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A NEW OPERATIONAL MAX-STABLE COPULA FOR ASYMMETRIC SPATIAL DEPENDENCE IN EXTREME VALUE ANALYSIS

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Abstract. Modeling the spatial dependence of extremes presents a dual challenge: guaranteeing max-stability while capturing the asymmetries observed in the field. We introduce a simplified version of an asymmetric Schlather copula, obtained by a clever reparametrization. This formulation offers both an explicit density, facilitating estimation by maximum likelihood and three interpretable parameters controlling the range, intensity and asymmetry of the dependence. Applied to a real dataset of extreme temperatures, our model outperforms several classical copulas (including Hüsler-Reiss and Schlather) in terms of fit. We also detail the analytical properties of the model (tail distribution, max-stability, density), demonstrating its flexibility and practical relevance for the analysis of extreme climate risks.

Keywords. Max-stable copula, Spatial extreme copula, Schlather copula, max-stables processes
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1. Introduction

In 2022, West Africa experienced extreme weather events on an unprecedented scale. In Nigeria, the worst floods in twenty years caused more than 660 deaths and displaced more than 2.4 million people, also affecting Chad, Niger and Burkina Faso (West Africa-Emergency Report 2023). These disasters are a reminder of how crucial it is to better understand and anticipate the extreme behavior of environmental variables, particularly their spatial propagation.

In this context, max-stable processes such as those of Smith (1990) [19], Schlather (2002) [17] or Brown-Resnick (Kabluchko et al., 2009) [9] have become the benchmark models for the dependence between extreme values at several sites. However, their rigid mathematical structure and inherent symmetry limit their practical applicability (Huser et al., 2024) [6] . Indeed, these processes assume that extremes behave identically whatever the direction or relative location between sites. However, many environmental data show directional effects, dominant wind

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regimes, topographical influences or climatic gradients that induce marked asymmetries in spatial dependence.

Similarly, the copulas classically used to model extreme dependencies, whether they are inspired by max-stable processes or other families, are often symmetrical, which prevents us from capturing differences between the behavior of marginals or directions of influence. However, the introduction of asymmetry makes it possible to represent realistic configurations, such as one site being strongly influenced by another without reciprocity, or stronger links in a given direction. This is particularly relevant in climatology and hydrology, where entrainment or propagation effects are rarely reversible.

Faced with these limitations, several approaches have been developed to introduce asymmetry into dependency models, in particular via asymmetric copulas (Khoudraji, 1995; Jaworski et al., 2010; Bacro et al., 2016) [10, 7, 2]. Alongside this work, we previously proposed an asymmetric extension of the Schlather copula to overcome the constraints of symmetric models (Bagré et al., 2025) [1]. However, the complexity of its mathematical form prevented maximum likelihood estimation, due to the lack of an explicit density.

This paper presents a simplified version of this copula, obtained by a judicious reparametrization. This new construction makes the density fully explicit, allowing efficient parameter estimation. It also retains the original max-stable framework while incorporating a directional flexibility that is crucial for modeling the asymmetric dependencies observed in extreme spatial data. To date, no asymmetric max-stable model simultaneously offers a closed density and a set of three easily interpretable parameters, which is the main contribution of this paper.

The paper is divided into three main sections: section 2 lays the theoretical foundations by defining copulas, recalling the dependence measures (Kendall's tau, Spearman's rho and tail dependence indices) and summarizing our earlier asymmetric extension of the Schlather copula; section 3 presents our major methodological contribution: the construction of the simplified asymmetric copula, the proof of its max-stability, the explicit expression of its density and the clear interpretation of its three parameters (range, intensity, asymmetry); finally, section 4 illustrates the operational interest of our approach on a dataset of extreme temperatures from five cities in Switzerland, provides the graphical and statistical diagnostics of the fit and compares the performances with those of the symmetric Hüsler-Reiss, and Schlather copulas to demonstrate a better consideration of directional asymmetry.

2. Materials and methods

2.1. Bivariate copula concepts.

Definition 2.1. [13, 20]

Let $I = [0, 1]$ and $I^2 = I \times I$.

A bivariate copula is a function $C : I^2 \longrightarrow I$ which is defined by the following conditions:

(1) C is grounded, i.e. $C(u, 0) = C(0, v) = 0$, for all $(u, v) \in I^2$

(2) The margins are uniform, i.e. $C(u,1) = u$ and $C(1,v) = v$, for all $(u,v) \in I^2$
(3) C is a 2-increasing function on I^2, i.e. for all $(u_1,u_2,v_1,v_2) \in I^4$ such that: $u_1 \leq u_2$ and $v_1 \leq v_2$, we have:
$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$

Remark 2.2. [21] If C is a symmetric copula then

$$C(u,v) = C(v,u) \qquad \forall u,v \in I \tag{2.1}$$

Otherwise, we say that C is asymmetric.

When modeling extreme values, not all copulas are suitable. The copulas that are suitable are copulas that verify the max-stability property. The following definition sheds light on this notion.

Definition 2.3. [13] Let n be a positive real constant. An extreme value copula C is a copula which satisfies the following relation :

$$C(u^n, v^n) = C^n(u,v). \tag{2.2}$$

2.2. **Associations and dependence measures.** In order to quantify the dependencies between random variables, several tools exist in the literature. However, not all tools are suitable for measuring non-linear and complex dependencies. Among the adapted measures, we will focus on Spearman's rho and Kendall's tau, which are measures of non-linear associations, and the upper and lower tail indices, which are also very practical measures of extreme dependence between two variables.
The expressions for these measures of association in terms of any copula C are given in the following definition:

Definition 2.4. [13] Let C be a bivariate copula. Spearman's rho and Kendall's tau as a function of the copula C are given respectively by:

$$\rho_S(C) = 12 \iint_{I^2} C(u,v)dudv - 3, \tag{2.3}$$

and

$$\tau_K(C) = 4 \iint_{I^2} C(u,v)dC(u,v) - 1. \tag{2.4}$$

Another measurement tool that can be used to effectively measure dependence in the tails of distributions is the tail dependence coefficient. There are two tail dependence coefficients λ_U and λ_L which are respectively the upper tail dependence coefficient and the lower tail dependence coefficient. These coefficients measure the probability of extreme values occurring simultaneously. Like Spearman's rho and Kendall's tau, these tail dependence coefficients can also be expressed in terms of a copula. The following lemma and theorem tell us more about these measures.

