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GENERALIZED FIXED POINT THEOREMS IN FUZZY NORMED LINEAR SPACE

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ABSTRACT. This article presents a comprehensive study of fixed point theorems in the context of complete Fuzzy normed linear spaces. By extending and generalizing the framework of fixed point theory, we establish some theorems for three and four self-mappings that offer deeper insights into the interplay of fuzzy structures. Several definitions are introduced to facilitate the formulation and understanding of these theorems, along with illustrative examples to validate and demonstrate the applicability of the results.

Keywords. Fuzzy normed linear space, Fixed points, R-weakly commuting maps
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1. INTRODUCTION

The introduction of fuzzy set theory by Zadeh [1] marked a pivotal advancement in mathematical analysis, providing a robust framework for representing imprecise and uncertain information often encountered in real-world scenarios. Since its emergence, fuzzy set theory has inspired extensive research across multiple areas of mathematical analysis, particularly in the development of fuzzy normed linear spaces, fuzzy metric spaces, and fuzzy operators.

Pioneering efforts by Felbin [7] and Katsaras [9] laid the groundwork for fuzzy normed and fuzzy topological vector spaces, respectively. These initial constructs were subsequently enhanced by contributions such as those of Cheng and Mordeson [6], who studied fuzzy linear operators within normed spaces, and Kaleva and Seikkala [9], who investigated the structural properties of fuzzy metric spaces. Together, these foundational studies established a basis for exploring analytical and topological behaviors under uncertainty.

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Further developments in the field were led by Bag and Samanta [2, 3, 4], whose research focused on finite-dimensional fuzzy normed linear spaces and intuitionistic fuzzy normed linear spaces. Their exploration of fuzzy bounded linear operators and their introduction of intuitionistic norms provided more refined approaches for handling degrees of truth and falsity, thereby deepening the theoretical underpinnings of fuzzy analysis.

Building upon these frameworks, Das and Saha [10] examined fixed point theorems in complete fuzzy normed linear spaces-a topic of considerable relevance due to its broad applications in nonlinear analysis and fuzzy optimization. Related work by Shrivastava et al [8] further advanced fixed point theory by establishing results for R-weakly commuting maps in fuzzy metric spaces.

In this article, we present an in-depth study of fixed point theorems within the setting of complete fuzzy normed linear spaces. By extending and generalizing the framework provided in previous works such as [10], we derive new fixed point results involving three and four self-mappings under specific conditions. These results offer broader applicability and deeper insights into the structure of fuzzy normed spaces. Additionally, we provide two illustrative examples and two corollaries to support and clarify the main theorems presented. The fixed point results developed in this work have promising applications in computer science, especially where uncertainty, approximation, and iterative methods are central. In artificial intelligence, fixed point principles help ensure the stability of fuzzy logic systems and learning algorithms. In fuzzy control systems—used in robotics and autonomous technologies—they provide a framework for consistent decision-making under imprecise data. Additionally, in data clustering and pattern recognition, especially with noisy or ambiguous inputs, these results support the design of adaptive, robust algorithms. Thus, our findings offer valuable theoretical tools with practical significance across various computational fields.

2. Preliminaries

In this section, we recall the basic concepts and results required for the present work.

2.1. Continuous t-Norms.

Definition 2.1. [8] A continuous t-norm is a binary operation $\oplus : [0,1]\times[0,1] \to [0,1]$ that satisfies the following properties:

(1) **Commutativity and associativity:** For all $x, y, z \in [0,1]$, it holds that $x \oplus y = y \oplus x$ and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.
(2) **Continuity:** The operation $\oplus$ is continuous on $[0,1] \times [0,1]$.
(3) **Neutral element:** For every $x \in [0,1]$, we have $x \oplus 1 = x$.
(4) **Monotonicity:** If $x \leq u$ and $y \leq v$, then $x \oplus y \leq u \oplus v$, for all $x, y, u, v \in [0,1]$.

