(\tau) = (1 \ 2 \ 1)$.

Hence, the order of multiplicities of is 2. i.e $|\mathbb{T}_{\alpha(x)}| = (1 \ 2 \ 1) = 2$

FIGURE 2. Tropical curve for $\Psi(x)$

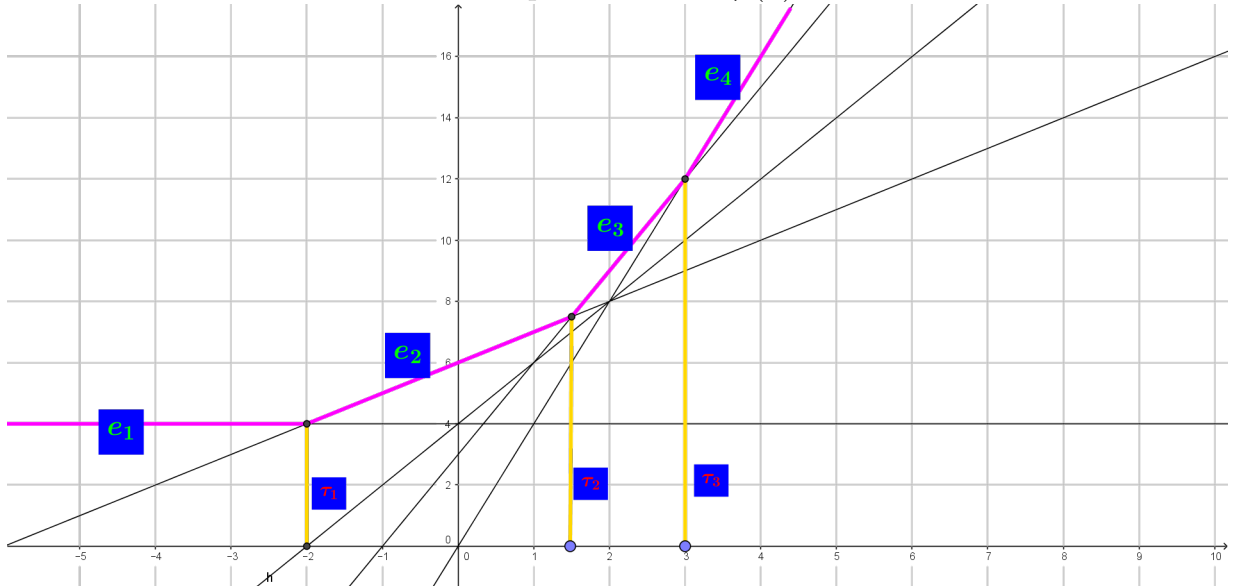


Similarly, from the tropical curve ii of $\mathbb{T}_{\Psi(x)}$ above, the roots were also obtained as;

$\tau_1 = -6$, $\tau_2 = 2$, and $\tau_3 = 3$, that is $|\tau| = 3$ having slopes $\varrho_1 = 0$, $\varrho_2 = 1$, $\varrho_3 = 3$ and $\varrho_3 = 4$ with multiplicities of $\rho(\tau) = (1 \ 2 \ 1)$.

Thus, the order of multiplicities of $|\mathbb{T}_{\Psi(x)}| = (1 \ 2 \ 1) = 2$

FIGURE 3. Tropical curve for $\mu(x)$



From above tropical curve, the roots of $\mathbb{T}_{\mu(x)}$ deduced as;
 $\tau_1 = -2$, $\tau_2 = 1.5$, and $\tau_3 = 3$, that is $|\tau| = 3$ having slopes $\rho_1 = 0, \rho_2 = 1, \rho_3 = 3$
 and $\rho_3 = 4$ with multiplicities of $\rho(\tau) = (1 \ 2 \ 1)$.

Theorem 2.3. *Every transformation in $\mathcal{DY}_\eta$ having the same order of the roots has a unique multiplicity of height one. $1 \leq \eta \leq 4$.*

Proof. Let $|\tau|$ be the order of the tropical polynomial for $1 \leq \eta \leq 4$. Then, let $\Lambda \in \mathcal{DY}_2$, $\Theta \in \mathcal{DY}_3$ and $\chi \in \mathcal{DY}_4$ such that

$\Lambda = \begin{pmatrix} \hbar_1 & \hbar_2 \\ \partial_1 & \partial_2 \end{pmatrix}$, $\Theta = \begin{pmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$, and $\chi = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \end{pmatrix}$ respectively.