Lemma 2.5. [13] *Let U and V be uniform random variables on I and C the associated copula. Then for all $u, v \in I$, we have:*

$$\mathbb{P}(U \leq u | V \leq v) = \frac{C(u,v)}{v} \tag{2.5}$$

and

$$\mathbb{P}(U > u | V > v) = \frac{1 - u - v + C(u,v)}{1 - v} \tag{2.6}$$

Theorem 2.6. [13] *Let X and Y be two random variables with joint distribution function F and C the copula associated with F. If the limits of the equations (2.5) and (2.6) exist then*

$$\lambda_U = 2 - \lim_{u \to 1^-} \frac{1 - C(u,u)}{1 - u} \qquad \textit{and} \qquad \lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u}. \tag{2.7}$$

The interpretation of these dependency index calculations depends on the value found. Details are given in the following definition:

Definition 2.7. [13] Let C be a copula.

i) If λ_U exists and its value is between $]0,1[$, then C has a right-tail dependence.
ii) If λ_L exists and its value is within the interval $]0,1[$, then C has a left-tail dependence.
The extreme values of I, i.e. 0 and 1, correspond respectively to the existence of perfect tail independence and perfect tail dependence. The more the value tends towards 1 (respectively 0), the more the dependence increases (decreases).

2.3. **Reminder of a class of bivariate asymmetric copulas.** In another paper [1], we constructed a new class of asymmetric bivariate copula based on the product of two symmetric copulas. The mathematical formulation of this copula is given in the following theorem:

Theorem 2.8. [1] *Let C_1 and C_2 be two symmetric copulas, $\alpha \neq \beta$ such as $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$, then:*

$$C(u,v) = \left[u^{2(1-\alpha-\beta)} C_1(u^\alpha, v^{1-\alpha}) C_2(u^\beta, v^{1-\beta})\right]^{\frac{1}{(2-\alpha-\beta)}}. \tag{2.8}$$

is an asymmetric copula.

We recall that it has also been shown that this class of asymmetric copulas retains the max-stability property if all the base copulas used are max-stable.
Let be the standard Schlather copula defined by:

$$C_\rho(u,v) = \exp\left(\frac{\log u + \log v}{2}\left\{1 + \sqrt{1 - \frac{2(1+\rho)\log u \log v}{(\log u + \log v)^2}}\right\}\right). \tag{2.9}$$

where $-1 < \rho < 1$ is a correlation coefficient.
By applying this theorem, if we replace C_1 and C_2 by two Schlather copulas with

parameters ρ_1 and ρ_2 respectively, we obtain:

$$\begin{aligned} &C\left[C_{\rho_1}, C_{\rho_2}\right](u,v) \\ &= \Bigg[u^{2(1-\alpha-\beta)} \exp\Bigg(\frac{\log u^\alpha + \log v^{1-\alpha}}{2}\left\{1+\sqrt{1-\frac{2(1+\rho_1)\log u^\alpha \log v^{1-\alpha}}{(\log u^\alpha + \log v^{1-\alpha})^2}}\right\} \\ &+ \frac{\log u^\beta + \log v^{1-\beta}}{2}\left\{1+\sqrt{1-\frac{2(1+\rho_2)\log u^\beta \log v^{1-\beta}}{(\log u^\beta + \log v^{1-\beta})^2}}\right\}\Bigg)\Bigg]^{1/(2-\alpha-\beta)}. \end{aligned} \tag{2.10}$$

$\alpha, \beta \in [0,1]$ *such that* $\alpha + \beta \leq 1$.
As this copula is a max-stable copula by construction, it is suitable for modeling extreme events. Its upper tail index is given by:

$$\lambda_U(C\left[C_{\rho_1}, C_{\rho_2}\right]) = 2 - \frac{1}{\zeta}\left(2(1-\alpha-\beta) + \frac{K_1}{2}\right) \tag{2.11}$$

where

$$K_1 = 2 + \sqrt{1-2\alpha(1+\rho_1)(1-\alpha)} + \sqrt{1-2\beta(1+\rho_2)(1-\beta)} \text{ and } \zeta = 2-\alpha-\beta \tag{2.12}$$

Although the copula given in (2.10) is extreme and spatial, its expression is complex and further complicates the calculation of the density, which is crucial in the modeling process. In the following we propose an asymmetric extension simplified by reparametrization of this copula which is even more practical and which also maintains the same dependency structure.

3. Main results

In this section, we propose a reparametrization of the copula presented previously in order to simplify it and make it applicable to real data. We express the parameters α and β in terms of an exponential correlation function defined by $\exp(-h/\theta)$, where h is the Euclidean distance between two sites and θ, a range parameter modulating the dependence between the two sites.

Remark 3.1. Before formulating the proposition 3.2, we consider a special case of the general framework of Theorem 2.8 by imposing a specific weight reparametrization: $\alpha = \frac{1}{1+\exp(-h/\theta)}$ and $\beta = 1-\alpha$.
This choice satisfies the condition $\alpha + \beta \leq 1$ and allows us to obtain a simplified and usable expression of the copula, leading in particular to an explicit density.

Proposition 3.2. *Let* $\alpha = \dfrac{1}{1+\exp(-h/\theta)}$ *and* $\beta = 1-\alpha$ *such that* $\theta > 0$ *and* $h \in [0, +\infty[$.

Given (2.10), then

$$C_{\theta,\rho_1,\rho_2}(u,v) = \exp\left[\frac{\log u^{\alpha}+\log v^{1-\alpha}}{2}\left(1+\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}\right)\right.$$
$$\left.+\frac{\log u^{1-\alpha}+\log v^{\alpha}}{2}\left(1+\sqrt{1-\frac{2(1+\rho_2)\log u^{1-\alpha}\log v^{\alpha}}{(\log u^{1-\alpha}+\log v^{\alpha})^2}}\right)\right], \tag{3.1}$$

is a copula.

Proof. In [1],

$$C\left[C_{\rho_1},C_{\rho_2}\right](u,v)$$
$$=\left[u^{2(1-\alpha-\beta)}\exp\left(\frac{\log u^{\alpha}+\log v^{1-\alpha}}{2}\left\{1+\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}\right\}\right.\right. \tag{3.2}$$
$$\left.\left.+\frac{\log u^{\beta}+\log v^{1-\beta}}{2}\left\{1+\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}\right\}\right)\right]^{1/(2-\alpha-\beta)}.$$

is a copula if the condition $\alpha+\beta\leq 1$ is satisfied.
By posing $\alpha=\dfrac{1}{1+\exp(-h/\theta)}=\dfrac{1}{1+a}$ and $\beta=\dfrac{\exp(-h/\theta)}{1+\exp(-h/\theta)}=\dfrac{a}{1+a}$, with $\theta\in\,]0,+\infty[$; $h\in[0,+\infty[$ and $a=\exp(-h/\theta)$, equation (3.2) become:

$$C\Big[C_{\rho_1},C_{\rho_2}\Big](u,v)$$
$$=\left[u^{2(1-\frac{1}{1+a}-\frac{a}{1+a})}\exp\left(\frac{\log u^{\frac{1}{1+a}}+\log v^{1-\frac{1}{1+a}}}{2}\left\{1+\sqrt{1-\frac{2(1+\rho_1)\log u^{\frac{1}{1+a}}\log v^{1-\frac{1}{1+a}}}{(\log u^{\frac{1}{1+a}}+\log v^{1-\frac{1}{1+a}})^2}}\right\}\right.\right.$$
$$\left.\left.+\frac{\log u^{\frac{a}{1+a}}+\log v^{1-\frac{a}{1+a}}}{2}\left\{1+\sqrt{1-\frac{2(1+\rho_2)\log u^{\frac{a}{1+a}}\log v^{1-\frac{a}{1+a}}}{(\log u^{\frac{a}{1+a}}+\log v^{1-\frac{a}{1+a}})^2}}\right\}\right)\right]^{1/(2-\frac{1}{1+a}-\frac{a}{1+a})}. \tag{3.3}$$