2.2. Fuzzy Sets and Fuzzy Norms.

Definition 2.2 (Fuzzy Set [10])**.** A fuzzy set $\mathcal{N}$ in $\mathcal{X}$ is a mapping $\mathcal{N} : \mathcal{X} \to [0,1]$. For $x \in \mathcal{X}$, $\mathcal{N}(x)$ is called the grade of membership of x.

Definition 2.3 (Fuzzy Norm [10])**.** Let $\mathcal{X}$ be a linear space over $\mathbb{R}$ and $\oplus$ be a continuous t-norm . A fuzzy subset $\mathcal{N}$ on $\mathcal{X} \times [0, \infty)$ is a fuzzy norm on $\mathcal{X}$ if for $a, b \in \mathcal{X}$ and $t, s > 0$, the following conditions hold:

(1) $0 \le \mathcal{N}(a, t) \le 1$.
(2) $\mathcal{N}(a, t) = 1$ for $t > 0$ if and only if $a = 0$.
(3) $\mathcal{N}(ca, t) = \mathcal{N}\left(a, \frac{t}{|c|}\right)$ for $c \neq 0$, $t > 0$.
(4) $\mathcal{N}(a, s) \oplus \mathcal{N}(b, t) \le \mathcal{N}(a + b, s + t)$ for $s, t \in \mathbb{R}$.
(5) $\mathcal{N}(a, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} \mathcal{N}(a, t) = 1$.

Further, $(\mathcal{X}, \mathcal{N}, \oplus)$ is a fuzzy normed linear space (FNLS). In the following, we have discussed two examples of fuzzy norm linear space, which will be very helpful to solve the examples 3.2 and 3.4 of our main results.

Example 2.4. [2] Let $(\mathcal{X}, ||.,.||)$ be a real normed linear space and $\oplus$ be a continuous t-norm. Define,

$$\mathcal{N}(x, t) = \begin{cases} \frac{t}{t+||x||}, & \text{if } t > 0, \quad x \in \mathcal{X} \\ 0, & \text{if } t \le 0, \quad x \in \mathcal{X}. \end{cases}$$

Then $(\mathcal{X}, \mathcal{N}, \oplus)$ is a fuzzy normed linear space.

Example 2.5. [2] Let $(\mathcal{X}, ||.,.||)$ be a real normed linear space and $\oplus$ be a continuous t-norm. Define,

$$\mathcal{N}(x, t) = \begin{cases} 0, & \text{if } t \le ||x|| \\ 1, & \text{if } t \ge ||x||. \end{cases}$$

Then $(\mathcal{X}, \mathcal{N}, \oplus)$ is a fuzzy normed linear space.

Definition 2.6. [4] A sequence $\{x_n\}$ in a FNLS $(\mathcal{X}, \mathcal{N}, \oplus)$ is said to converge to $x \in \mathcal{X}$ if

$$\lim_{n\to\infty} \mathcal{N}(x_n - x, t) = 1.$$

Definition 2.7. [8] A sequence $\{x_n\}$ in a FNLS $(\mathcal{X}, \mathcal{N}, \oplus)$ is called a *Cauchy sequence* if

$$\lim_{n\to\infty} \mathcal{N}(a_{n+p} - a_n, t) = 1, \text{for all } t > 0 \text{ and } p = 1, 2, 3, \dots.$$

Definition 2.8. [8] A pair of self-maps (F, G) on a FNLS $(\mathcal{X}, \mathcal{N}, \oplus)$ is said to be *R-weakly commuting* if there exists a positive real number κ such that

$$\mathcal{N}(FGx - GFx, t) \ge \mathcal{N}(Fx - Gx, \tfrac{t}{\kappa}).$$

Lemma 2.9. [10] *In a FNLS $(\mathcal{X}, \mathcal{N}, \oplus)$, if for $x \in X$, $\mathcal{N}(x, kt) \ge \mathcal{N}(x, t)$, for every $t > 0$ and some $0 < k < 1$, then $x = 0$.*

Lemma 2.10. [10] *A sequence $\{a_n\}$ in a FNLS $(\mathcal{X}, \mathcal{N}, \oplus)$ satisfying*

$$\mathcal{N}(x_{n+1} - x_n, kt) \ge \mathcal{N}(x_n - x_{n-1}, t)$$

for some $k \in (0,1)$ and for all $t > 0$, is a Cauchy sequence.

