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NONLOCAL DISCRETE PROBLEM INVOLVING THE ANISOTROPIC $p(k)$-CAPILLARITY DIFFERENTIAL OPERATOR

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This paper is dedicated to the memory of Professor Charles E. Chidume

ABSTRACT. In this paper, we investigate the existence and multiplicity of solutions for a class of nonlocal discrete problems governed by a $p(k)$-capillarity differential operator in a T-dimensional Banach space. Our technical approach is based on a minimization method combined with adequate variational techniques, particularly the mountain pass theorem of A. Ambrosetti and P.H. Rabinowitz and the (S_+) mapping theory.

Keywords. Kirchhoff type equation, nonlocal discrete problem, $p(k)$-capillarity differential operator, boundary value problem, multiple solutions, mountain pass theorem, (S_+) mapping theory
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1. INTRODUCTION

In recent years, significant effort has been devoted to the study of discrete problems under various types of boundary conditions. A range of techniques has been utilized to tackle such problems, including fixed-point methods, the method of lower and upper solutions, and applications of the Brouwer degree (see [1, 3, 6, 7, 8, 9, 10, 12, 14] and the references therein). It is widely recognized that critical point theory and variational techniques constitute powerful tools for analyzing problems governed by differential equations.

The main goal of this paper is to investigate, under various conditions on the data, the existence and multiplicity of solutions for the following nonlocal discrete boundary value problem.

$$\begin{cases} M_1(L_1(u))\left(-\triangle\varphi(k-1,\triangle u(k-1)) + q(k)|u(k)|^{p(k)-2}u(k)\right) = \lambda M_2(L_2(u))f(k,u(k)) \\ +\mu M_3(L_3(u))g(k,u(k)), \quad k\in\mathbb{Z}[1,T], \\ \triangle u(0) = \triangle u(T) = 0, \end{cases} \tag{1.1}$$

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where $T \geq 2$ is a fixed positive integer, and $\mathbb{Z}[a,b]$ denotes the discrete set $\{a, a+1, \ldots, b-1, b\}$for integers $a < b$. The functions $f, g : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ are Carathéodory mappings, while $M_1, M_2, M_3 : [0, \infty) \to \mathbb{R}$ are continuous and non-decreasing. The constants $\lambda, \mu > 0$ are real parameters. Moreover, $p : \mathbb{Z}[0,T] \to [2, \infty)$, $q : \mathbb{Z}[1,T] \to [1, \infty)$are given functions. The forward difference operator is defined by $\triangle u(k) = u(k+1) - u(k)$, and the nonlinear term φ is given by $\varphi(k,v) = h(|v|^{p(k)})|v|^{p(k)-2}v$, where $h : \mathbb{R} \to \mathbb{R}$ is an increasing function. In addition, the terms L_1, L_2 and L_3 are defined by the following expressions

$$L_1(u) = \sum_{k=1}^{T+1} \frac{H(|\triangle u(k-1)|^{p(k-1)})}{p(k-1)} + \sum_{k=1}^{T} \frac{q(k)}{p(k)} |u(k)|^{p(k)}, \tag{1.2}$$

H is the primitive function of h, i.e., $H(t) = \displaystyle\int_0^t h(s)\, ds$,

$$L_2(u) = \sum_{k=1}^{T} F(k,u), \tag{1.3}$$

$$L_3(u) = \sum_{k=1}^{T} G(k,u), \tag{1.4}$$

where $F(k,u) = \displaystyle\int_0^u f(k,s)\, ds$ and $G(k,u) = \displaystyle\int_0^u g(k,s)\, ds$.

For the rest of this paper, we will use the following notations.

$$p^+ := \max_{k \in \mathbb{Z}[0,T]} p(k), \quad p^- := \min_{k \in \mathbb{Z}[0,T]} p(k),$$

$$q^+ := \max_{k \in \mathbb{Z}[1,T]} q(k), \quad q^- := \min_{k \in \mathbb{Z}[1,T]} q(k), \quad \overline{q} := \sum_{k=1}^{T} q(k).$$

Because difference equations naturally capture a wide range of discrete phenomena encountered in practice, in areas such as statistics, computer science, electrical circuit analysis, biology, neural networks, optimal control, economics, and finance; it is essential to investigate the existence of solutions for these equations and the associated discrete boundary-value problems. For a comprehensive survey of these topics, see, for example, the monograph [16] and the literature cited therein.