As $\dfrac{1}{1+a}+\dfrac{a}{1+a}=\dfrac{1}{1+\exp(-h/\theta)}+\dfrac{\exp(-h/\theta)}{1+\exp(-h/\theta)}=1$, then (3.3) is indeed a copula. By simplification of (3.3), we find (3.1) :

$$C_{\theta,\rho_1,\rho_2}(u,v) = \exp\left[\frac{\log u^{\alpha}+\log v^{1-\alpha}}{2}\left(1+\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}\right)\right.$$
$$\left.+\frac{\log u^{\beta}+\log v^{1-\beta}}{2}\left(1+\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}\right)\right]$$

The modification therefore respects the structure of a copula □

Definition 3.3. Let $\alpha=\dfrac{1}{1+\exp(-h/\theta)}$ and $\beta=1-\alpha$ such that $\theta>0$ and $h\in[0,+\infty)$. Then the asymmetric extension simplified by reparametrization of the Schlather copula is given by:

$$C_{\theta,\rho_1,\rho_2}(u,v) = \exp\left[\frac{\log u^\alpha + \log v^{1-\alpha}}{2}\left(1+\sqrt{1-\frac{2(1+\rho_1)\log u^\alpha \log v^{1-\alpha}}{(\log u^\alpha + \log v^{1-\alpha})^2}}\right)\right.$$
$$\left. + \frac{\log u^{1-\alpha} + \log v^\alpha}{2}\left(1+\sqrt{1-\frac{2(1+\rho_2)\log u^{1-\alpha} \log v^\alpha}{(\log u^{1-\alpha} + \log v^\alpha)^2}}\right)\right] \tag{3.4}$$

Without ambiguity of terminology, in remainder of this document, we will sometimes call the asymmetric extension simplified by reparametrization of the Schlather copula as the extended Schlather copula.

Proposition 3.4. *The extended Schlather copula is a max-stable copula.*

Proof. Let C_{θ,ρ_1,ρ_2} the extended Schlather copula.
Let $n \geq 0$. Then

$$C_{\theta,\rho_1,\rho_2}(u^n,v^n) = \exp\left[\frac{\log u^{n\alpha} + \log v^{n(1-\alpha)}}{2}\left(1+\sqrt{1-\frac{2(1+\rho_1)\log u^{n\alpha} \log v^{n(1-\alpha)}}{(\log u^{n\alpha} + \log v^{n(1-\alpha)})^2}}\right)\right.$$
$$\left. + \frac{\log u^{n(1-\alpha)} + \log v^{n\alpha}}{2}\left(1+\sqrt{1-\frac{2(1+\rho_2)\log u^{n(1-\alpha)} \log v^{n\alpha}}{(\log u^{n(1-\alpha)} + \log v^{n\alpha})^2}}\right)\right]$$
$$C_{\theta,\rho_1,\rho_2}(u^n,v^n) = \exp\left[n\times\frac{(\log u^\alpha + \log v^{1-\alpha})}{2}\left(1+\sqrt{1-\frac{2(1+\rho_1)\log u^\alpha \log v^{1-\alpha}}{(\log u^\alpha + \log v^{1-\alpha})^2}}\right)\right.$$
$$\left. + n\times\frac{(\log u^{1-\alpha} + \log v^\alpha)}{2}\left(1+\sqrt{1-\frac{2(1+\rho_2)\log u^{1-\alpha} \log v^\alpha}{(\log u^{1-\alpha} + \log v^\alpha)^2}}\right)\right]$$
$$C_{\theta,\rho_1,\rho_2}(u^n,v^n) = (C_{\theta,\rho_1,\rho_2}(u,v))^n \tag{3.5}$$

From (3.5), we conclude that the extended Schlather copula is max-stable □

Remark 3.5. i) Unlike the Schlather copula, which never achieves independence, this new copula, which is not only an asymmetric extension of the Schlather copula, can capture independence between two variables if $h \longrightarrow +\infty$.

ii For $h \longrightarrow 0$, *i.e.*, $\alpha = \beta = 1/2$ and $\rho_1 = \rho_2$, then the extended Schlather copula reduces to the standard Schlather copula.

Remark 3.6. The parameters ρ_1 and ρ_2 respectively control the local dependence intensity in the two asymmetric components of the copula. When $\rho_1 = \rho_2$, the expression becomes symmetric in u and v, and the copula itself is symmetric. In contrast, if $\rho_1 \neq \rho_2$, asymmetry is introduced: for instance, when $\rho_1 > \rho_2$, the dependence structure leans toward the variable u, whereas when $\rho_2 > \rho_1$, the asymmetry shifts toward v. This asymmetric behavior is illustrated and quantified in the density contour plots and through sensitivity analyses in Figures 2 and 3.

Figure 1 shows us the scatter plot of the copula and its contour for a certain number of parameters.

FIGURE 1. Surface and contour plot of the extended Schlather copula with $\theta = 3$, $\rho_1 = 0.9$, and $\rho_2 = 0.7$. The contour levels indicate value of the copula $C_{\theta,\rho_1,\rho_2}(u, v)$.

The outline of this copula shows that it does indeed capture a positive dependency. This is essential for extreme copulas.

3.1. **Extended Schlather copula density.** It should be noted that even the density of bivariate copulas derived from max-stable processes such as Brown-Resnik, Schlather, extremal-t and others are sometimes incalculable in view of their complex expressions. As a result, these classes of copulas are less widely used in the literature, to the detriment of max-stable processes, which also sometimes suffer from flexibility in modeling.
One of the aims of this paper was also to be able to give an explicit expression for the copula density that we constructed in (3.1). The closed expression of the density c_{θ,ρ_1,ρ_2} is given in the following proposition.