3. Main Results

In our main results we first show a common fixed point theorem for three self mappings divided into two pairs of R-weakly compatible mappings in fuzzy complete normed linear space. Next in theorem 3.3 we show a common fixed point theorem for four self mappings divided into two pairs of R-weakly compatible mappings. This is a larger version of theorem 3.1, where the theorem is proved by taking three self mappings divide into two pairs of R-weakly compatible mappings in Fuzzy complete normed linear space. Finally, we give two examples which illustrate our main results.

Theorem 3.1. *Let P, Q and R be three self-mappings on a complete fuzzy normed linear space $(\mathcal{X}, \mathcal{N}, \oplus)$. Suppose the following conditions are satisfied:*

(a) $P(\mathcal{X}) \subseteq Q(\mathcal{X})$ *and* $R(\mathcal{X}) \subseteq Q(\mathcal{X})$,

(b) $\mathcal{N}(Px - Ry, kt) \geq \mathcal{N}(Qx - Qy, t)$ *for all* $x, y \in \mathcal{X}$, $t > 0$, *and some* $0 < k < 1$,

(c) *Either P or R is continuous,*

(d) *The pairs (P, Q) and (R, Q) are R-weakly commuting,*

(e) *If $x_n \to x$ and $y_n \to y$, then $\mathcal{N}(x_n - y_n, t) \to \mathcal{N}(x - y, t)$ for all $t > 0$.*

Then P, Q, and R have a unique common fixed point in $\mathcal{X}$.

Proof. Choose an arbitrary point $x_0 \in \mathcal{X}$. Since $P(\mathcal{X}) \subseteq Q(\mathcal{X})$, there exists $x_1 \in \mathcal{X}$ such that $y_1 = Px_0 = Qx_1$. Similarly, since $R(\mathcal{X}) \subseteq Q(\mathcal{X})$, there exists $x_2 \in \mathcal{X}$ such that $y_2 = Rx_1 = Qx_2$, $y_3 = Px_2 = Qx_3$, ... Continuing this process, we construct a sequence defined by:
$y_{n+1} = Px_n = Qx_{n+1}$ and $y_{n+2} = Rx_{n+1} = Qx_{n+2}$.

$$N(Qx_{n+1} - Qx_n, kt) = N(Px_n - Rx_{n-1}) \geq N(Qx_n - Qx_{n-1}, t)$$

By Lemma 2.10, it follows that $\{Qx_n\}$ is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete, there exists $z \in \mathcal{X}$ such that:

$$Px_n = Qx_{n+1} = Rx_n \to z.$$

As $P(\mathcal{X}) \subseteq Q(\mathcal{X})$, there exists $u \in \mathcal{X}$ such that $z = Qu$. Then $\mathcal{N}(Qu - z, t) = 1$ for all $t > 0$. Now consider:

$$\begin{aligned}
\mathcal{N}(z - Ru, t) &\geq \mathcal{N}(z - Qx_n, \tfrac{t}{2}) \oplus \mathcal{N}(Qx_n - Ru, \tfrac{t}{2}) \\
&= \mathcal{N}(z - Qx_n, \tfrac{t}{2}) \oplus \mathcal{N}(Px_{n-1} - Ru, \tfrac{t}{2}) \\
&\geq \mathcal{N}(z - Qx_n, \tfrac{t}{2}) \oplus \mathcal{N}(Qx_{n-1} - Qu, \tfrac{t}{2k}) \\
&\to 1 \oplus 1 = 1.
\end{aligned}$$

as $n \to \infty$. Therefore,

$$Qu = Ru = z$$

Since (R,Q) is R-weakly commuting, it follows that:

$$\mathcal{N}(RQu - QRu, t) \geq \mathcal{N}(Ru - Qu, \tfrac{t}{\kappa}) = 1 \quad \Rightarrow \quad RQu = QRu$$