Also let $\Lambda(z) \in \mathcal{DY}_2$, $\Theta(z) \in \mathcal{DY}_3$ and $\chi(z) \in \mathcal{DY}_4$ be the polynomials defined by:

$$\Lambda(z) = \hbar_1 z^2 + \hbar_2 z + \partial_1 z + \partial_2$$

$$\Theta(z) = \Upsilon_1 z^3 + \Upsilon_2 z^2 + \Upsilon_3 z + \sigma_1 z^2 + \sigma_2 z + \sigma_3,$$

$$\chi(z) = v_1 z^4 + v_2 z^3 + v_3 z^2 + v_4 z + \ell_1 z^3 + \ell_2 z^2 + \ell_3 z + \ell_4,$$

having their tropical polynomials as:

$$\mathbb{T}(\Lambda(z)) = "\hbar_1 z^2 + \hbar_2 z + \partial_1 z + \partial_2"$$

$$\mathbb{T}(\Theta(z)) = "\Upsilon_1 z^3 + \Upsilon_2 z^2 + \Upsilon_3 z + \sigma_1 z^2 + \sigma_2 z + \sigma_3",$$

$$\mathbb{T}(\chi(z)) = "v_1 z^4 + v_2 z^3 + v_3 z^2 + v_4 z + \ell_1 z^3 + \ell_2 z^2 + \ell_3 z + \ell_4",$$

which given

$$\mathbb{T}(\Lambda(z)) = \max\{2z + \hbar_1, z + \hbar_2, z + \partial_1, \partial_2\}$$

$$\mathbb{T}(\Theta(z)) = \max\{3z + \Upsilon_1, 2z + \Upsilon_2, z + \Upsilon_3, 2z + \sigma_1, z + \sigma_2, \sigma_3, \}$$

$$\mathbb{T}(\chi(z)) = \max\{4z + v_1, 3z + v_2, 2z + v_3, z + v_4, 3z + \ell_1, 2z + \ell_2, z + \ell_3, \ell_4, \}$$

Then, from the tropical curve we observed that, for $\Lambda(z) \in \mathcal{DY}_2$, $\Theta(z) \in \mathcal{DY}_3$, and $\chi(z) \in \mathcal{DY}_4$ respectively having the same order of the tropical roots has a unique multiplicity of height one with the different slopes and values of the roots. i.e $|\rho(\Lambda(z))| = |\rho(\Theta(z))| = |\rho(\chi(z))| = 1$ $\square$

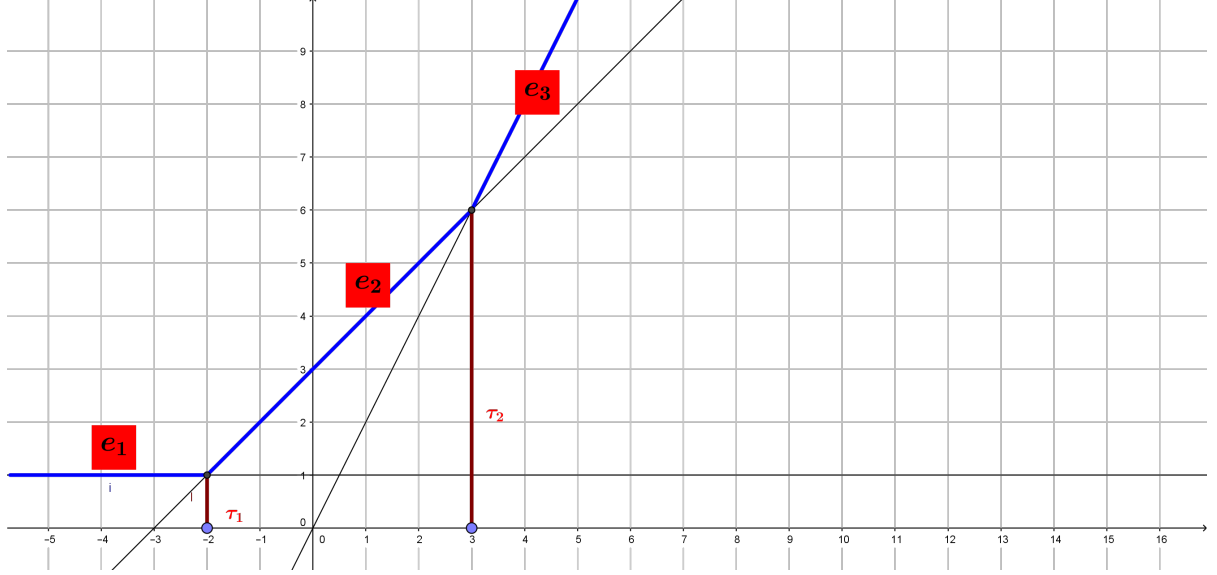
Example 2.4. Consider $\Lambda \in \mathcal{DY}_2$, such that

$\Lambda = (1)(21]$ with polynomial function;

$$\begin{aligned} \Lambda(x) &= x^2 + 2x + x + 1 \\ &= x^2 + 3x + 1 \end{aligned}$$

Then, we have tropical polynomial as;

$$"x^2 + 3x + 2" = \max(2x, x + 3, 1)$$

FIGURE 4. Tropical curve for $\Lambda(x)$ 

The tropical curve $\mathbb{T}_{\Lambda(x)}$ above gives the tropical roots; $\tau_1 = -2$, and $\tau_2 = 3$ with slopes $\varrho_1 = 0, \varrho_2 = 1$ and $\varrho_3 = 2$ with the order of multiplicities of $|\rho(\mathbb{T}_{\Lambda(x)})| = (1 \ 1) = 1$