Proposition 3.7. *The density associated with the extended Schlather copula is defined by:*

$$c_{\theta,\rho_1,\rho_2}(u, v) = \frac{C_{\theta,\rho_1,\rho_2}(u, v)}{uv} \left[A(u, v) \times B(u, v) + v \times \frac{\partial B(u, v)}{\partial v} \right] \tag{3.6}$$

Where:

$$
\left\{
\begin{aligned}
\alpha &= \frac{1}{1+\exp(-h/\theta)} \qquad \textit{and} \qquad \beta = 1-\alpha \\
C_{\theta,\rho_1,\rho_2}(u,v) &= \exp\left[\frac{\log u^{\alpha}+\log v^{1-\alpha}}{2}\left(1+\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}\right)\right. \\
&\quad \left. + \frac{\log u^{\beta}+\log v^{1-\beta}}{2}\left(1+\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}\right)\right] \\
B(u,v) &= \frac{\alpha}{2}\left(1+\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}\right) + \frac{\log u^{\alpha}+\log v^{1-\alpha}}{2} \\
&\quad \times \frac{-2\alpha(1+\rho_1)\log v^{1-\alpha}(\log u^{\alpha}+\log v^{1-\alpha}) + 4\alpha(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{2(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}} \\
&\quad + \frac{\beta}{2}\left(1+\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}\right) + \frac{\log u^{\beta}+\log v^{1-\beta}}{2} \\
&\quad \times \frac{-2\beta(1+\rho_2)\log v^{1-\beta}(\log u^{\beta}+\log v^{1-\beta}) + 4\beta(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{2(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}}
\end{aligned}
\right. \tag{3.7}
$$

$$
\left\{
\begin{aligned}
A(u,v) &= \Bigg[\frac{1-\alpha}{2}\left(1+\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}\right)+\frac{\log u^{\alpha}+\log v^{1-\alpha}}{2}\\
&\quad\times\frac{-2(1-\alpha)(1+\rho_1)\log u^{\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4(1-\alpha)(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{2(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}}\\
&\quad+\frac{1-\beta}{2}\left(1+\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}\right)+\frac{\log u^{\beta}+\log v^{1-\beta}}{2}\\
&\quad\times\frac{-2(1-\beta)(1+\rho_2)\log u^{\beta}(\log u^{\beta}+\log v^{1-\beta})+4(1-\beta)(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{2(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}}\Bigg]\\
&\textit{and}\\
\frac{\partial B(u,v)}{\partial v} &= \frac{\alpha}{2}.\frac{-2(1-\alpha)(1+\rho_1)\log u^{\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4(1-\alpha)(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{2v(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}}\\
&\quad+\frac{1-\alpha}{2v}.\frac{-2\alpha(1+\rho_1)\log v^{1-\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4\alpha(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{2(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}}\\
&\quad+\frac{\log u^{\alpha}+\log v^{1-\alpha}}{2}\times\frac{N_1^{'}(u,v)\times D_1(u,v)-N_1(u,v)\times D_1^{'}(u,v)}{(D_1(u,v))^2}\\
&\quad+\frac{\beta}{2}.\frac{-2(1-\beta)(1+\rho_2)\log u^{\beta}(\log u^{\beta}+\log v^{1-\beta})+4(1-\beta)(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{2v(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}}\\
&\quad+\frac{1-\beta}{2v}.\frac{-2\beta(1+\rho_2)\log v^{1-\beta}(\log u^{\beta}+\log v^{1-\beta})+4\beta(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{2(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}}\\
&\quad+\frac{\log u^{\beta}+\log v^{1-\beta}}{2}\times\frac{N_2^{'}(u,v)\times D_2(u,v)-N_2(u,v)\times D_2^{'}(u,v)}{(D_2(u,v))^2}\\
&\textit{with}\\
N_1(u,v) &= -2\alpha(1+\rho_1)\log v^{1-\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4\alpha(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}\\
N_2(u,v) &= -2\beta(1+\rho_2)\log v^{1-\beta}(\log u^{\beta}+\log v^{1-\beta})+4\beta(1+\rho_2)\log u^{\beta}\log v^{1-\beta}\\
D_1(u,v) &= 2(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}\\
D_2(u,v) &= 2(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}\\
N_1^{'}(u,v) &= \frac{1}{v}\Big[-2\alpha(1+\rho_1)(1-\alpha)(\log u^{\alpha}+\log v^{1-\alpha})^2-2\alpha(1+\rho_1)(1-\alpha)\log v^{1-\alpha}\\
&\quad+4\alpha(1+\rho_1)(1-\alpha)\log u^{\alpha}\Big]\\
N_2^{'}(u,v) &= \frac{1}{v}\Big[-2\beta(1+\rho_2)(1-\beta)(\log v^{\beta}+\log v^{1-\beta})^2-2\beta(1+\rho_2)(1-\beta)\log v^{1-\beta}\\
&\quad+4\beta(1+\rho_2)(1-\alpha)\log u^{\beta}\Big]
\end{aligned}
\right.
\tag{3.8}
$$

$$\begin{cases} D_1^{'}(u,v) = 2\Bigg[3(1-\alpha)\times\frac{1}{v}(\log u^{\alpha}+\log v^{1-\alpha})^2\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}} \\ \quad +(\log u^{\alpha}+\log v^{1-\alpha})^3 \\ \quad \times\frac{-2(1-\alpha)(1+\rho_1)\log u^{\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4(1-\alpha)(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{2v(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}}\Bigg] \\ D_2^{'}(u,v) = 2\Bigg[3(1-\beta)\times\frac{1}{v}(\log u^{\beta}+\log v^{1-\beta})^2\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}} \\ \quad +(\log u^{\beta}+\log v^{1-\beta})^3 \\ \quad \times\frac{-2(1-\beta)(1+\rho_2)\log u^{\beta}(\log u^{\beta}+\log v^{1-\beta})+4(1-\beta)(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{2v(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}}\Bigg] \end{cases} \tag{3.9}$$

Proof. Given C_{θ,ρ_1,ρ_2} the extended Schlather copula,
the derivative with respect to the variable u is given by:

$$\frac{\partial C_{\theta,\rho_1,\rho_2}(u,v)}{\partial u} = \frac{1}{u}\Bigg[C_{\theta,\rho_1,\rho_2}(u,v)\times B(u,v)\Bigg] \tag{3.10}$$

with

$$\begin{aligned} B(u,v) = {} & \frac{\alpha}{2}\left(1+\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}\right)+\frac{\log u^{\alpha}+\log v^{1-\alpha}}{2} \\ & \times\frac{-2\alpha(1+\rho_1)\log v^{1-\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4\alpha(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{2(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}} \\ & +\frac{\beta}{2}\left(1+\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}\right)+\frac{\log u^{\beta}+\log v^{1-\beta}}{2} \\ & \times\frac{-2\beta(1+\rho_2)\log v^{1-\beta}(\log u^{\beta}+\log v^{1-\beta})+4\beta(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{2(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}} \end{aligned} \tag{3.11}$$