Hence,

$$Rz = Qz$$

As $R(\mathcal{X}) \subseteq Q(\mathcal{X})$, there exists $v \in \mathcal{X}$ such that $z = Qv$, i.e., $\mathcal{N}(Qv - z, t) = 1$ for all $t > 0$. Consider:

$$\begin{aligned}
\mathcal{N}(z - Pv, t) &\geq \mathcal{N}(z - Qx_{n+2}, \tfrac{t}{2}) \oplus \mathcal{N}(Qx_{n+2} - Pv, \tfrac{t}{2}) \\
&= \mathcal{N}(z - Qx_{n+2}, \tfrac{t}{2}) \oplus \mathcal{N}(Rx_{n+1} - Pv, \tfrac{t}{2}) \\
&\geq \mathcal{N}(z - Qx_{n+2}, \tfrac{t}{2}) \oplus \mathcal{N}(Qx_{n+1} - Qv, \tfrac{t}{2k}) \\
&\to 1 \oplus 1 = 1 \quad as \quad n \to \infty.
\end{aligned}$$

Thus,

$$Qv = Pv = z.$$

Since (P,Q) is R-weakly commuting, we get:

$$\mathcal{N}(PQv - QPv, t) \geq \mathcal{N}(Pv - Qv, \tfrac{t}{\kappa}) = 1 \quad \Rightarrow \quad PQv = QPv$$

Therefore,

$$Pz = Qz$$

From earlier results, we now have:

$$Pz = Qz = Rz$$

Finally, using condition (b):

$$\begin{aligned}
\mathcal{N}(z - Rz, kt) &= \mathcal{N}(Pv - Rz, kt) \\
&\geq \mathcal{N}(Qv - Qz, t) = \mathcal{N}(z - Rz, t)
\end{aligned}$$

By Lemma 2.9, this implies $Rz = z$, and hence:

$$Pz = Qz = Rz = z$$

So z is a common fixed point of P, Q, and R. To prove uniqueness, suppose there exists another common fixed point $w \neq z$, such that $Pw = Qw = Rw = w$. Then,

$$\begin{aligned}
\mathcal{N}(z - w, kt) &= \mathcal{N}(Pz - Rw, kt) \\
&\geq \mathcal{N}(Qz - Qw, t) = \mathcal{N}(z - w, t)
\end{aligned}$$

Again, by Lemma 2.9, we conclude that $z = w$. Thus, the common fixed point is unique.

□

Example 3.2. Consider the fuzzy normed linear space $(\mathbb{R}, N, \oplus)$ derived from the classical normed space $(\mathbb{R}, \|\cdot\|)$, where the norm is given by the absolute value.

Define the self-mappings P, Q, and R on $\mathbb{R}$ as follows:

$$P(x) = 1, \quad Q(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}, \quad R(x) = 1.$$

These mappings satisfy the following conditions:

(a) The images of P and R are subsets of the image of Q, i.e., $P(\mathbb{R}) \subseteq Q(\mathbb{R})$ and $R(\mathbb{R}) \subseteq Q(\mathbb{R})$.
(b) For all $x, y \in \mathbb{R}$, $t > 0$, and for some $k \in (0,1)$, we have $\mathcal{N}(P(x) - R(y), kt) = 1 \geq \mathcal{N}(Qx - Qy, t)$.
(c) At least one of the mappings P or R is continuous.
(d) The pairs (P,Q) and (R,Q) are R-weakly commuting because the compositions commute in the sense that $PQ = QP$ and $RQ = QR$. Therefore $\mathcal{N}(PQx - QPx, t) \geq \mathcal{N}(Px - Qx, \frac{t}{\kappa})$ and $\mathcal{N}(RQx - QRx, t) \geq \mathcal{N}(Rx - Qx, \frac{t}{\kappa})$ holds successfully.
(e) If sequences $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively in $\mathbb{R}$, then $\mathcal{N}(x_n - y_n, t) \to \mathcal{N}(x - y, t)$ for every $t > 0$.