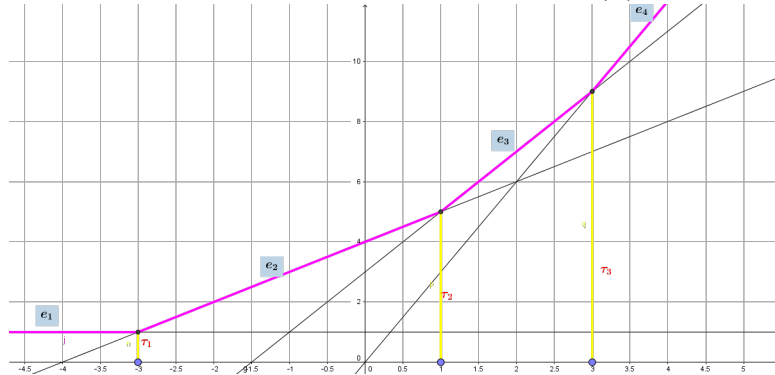
Example 2.5. Consider $\Theta \in \mathcal{DT}_3$, such that

$\Theta = (1)(21)(31)$ with polynomial function;

$$\Theta = x^3 + 3x^2 + 4x + 1$$

Then, we have the tropical polynomial as;

$$"x^3 + 3x^2 + 4x + 1" = \max(3x, 2x + 3, 4 + x, 1)$$

FIGURE 5. Tropical curve for $\Theta(x)$ 

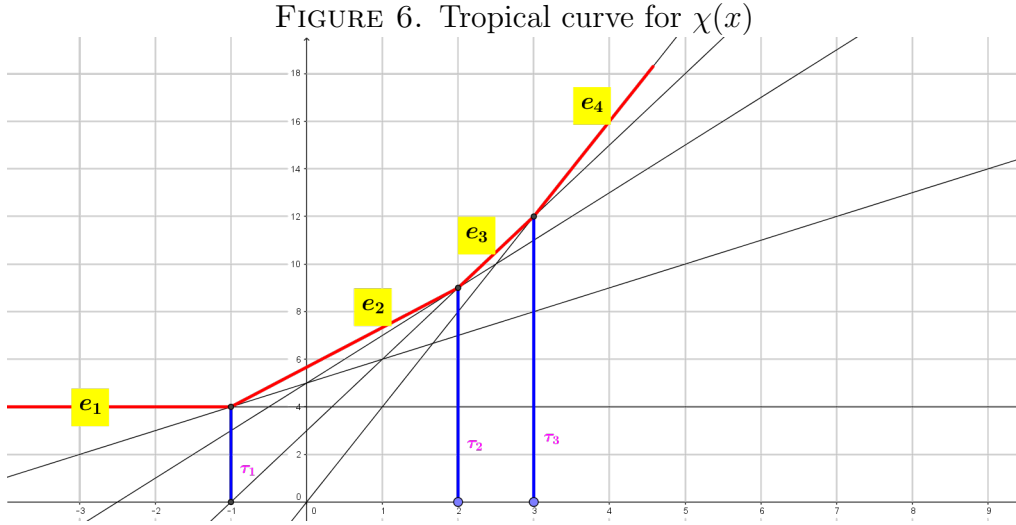
From the above tropical curve $\mathbb{T}_{\Theta(x)}$ the tropical roots were got as; $\tau_1 = -3$, $\tau_2 = 1$, and $\tau_3 = 3$ with slopes $\varrho_1 = 0, \varrho_2 = 1, \varrho_3 = 2$ and $\varrho_4 = 3$ with the order of multiplicities $|\rho(\mathbb{T}_{\Theta(x)})| = (1 \ 1 \ 1) = 1$

Example 2.6. Consider $\chi \in \mathcal{DY}_4$, such that
 $\chi = (1)(2)(31](4)$ with polynomial functions;

$$\chi = x^4 + 3x^3 + 5x^2 + 5x + 4$$

With tropical polynomial;

$${}^{\text{tr}}x^4 + 3x^3 + 5x^2 + 5x + 4 = \max(4x, 3x + 3, 2x + 5, x + 5, 4)$$



From the above tropical curve $\mathbb{T}_{\chi(x)}$, the tropical roots were got as; $\tau_1 = -2$, $\tau_2 = 2$ and $\tau_3 = 3$ with slopes $\varrho_1 = 0$, $\varrho_2 = 1$, $\varrho_3 = 2$ and $\varrho_4 = 3$ with the order of multiplicities $|\rho(\mathbb{T}_{\chi(x)})| = (1 \ 1 \ 1) = 1$

Conclusion

In this paper, we have shown that all the elements in $\mathcal{DY}_4$ satisfying $|\tau| < 4$ has a unique multiplicity of height two as shown in the Theorem 2.1 and illustrated in the Example 2.2. Moreover, every transformation in $\mathcal{DY}_\eta$, such that $1 \leq \eta \leq 4$ having the same order of the roots also has a unique multiplicity of height one as shown in the Theorem 2.3 and illustrated in the Examples 2.4, 2.5 and 2.6 respectively.

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