By differentiating equation (3.10) with respect to the variable v, we obtain the expression of the density, which has form:

$$\begin{aligned} \frac{\partial^2 C_{\theta,\rho_1,\rho_2}(u,v)}{\partial u\partial v} &= \frac{1}{u}\Bigg[\frac{\partial C_{\theta,\rho_1,\rho_2}(u,v)}{\partial v}\times B(u,v)+C(u,v)\times\frac{\partial B(u,v)}{\partial v}\Bigg] \\ &= \frac{C_{\theta,\rho_1,\rho_2}(u,v)}{uv}\Bigg[A(u,v)\times B(u,v)+v.\frac{\partial B(u,v)}{\partial v}\Bigg] \end{aligned} \tag{3.12}$$

where

$$
\begin{aligned}
A(u,v) = \Bigg[& \frac{1-\alpha}{2}\left(1+\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}\right)+\frac{\log u^{\alpha}+\log v^{1-\alpha}}{2} \\
& \times \frac{-2(1-\alpha)(1+\rho_1)\log u^{\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4(1-\alpha)(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{2(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}} \\
& +\frac{1-\beta}{2}\left(1+\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}\right)+\frac{\log u^{\beta}+\log v^{1-\beta}}{2} \\
& \times \frac{-2(1-\beta)(1+\rho_2)\log u^{\beta}(\log u^{\beta}+\log v^{1-\beta})+4(1-\beta)(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{2(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}}\Bigg]
\end{aligned}
\tag{3.13}
$$

It remains now to compute $\dfrac{\partial B(u,v)}{\partial v}$, the derivative of B with respect to the variable v.

Let,

$$N_1(u,v) = -2\alpha(1+\rho_1)\log v^{1-\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4\alpha(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha} \tag{3.14}$$

$$N_2(u,v) = -2\beta(1+\rho_2)\log v^{1-\beta}(\log u^{\beta}+\log v^{1-\beta})+4\beta(1+\rho_2)\log u^{\beta}\log v^{1-\beta} \tag{3.15}$$

$$D_1(u,v) = 2(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}} \tag{3.16}$$

$$D_2(u,v) = 2(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}} \tag{3.17}$$

The derivatives with respect to v are given by:

$$
\begin{aligned}
N_1^{'}(u,v) = \frac{1}{v}\Big[& -2\alpha(1+\rho_1)(1-\alpha)(\log u^{\alpha}+\log v^{1-\alpha})^2-2\alpha(1+\rho_1)(1-\alpha)\log v^{1-\alpha} \\
& +4\alpha(1+\rho_1)(1-\alpha)\log u^{\alpha}\Big]
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
N_2^{'}(u,v) = \frac{1}{v}\Big[& -2\beta(1+\rho_2)(1-\beta)(\log v^{\beta}+\log v^{1-\beta})^2-2\beta(1+\rho_2)(1-\beta)\log v^{1-\beta} \\
& +4\beta(1+\rho_2)(1-\alpha)\log u^{\beta}\Big]
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
D_1^{'}(u,v) = 2\Bigg[& 3(1-\alpha)\times\frac{1}{v}(\log u^{\alpha}+\log v^{1-\alpha})^2\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}} \\
& +(\log u^{\alpha}+\log v^{1-\alpha})^3 \\
& \times\frac{-2(1-\alpha)(1+\rho_1)\log u^{\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4(1-\alpha)(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{2v(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}}\Bigg]
\end{aligned}
\tag{3.20}
$$

$$D_2^{'}(u,v) = 2\Bigg[3(1-\beta)\times\frac{1}{v}(\log u^{\beta}+\log v^{1-\beta})^2\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}} \\ +(\log u^{\beta}+\log v^{1-\beta})^3 \\ \times\frac{-2(1-\beta)(1+\rho_2)\log u^{\beta}(\log u^{\beta}+\log v^{1-\beta})+4(1-\beta)(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{2v(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}}\Bigg] \tag{3.21}$$

And therefore, we can have the expression of the derivative of $B(u,v)$ with respect to v such that:

$$\frac{\partial B(u,v)}{\partial v} = \frac{\alpha}{2}.\frac{-2(1-\alpha)(1+\rho_1)\log u^{\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4(1-\alpha)(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{2v(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}} \\ +\frac{1-\alpha}{2v}.\frac{-2\alpha(1+\rho_1)\log v^{1-\alpha}(\log u^{\alpha}+\log v^{1-\alpha})+4\alpha(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{2(\log u^{\alpha}+\log v^{1-\alpha})^3\sqrt{1-\frac{2(1+\rho_1)\log u^{\alpha}\log v^{1-\alpha}}{(\log u^{\alpha}+\log v^{1-\alpha})^2}}} \\ +\frac{\log u^{\alpha}+\log v^{1-\alpha}}{2}\times\frac{N_1^{'}(u,v)\times D_1(u,v)-N_1(u,v)\times D_1^{'}(u,v)}{(D_1(u,v))^2} \\ +\frac{\beta}{2}.\frac{-2(1-\beta)(1+\rho_2)\log u^{\beta}(\log u^{\beta}+\log v^{1-\beta})+4(1-\beta)(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{2v(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}} \\ +\frac{1-\beta}{2v}.\frac{-2\beta(1+\rho_2)\log v^{1-\beta}(\log u^{\beta}+\log v^{1-\beta})+4\beta(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{2(\log u^{\beta}+\log v^{1-\beta})^3\sqrt{1-\frac{2(1+\rho_2)\log u^{\beta}\log v^{1-\beta}}{(\log u^{\beta}+\log v^{1-\beta})^2}}} \\ +\frac{\log u^{\beta}+\log v^{1-\beta}}{2}\times\frac{N_2^{'}(u,v)\times D_2(u,v)-N_2(u,v)\times D_2^{'}(u,v)}{(D_2(u,v))^2} \tag{3.22}$$

Thus, we now obtain the explicit expression of the density of the extended Schlather copula.

Remark 3.8. Despite the complexity of the density expression, we emphasize that it was derived manually and implemented without difficulty. The evaluation of the likelihood function remains numerically stable across the tested parameter ranges, and the optimization of the log-likelihood is both rapid and efficient in practice. These computational properties make the model suitable for practical inference.

□

Figures 2 and 3 show graphical representations of the density and contours of the copula C_{θ,ρ_1,ρ_2} defined above for different fixed parameters.

3.2. **Dependence properties of the extended Schlather Copula.** In this section, we will focus specifically on the Spearman's Rho and the upper and lower tail dependence coefficients associated with the copula given in definition 2.4.

FIGURE 2. Surface and contour plot of the extended Schlather density with $\theta = 1.5$, $\rho_1 = 0.5$, and $\rho_2 = 0.7$. The contour levels correspond to values of the copula density $c_{\theta,\rho_1,\rho_2}(u, v)$.