Since all the hypotheses of Theorem 3.1 are satisfied, it follows that $1 \in \mathbb{R}$ is the unique common fixed point of the mappings P, Q and R.

Theorem 3.3. *Let P, Q, T and S be self-mappings on a complete fuzzy normed linear space $(\mathcal{X}, \mathcal{N}, \oplus)$. Suppose the following conditions hold:*

(a) *The image of S is contained in the image of Q and the image of T is contained in the image of P, i.e., $S(\mathcal{X}) \subseteq Q(\mathcal{X})$ and $T(\mathcal{X}) \subseteq P(\mathcal{X})$.*
(b) *For all $x, y \in \mathcal{X}$, $t > 0$, and some constant $k \in (0,1)$, the inequality*
$$\mathcal{N}(Sx - Ty, kt) \geq \mathcal{N}(Px - Qy, t)$$
holds.
(c) *At least one of the mappings S or T is continuous.*
(d) *The pairs (S,P) and (T,Q) are R-weakly commuting.*
(e) *If sequences $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively in $\mathcal{X}$, then $\mathcal{N}(x_n - y_n, t) \to \mathcal{N}(x - y, t)$ for every $t > 0$.*

Then the mappings P, Q, T and S have a unique common fixed point in $\mathcal{X}$.

Proof. Let $x_0 \in \mathcal{X}$ be an arbitrary element. Then there exists $x_1 \in \mathcal{X}$ such that $y_1 = Sx_0 = Qx_1$ and $y_2 = Tx_1 = Px_2, \ldots$. Continuing this process, we construct sequences $\{x_n\}$ and $\{y_n\}$ such that

$$y_{n+1} = Sx_n = Qx_{n+1}, \quad y_{n+2} = Tx_{n+1} = Px_{n+2}, \quad \text{for } n = 0, 1, 2, \ldots \tag{3.1}$$

Now, consider:

$$\begin{aligned} \mathcal{N}(Qx_{n+1} - Qx_n, kt) = \mathcal{N}(y_{n+1} - y_n, kt) &= \mathcal{N}(Sx_n - Tx_{n-1}, kt) \\ &\geq \mathcal{N}(Px_n - Qx_{n-1}, t) \quad \text{(by condition (b))} \\ &= \mathcal{N}(y_n - y_{n-1}, t) \\ &= \mathcal{N}(Qx_n - Qx_{n-1}, t) \end{aligned}$$

Hence, by Lemma 2.10, the sequence $\{Qx_n\}$ is Cauchy in the complete fuzzy normed space $\mathcal{X}$. Therefore, there exists $z \in \mathcal{X}$ such that

$$y_{n+1} = Sx_n = Qx_{n+1} = Tx_{n+1} = Px_{n+2} \to z \quad \text{as } n \to \infty.$$

Since $S(\mathcal{X}) \subseteq Q(\mathcal{X})$, there exists $u_1 \in \mathcal{X}$ such that $z = Qu_1$, i.e., $\mathcal{N}(Qu_1 - z, t) = 1$ for all $t > 0$.

Now observe:

$$\begin{aligned}
\mathcal{N}(z - Tu_1, t) &\geq \mathcal{N}(z - Qx_n, \tfrac{t}{2}) \oplus \mathcal{N}(Qx_n - Tu_1, \tfrac{t}{2}) \\
&= \mathcal{N}(z - Qx_n, \tfrac{t}{2}) \oplus \mathcal{N}(Sx_{n-1} - Tu_1, \tfrac{t}{2}) \\
&\geq \mathcal{N}(z - Qx_n, \tfrac{t}{2}) \oplus \mathcal{N}(Px_{n-1} - Qu_1, \tfrac{t}{2k}) \quad \text{(by condition (b))} \\
&= \mathcal{N}(z - y_n, \tfrac{t}{2}) \oplus \mathcal{N}(y_{n-1} - Qu_1, \tfrac{t}{2k}) \\
&\to 1 \oplus 1 = 1 \quad \text{as } n \to \infty
\end{aligned}$$