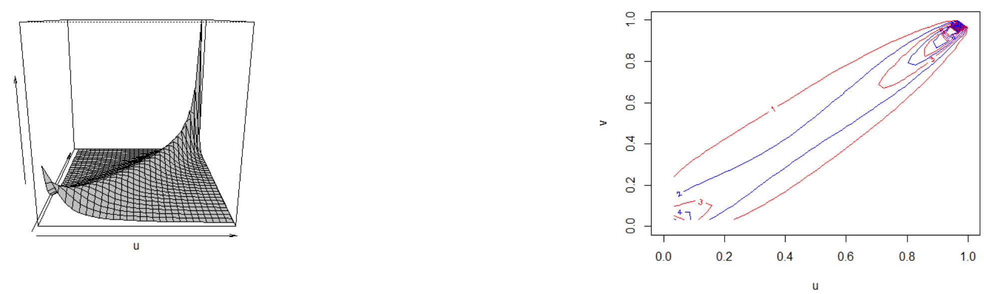


FIGURE 3. Surface and contour plot of the extended Schlather density with $\theta = 3$, $\rho_1 = 0.9$, and $\rho_2 = 0.7$. The contour levels correspond to values of the copula density $c_{\theta,\rho_1,\rho_2}(u, v)$.

3.2.1. *Spearman's Rho.* Although the analytical expression of Spearman's Rho is complex to find in our case, a numerical alternative allows us to see the evolution of this measure as a function of the dependence parameters ρ_1 and ρ_2. The results are given in the following table (1):

3.2.2. *Upper and lower tail dependency.* The following proposition tells us about the tail dependencies of the extended Schlather.

Proposition 3.9. *Let C_{θ,ρ_1,ρ_2} be the extended Schlather copula. Then the upper and lower tail dependence are given respectively by:*

$$\lambda_U(C_{\theta,\rho_1,\rho_2}) = 2 - \frac{2 + \sqrt{1 - 2(1+\rho_1)\frac{\exp(-h/\theta)}{(1+\exp(-h/\theta))^2}} + \sqrt{1 - 2(1+\rho_2)\frac{\exp(-h/\theta)}{(1+\exp(-h/\theta))^2}}}{2}. \tag{3.23}$$

and

$$\lambda_L(C_{\theta,\rho_1,\rho_2}) = 0 \tag{3.24}$$

TABLE 1. Table of values of ρ_S as a function of ρ_1 and ρ_2 for $\theta = 3$

$\rho_2 \diagdown \rho_1$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
0	0.306	0.326	0.347	0.369	0.392	0.418	0.445	0.476	0.511	0.551	0.603
0.1	0.326	0.346	0.367	0.389	0.413	0.439	0.467	0.497	0.532	0.574	0.626
0.2	0.347	0.367	0.388	0.411	0.435	0.461	0.489	0.520	0.556	0.598	0.651
0.3	0.369	0.389	0.411	0.434	0.458	0.485	0.513	0.545	0.581	0.624	0.677
0.4	0.392	0.413	0.435	0.458	0.483	0.510	0.539	0.571	0.607	0.651	0.705
0.5	0.418	0.439	0.461	0.485	0.510	0.537	0.566	0.599	0.636	0.680	0.735
0.6	0.445	0.467	0.489	0.513	0.539	0.566	0.596	0.629	0.667	0.712	0.768
0.7	0.476	0.497	0.520	0.545	0.571	0.599	0.629	0.663	0.702	0.747	0.805
0.8	0.511	0.532	0.556	0.581	0.607	0.636	0.667	0.702	0.741	0.788	0.846
0.9	0.551	0.574	0.598	0.624	0.651	0.680	0.712	0.747	0.788	0.836	0.896
0.99	0.603	0.626	0.651	0.677	0.705	0.735	0.768	0.805	0.846	0.896	0.959

Proof. From the preliminary results we know that:

$$\lambda_U = 2 - \lim_{u \to 1^-} \frac{1 - C_{\theta,\rho_1,\rho_2}(u,u)}{1-u} \qquad \textit{and} \qquad \lambda_L = \lim_{u \to 0^+} \frac{C_{\theta,\rho_1,\rho_2}(u,u)}{u}. \tag{3.25}$$

To calculate this limit we will use the L'Hôspital's rule which consists of deriving the numerator and denominator before applying the limit.

$$\begin{aligned} C_{\theta,\rho_1,\rho_2}(u,v) &= \exp\Bigg[\frac{\log u^\alpha + \log u^{1-\alpha}}{2}\left(1 + \sqrt{1 - \frac{2(1+\rho_1)\log u^\alpha \log u^{1-\alpha}}{(\log u^\alpha + \log u^{1-\alpha})^2}}\right) \\ &+ \frac{\log u^{1-\alpha} + \log v^\alpha}{2}\left(1 + \sqrt{1 - \frac{2(1+\rho_2)\log u^{1-\alpha}\log u^\alpha}{(\log u^{1-\alpha} + \log u^\alpha)^2}}\right)\Bigg] \\ &= \exp\left[\frac{\log u}{2}\left(2 + \sqrt{1 - 2\alpha(1-\alpha)(1+\rho_1)} + \sqrt{1 - 2\alpha(1-\alpha)(1+\rho_2)}\right)\right] \end{aligned} \tag{3.26}$$

Let:

$$K = \frac{2 + \sqrt{1 - 2\alpha(1-\alpha)(1+\rho_1)} + \sqrt{1 - 2\alpha(1-\alpha)(1+\rho_2)}}{2} \tag{3.27}$$

The equation (3.26) become:

$$C_{\theta,\rho_1,\rho_2}(u,v) = \exp\big(K \log u\big) = \exp\big(\log u^K\big) = u^K \tag{3.28}$$

Let's move on to the partial derivative with respect to u.

$$\frac{\partial C_{\theta,\rho_1,\rho_2}(u,u)}{\partial u} = K u^{K-1}$$

Then,

$$\lambda_U(C_{\theta,\rho_1,\rho_2}) = 2 - \lim_{u\to 1^-} \frac{1 - C_{\theta,\rho_1,\rho_2}(u,u)}{1-u} = 2 - K \tag{3.29}$$

Posing $\alpha = \dfrac{1}{1+\exp(-h/\theta)}$, we finally obtain:

$$\lambda_U(C_{\theta,\rho_1,\rho_2}) = 2 - \frac{2 + \sqrt{1 - 2(1+\rho_1)\frac{\exp(-h/\theta)}{(1+\exp(-h/\theta))^2}} + \sqrt{1 - 2(1+\rho_2)\frac{\exp(-h/\theta)}{(1+\exp(-h/\theta))^2}}}{2}. \tag{3.30}$$

For the lower tail dependence index $\lambda_L(C_{\theta,\rho_1,\rho_2})$, If we follow the same procedure, we obtain $\lambda_L(C_{\theta,\rho_1,\rho_2}) = 0$ □

4. Application of the extended Schlather copula to extreme temperatures in Switzerland

4.1. **Data.** The data we are going to process correspond to the monthly maximum temperatures (in degrees Celsius) collected in five Swiss cities from 01 January 1961 to 01 June 2006. The cities (variables) concerned are : Neuchâtel, Davos, Basel, Locarno and Arosa.
In order to gain a better understanding of the statistical characteristics of the extreme monthly temperatures observed in various Swiss cities, we present below a descriptive table. This table (2) provides the usual indicators (mean, standard deviation, minimum, maximum, skewness and kurtosis), making it possible to assess the variability and shape of the marginal distributions.

Table 2. Statistical data for different cities

Cities	N	**Mean**	**SD**	**Min**	**Max**	**Skewness**	**Kurtosis**
NEUCHATEL	546	20.56502	7.829619	3.4	36.20	−0.0900549	1.801686
DAVOS	546	16.03901	6.995068	1.5	29.00	−0.116341	1.788227
BASEL	546	22.6804	7.525348	3.10	38.60	−0.1451495	1.94121
LOCARNO	546	22.66392	6.326255	8.40	37.90	−0.1824485	1.990936
AROSA	546	17.03095	6.697573	1.90	30.60	−0.05906555	1.859813

The data being the maximum temperatures, the theory of extreme values tells us that these data are the realization of variables which follow a GEV law. So each of the 5 variables will be fitted according to the GEV law given below:

$$H_{\xi,\mu,\sigma}(x) = \exp\left(-\left(1+\xi\times\frac{(x-\mu)}{\sigma}\right)^{-\frac{1}{\xi}}\right), \xi\neq 0,\ 1+\xi\times\frac{(x-\mu)}{\sigma} > 0. \tag{4.1}$$

where μ, σ and ξ are the location, scale and shape parameters respectively.
The following table provides the results of the GEV parameters estimated for the first 20 variable sites in our database.

TABLE 3. Adjustment of GEV parameters at each site

Cities	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\xi}$
$NEUCHATEL$	18.4227	8.251605	−0.4451845
$DAVOS$	14.24092	7.471961	−0.477337
$BASEL$	20.4869632	7.8438649	−0.4109988
$LOCARNO$	20.6542880	6.4948662	−0.3618752
$AROSA$	15.1246574	6.9945673	−0.4225987

Figure 4 illustrates the dependency structure between pairs of cities using scatter plots, after transforming the margins using the function *pgev(.)*. This visualization provides a visual indication of the nature and intensity of the dependencies before the copulas are adjusted.

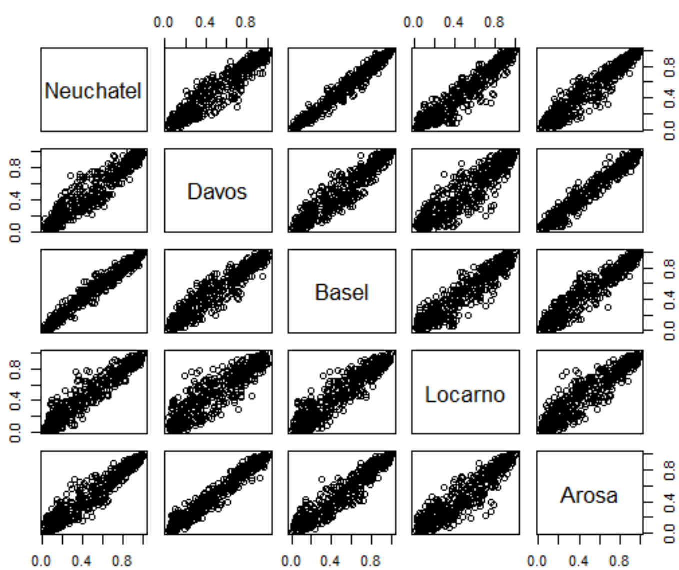


FIGURE 4. Scatter plots of points clouds by city pair for a sample size of n=546 points.

4.2. **Copula Model Selection.** After adjusting the margins, we model the dependence using three copulas: Hüsler-Reiss, Schlather and our extended copula. The parameters have been estimated by composite likelihood on data transformed to unit margins (uniform probability). Model selection was carried out using the Composite Likelihood Akaike Information Criterion (CLAIC), adapted to this estimation framework. The results are summarized in Table (4) below:

The extended Schlather copula showed the best performance according to CLAIC (smallest CLAIC value), and was therefore retained for the rest of the

TABLE 4. Comparison of the performance of copula models

Copule	Parameters	Loglik	CLAIC
Hüsler-Reiss	$\alpha = 3.871463$	5408.9	-10817.8
Schlather	$\rho = 0.9171902$	5512.0	-11024.1
Extended Schlather	$\rho_1 = 0.9431713$ $\rho_2 = 0.9295419$ $\theta = 8.7559888$	**5564.0**	**-11128.27**

analysis.
Figure 5 shows the scatter plots obtained from the selected Schlather copula, applied to 10 pairs of cities, each pair corresponding to a simulated sample. This visualization allows a visual comparison of the simulated dependency structure with that of the real data, previously uniformly transformed. The aim is to assess whether the Schlather copula accurately captures the dependencies between the extreme variables of the different cities.

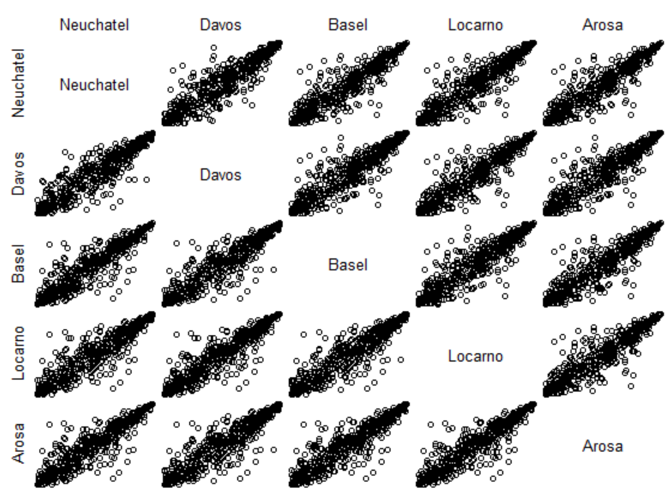


FIGURE 5. Point clouds obtained from the extended Schlather copula applied to 10 pairs of cities.

These simulated samples are comparable to those obtained from uniformly transformed data. The aim is to check visually whether the dependency structure of the extremes is well captured by the copula. As a note, it shows that the dependency structure between cities is reproduced well overall, although some pairs show slightly different concentrations, particularly in the tails.
Again, to visually assess the ability of our copula model to reproduce the observed dependencies between variables, we present in figure 6 the QQ-plots for

the ten city pairs. These plots compare the empirical quantiles of the transformed data with those simulated from each copula model. A good fit is shown by an alignment close to the diagonal, indicating that the model correctly captures the dependency structure.