Thus, we conclude that

$$Tu_1 = Qu_1 = z \tag{3.2}$$

Since the pair (T, Q) is R-weakly commuting, we have

$$\mathcal{N}(TQu_1 - QTu_1, t) \geq \mathcal{N}(Tu_1 - Qu_1, t/\kappa_1) = 1 \quad \text{for any } \kappa_1 > 0,$$

which implies $TQu_1 = QTu_1$, and hence

$$Tz = Qz \tag{3.3}$$

Since $T(\mathcal{X}) \subseteq P(\mathcal{X})$, and so, there exists $v_1 \in \mathcal{X}$ such that $z = Pv_1$, i.e., $\mathcal{N}(Pv_1 - z, t) = 1$ for all $t > 0$.

Next, we have:

$$\begin{aligned}
\mathcal{N}(z - Sv_1, t) &\geq \mathcal{N}(z - Px_{n+2}, \tfrac{t}{2}) \oplus \mathcal{N}(Px_{n+2} - Sv_1, \tfrac{t}{2}) \\
&= \mathcal{N}(z - Px_{n+2}, \tfrac{t}{2}) \oplus \mathcal{N}(Sv_1 - Tx_{n+1}, \tfrac{t}{2k}) \\
&\geq \mathcal{N}(z - Px_{n+2}, \tfrac{t}{2}) \oplus \mathcal{N}(Pv_1 - Qx_{n+1,} \tfrac{t}{2k}) \\
&= \mathcal{N}(z - Px_{n+2}, \tfrac{t}{2}) \oplus \mathcal{N}(Pv_1 - y_{n+1}, \tfrac{t}{2k}) \\
&\to 1 \oplus 1 = 1 \quad \text{as } n \to \infty
\end{aligned}$$

Thus,

$$Sv_1 = Pv_1 = z \tag{3.4}$$

Since (S, P) is R-weakly commuting, we get

$$\mathcal{N}(SPv_1 - PSv_1, t) \geq \mathcal{N}(Sv_1 - Pv_1, t/\kappa_1) = 1,$$

which implies $SPv_1 = PSv_1$, and therefore

$$Sz = Pz \tag{3.5}$$

Now consider:

$$\begin{aligned}
\mathcal{N}(z - Tz, kt) &= \mathcal{N}(Sv_1 - Tz, kt) \quad \text{(by equation (3.4))} \\
&\geq \mathcal{N}(Pv_1 - Qz, t) \\
&= \mathcal{N}(z - Tz, t) \quad \text{(by equation (3.4) and (3.3))}
\end{aligned}$$

By Lemma 2.9, it follows that $Tz = z$, and using equation (3.3), we obtain

$$Tz = Qz = z$$

Similarly,

$$\begin{aligned}\mathcal{N}(Sz - z, kt) &= \mathcal{N}(Sz - Tu_1, kt) \quad \text{(by equation 3.2)}\\ &\geq \mathcal{N}(Pz - Qu_1, t) \quad \text{(by condition (b))}\\ &= \mathcal{N}(Pz - z, t) = \mathcal{N}(Sz - z, t) \quad \text{(by equation (3.2) and equation (3.5))}\end{aligned}$$

Again by Lemma 2.9, we conclude that $Sz = z$, and thus, from equation (3.5),

$$Sz = Pz = z$$

Therefore, we have

$$Pz = Qz = Tz = Sz = z,$$

which shows that z is a common fixed point of P, Q, T, and S.

To prove uniqueness, suppose there exists another common fixed point $w \in \mathcal{X}$ such that

$$Pw = Qw = Tw = Sw = w.$$

Then,

$$\begin{aligned}\mathcal{N}(z - w, kt) &= \mathcal{N}(Sz - Tw, kt)\\ &= \mathcal{N}(Pz - Qw, t) \quad \text{(by condition (b))}\\ &= \mathcal{N}(z - w, t)\end{aligned}$$

By Lemma 2.9, it follows that $z = w$.

Hence, z is the unique common fixed point of the mappings P, Q, T and S.

$\square$

Example 3.4. Consider the fuzzy normed linear space $(\mathbb{R}, \mathcal{N}, \oplus)$, which is derived from the complete normed linear space $(\mathbb{R}, \|.\|)$ equipped with the usual absolute value norm.

Define four self-mappings P, Q, T and S on $\mathbb{R}$ as follows:

$$T(x) = 1, \quad Q(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q},\\ 0, & \text{if } x \notin \mathbb{Q},\end{cases}$$

$$S(x) = 1, \quad P(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q},\\ 0, & \text{if } x \notin \mathbb{Q}.\end{cases}$$

We observe the following:

(a) The range of T is contained in the range of P, and the range of S is contained in the range of Q;
(b) For all $x, y \in \mathbb{R}$, all $t > 0$, and some constant $0 < k < 1$, we have $\mathcal{N}(Sx - Ty, kt) = 1 \geq \mathcal{N}(Px - Qy, t)$;
(c) At least one of the mappings S or T is continuous;
(d) The compositions satisfy $SP = PS$ and $TQ = QT$. Therefore $\mathcal{N}(SPx - PSx, t) \geq \mathcal{N}(Sx - Px, \frac{t}{\kappa})$ and $\mathcal{N}(TQx - QTx, t) \geq \mathcal{N}(Tx - Qx, \frac{t}{\kappa})$ holds, which indicates that the pairs (S, P) and (T, Q) are R-weakly commuting;

(e) If $x_n \to x$ and $y_n \to y$, then $\mathcal{N}(x_n - y_n, t) \to \mathcal{N}(x - y, t)$ for every $t > 0$.

Thus, all the conditions of Theorem 3.3 are satisfied, and it follows that the number $1 \in \mathbb{R}$ is the unique common fixed point of the mappings P, Q, T and S.

Corollary 3.5. *Let P, Q and R be three self-maps on a complete fuzzy normed linear space $(\mathcal{X}, \mathcal{N}, \oplus)$ satisfying the following conditions:*

(a) *$P(\mathcal{X}) \subseteq Q(\mathcal{X})$ and $R(\mathcal{X}) \subseteq Q(\mathcal{X})$;*
(b) *For all $x, y \in \mathcal{X}$, all $t > 0$, and some $0 < k < 1$, we have $\mathcal{N}(Px - Ry, kt) \geq \mathcal{N}(Qx - Qy, t)$;*
(c) *At least one of the mappings P or R is continuous;*
(d) *The pairs (P, Q) and (R, Q) are R-weakly commuting;*
(e) *If $x_n \to x$ and $y_n \to y$, then $\mathcal{N}(x_n - y_n, t) \to \mathcal{N}(x - y, t)$ for every $t > 0$.*

Then the mappings P, Q and R have a unique common fixed point.

Proof. Setting $S = P = Q$ and $T = R$ in Theorem 3.3, then

(a) $S(\mathcal{X}) \subseteq Q(\mathcal{X})$ and $T(\mathcal{X}) \subseteq P(\mathcal{X})$ becomes $P(\mathcal{X}) \subseteq Q(\mathcal{X})$ and $R(\mathcal{X}) \subseteq Q(\mathcal{X})$;
(b) For all $x, y \in \mathcal{X}$, all $t > 0$, and some $0 < k < 1$, we have $\mathcal{N}(Px - Ry, kt) \geq \mathcal{N}(Qx - Qy, t)$;
Similarlly, the other conditions (c), (d) and (e) arrives easily.

Hence the proof is complete.