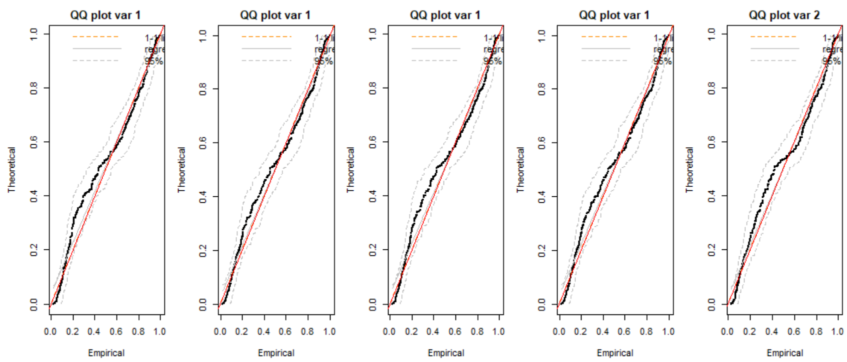


FIGURE 6. QQ-plots comparing the empirical quantiles and those simulated using the extended Schlather copula, for the first 5 pairs of cities. The proximity of the points to the diagonal indicates a good fit between the simulated distributions and the observed data.

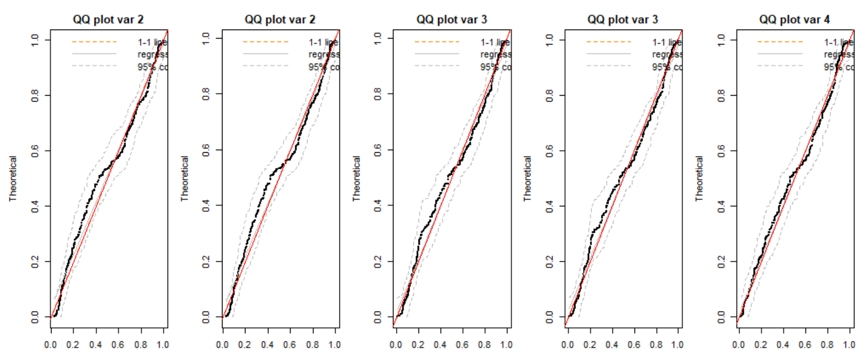
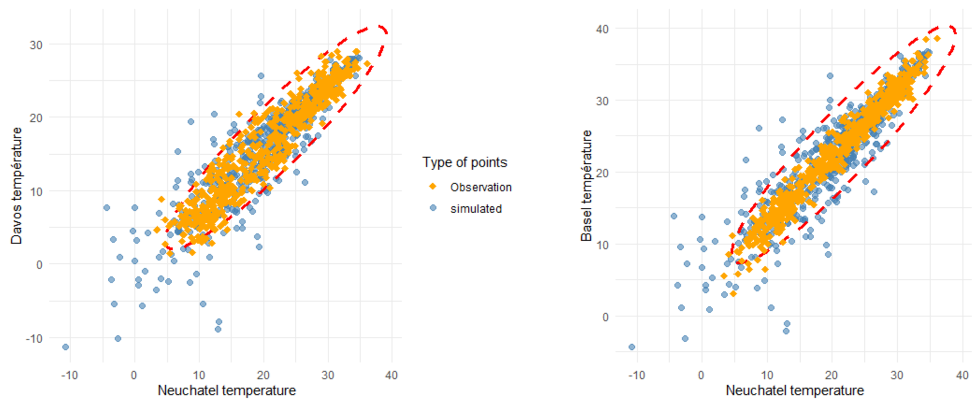


FIGURE 7. QQ-plots comparing the empirical quantiles and those simulated using extended Schlather copula, for the other 5 pairs of cities. The proximity of the points to the diagonal indicates a good fit between the simulated distributions and the observed data.

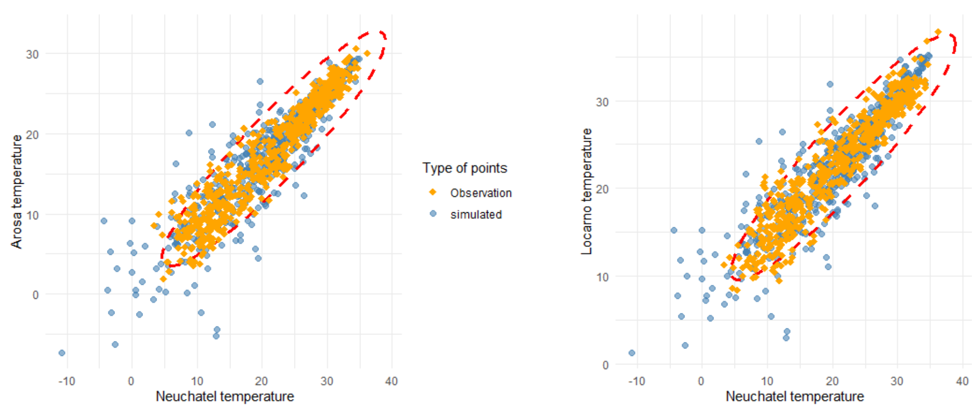
Figures 8 and 9 below compare the extreme temperatures observed (x-axis) with those simulated using our extended copula (y-axis), for four pairs of cities including Neuchâtel. Each graph is accompanied by an ellipse representing a 95% confidence band, constructed from the simulations. These visualization allow us to assess the copula's ability to faithfully reproduce the dependency structures present in the real data.

For all the pairs analysed, the simulated points are generally well contained within the confidence ellipses, confirming the statistical validity of the model.



(a): Neuchâtel - Davos (b): Neuchâtel - Basel

Figure 8. Comparison between observed temperatures and those simulated by the copula, for two pairs of cities including Neuchâtel. The ellipse represents a 95% confidence band.



(c): Neuchâtel -Arosa (d): Neuchâtel - Locarno

Figure 9. Comparison between observed temperatures and those simulated by the copula, for two other pairs of cities including Neuchâtel. The upper extremes are well captured by the modelling.

The upper extremes, which are of major interest in climate risk analysis, are particularly well fitted, demonstrating the ability of the extended copula to capture co-dependencies in the tails. On the other hand, slight deviations can be observed in the central zones, which is still common and acceptable in asymmetric dependency models.
All these results confirm the capacity of our new asymmetric copula in a context of modelling environmental extremes, to the detriment of certain copulas which are no less important in this field.

5. Conclusion

In this paper, we have introduced a simplified version of the extended Schlather copula, obtained by reparametrization, and demonstrated that it retains the max-stable structure while offering an explicit density favorable to maximum likelihood estimation. After studying its analytical properties and illustrating its behavior with graphical simulations, we applied it to a real maximum temperature dataset. A comparison based on CLAIC and log-likelihood criteria, against two competing copulas, highlighted the superiority of the extended Schlather copula for representing the spatial dependence of extremes. There are many prospects for the future: Deepen the theoretical analysis and statistical robustness of this copula (sensitivity studies, high-dimensional behavior). Extend the methodology to other max-stable copulas (Brown-Resnick, Smith, Extremal-t) and their asymmetric variants, in order to offer a complete range of models adapted to extreme phenomena. Integrate climatic or topographical covariates to increase the model's predictive capacity and operational relevance. We hope that this contribution will enrich the literature on extreme spatial copulas and encourage their adoption in the study of climate and environmental risks.

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