□

Corollary 3.6. [10] *Let A and B be two self-mappings on a complete fuzzy normed linear space $(\mathcal{X}, \mathcal{N}, \oplus)$ such that:*

(a) *The range of A is contained in the range of B;*
(b) *For all $x, y \in \mathcal{X}$, all $t > 0$, and some $0 < k < 1$, it holds that $\mathcal{N}(Ax - Ay, kt) \geq \mathcal{N}(Bx - By, t)$;*
(c) *At least one of the mappings A or B is continuous;*
(d) *The pair (A, B) is R-weakly commuting;*
(e) *If $x_n \to x$ and $y_n \to y$, then $\mathcal{N}(x_n - y_n, t) \to \mathcal{N}(x - y, t)$ for all $t > 0$.*

Then the mappings A and B have a unique common fixed point.

Proof. This follows immediately by choosing $T = S = A$ and $P = Q = B$ in Theorem 3.3

□

4. Conclusion

Building on this established foundation, fixed point theory in fuzzy normed linear spaces has emerged as a significant area of research, with important implications in nonlinear analysis and optimization. The work of Das and Saha [10], along with that of Shrivastava et al [8], has contributed meaningfully to this domain. In the present study, we have extended these developments by establishing new fixed point theorems for three as well as four self mappings in the context of complete fuzzy normed linear spaces. The theorems are supported by concrete conditions that enhance their scope and applicability. The inclusion of illustrative examples and corollaries further clarifies and validates the theoretical findings. These results not only deepen our understanding of fixed point behavior in fuzzy normed linear spaces but also open pathways for future research in more generalized or hybrid fuzzy frameworks.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1. Zadeh, L. A. (1965). Information and control. Fuzzy sets, 8(3), 338-353 https://doi.org/10.1016/S0019-9958(65)90241-X
2. Bag, T., and Samanta, S. K. (2003). Finite dimensional fuzzy normed linear spaces. Journal of Fuzzy Mathematics, 11(3), 687-706.
3. Bag, T., and Samanta, S. K. (2005). Fuzzy bounded linear operators. Fuzzy sets and Systems, 151(3),513-547. https://doi.org/10.1016/j.fss.2004.05.004
4. Bag, T., and Samanta, S. K. (2013). Finite dimensional intuitionistic fuzzy normed linear spaces. Ann. Fuzzy Math. Inform, 6(2), 45-57.
5. Bera, T., and Mahapatra, N. K. (2018). Neutrosophic soft normed linear spaces. Neutrosophic Sets and Systems. 23, 52-71.
6. Cheng, S. C., and Mordeson, J. N. (1992). Fuzzy linear operators and fuzzy normed linear spaces. In First International Conference on Fuzzy Theory and Technology Proceedings, Abstracts and Summaries (pp.193-197).
7. Felbin, C. (1992). Finite dimensional fuzzy normed linear space. Fuzzy sets and Systems, 48(2), 239-248. https://doi.org/10.1016/0165-0114(92)90338-5
8. Shrivastava, R., Vijyakumar, M., Kohli, M., Singhvi, J., & Bhardwaj, R. (2011). Some Common Fixed Points for R-Weakly Commuting Maps in Fuzzy Metric Spaces. Int. J. Contemp. Math. Sciences, 6(33), 1629-1634.
9. Kaleva, O. and Seikkala, S., (1984) On fuzzy metric spaces, Fuzzy sets and systems 12, 215-229. https://doi.org/10.1016/0165-0114(84)90069-1
10. Das, R. N., Saha L. M., (2015), On fixed points in complete fuzzy normed linear space, Annals of fuzzy mathematics and Informatics, 10(4), 515-524.
11. Issac, P., and Maya, K. (2012). On the Intuitionistic fuzzy normed linear space (RnA), Inter. J. Fuzzy Math. And Systems, 2(2), 95-110.
12. Katsaras, A. K. (1984). Fuzzy topological vector spaces II. Fuzzy sets and systems, 12(2), 143-154. https://doi.org/10.1016/0165-0114(84)90034-4

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