The $p(k)$-capillarity operator models capillary phenomena in heterogeneous media with spatially varying properties. It is applicable to non-Newtonian fluid dynamics, where local rheology depends on variable shear rates. In image processing and material science, it captures anisotropic diffusion and complex phase transitions. Biological systems, such as elastic tissues with nonuniform properties, also motivate its study. Overall, the $p(k)$-framework provides a flexible tool for describing nonstandard interface behaviors across diverse fields.

The discrete $p(k)$-Laplacian operator exhibits more intricate nonlinear behavior compared to the classical discrete p-Laplacian operator; notably, it lacks homogeneity.

The significance of studying problem (1.1) stems from the presence of a non-homogeneous differential operator

$$\triangle(\varphi(k-1,\triangle u(k-1))), \tag{1.5}$$

which incorporates variable exponents. This framework encompasses, in particular, the discrete analogs of the classical $p(k)$-Laplace or $p(k)$-mean curvature curvature operators, given respectively by

$$\triangle\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right),$$

and

$$\triangle\left[\left(1+\frac{|\triangle u(k-1)|^{p(k-1)}}{\sqrt{1+|\triangle u(k-1)|^{2p(k-1)}}}\right)|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right].$$

Thus, problem (1.1) can be regarded as the discrete counterpart of the following functional differential equation

$$\begin{cases} -\left(\left(1+\dfrac{|u'(t)|^{p(t)}}{\sqrt{1+|u'(t)|^{2p(t)}}}\right)|u'(t)|^{p(t)-2}u'(t)\right)' = \lambda f(t,u(t)), \quad t\in(0,1), \\ u'(0)=u'(1)=0, \end{cases} \tag{1.6}$$

By taking $h(t)=1$ and $M_1(t)=M_2(t)=M_3(t)=1$, problem (1.1) problem (1.1) simplifies to the classical discrete $p(k)$-Laplacian problem:

$$\begin{cases} -\triangle\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right)+q(k)|u(k)|^{p(k)-2}u(k)=\lambda f(k,u(k)) \\ \qquad\qquad +\mu g(k,u(k)), \quad k\in\mathbb{Z}[1,T], \\ \triangle u(0)=\triangle u(T)=0. \end{cases} \tag{1.7}$$

Thus, problem (1.1) can be viewed as a generalization of (1.7).

It is important to emphasize that, due to the presence of the nonconstant functions M_i for $i=1,2,3$ in the first equation of (1.1), the relation no longer holds pointwise, characterizing the problem as nonlocal. Such nonlocal problems arise naturally in modeling various physical and biological phenomena, where the state variable uu represents quantities influenced by their average behavior, such as the distribution of a population. Furthermore, problem (1.1) is closely related to the stationary form of the classical Kirchhoff equation (see [17]).

$$\rho\frac{\partial^2 u}{\partial t^2}=\left(T_0+\frac{Ea}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}, \tag{1.8}$$

where $\rho>0$ the mass per unit length, T_0 the base tension, E the Young modulus, a the area of the cross-section and L the initial length of the string.

Many authors have recently studied nonlocal discrete problems involving the Kirchhoff-type operator, yielding several important and interesting results. We refer the reader to [13, 18, 24, 29, 30]. The study of discrete anisotropic problems of the $p(k)$-Laplacian tpe was introduced by Mihăilescu et *al.* in [21] (see also [19]).

To the best of our knowledge, the first work addressing anisotropic discrete boundary value problems involving $p(\cdot)$-Kirchhoff-type operators was conducted by Yücedag (see [29]).

In [11], Fan initiated the study of a class of bi-nonlocal $p(x)$-Kirchhoff-type problems with Dirichlet boundary conditions, by considering the following Dirichlet boundary problem:

$$\begin{cases} -a\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx\right) \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = b\left(\int_\Omega F(x,u)\,dx\right) f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where a and b are continuous functions. Under suitable conditions on a, b and the Ambrosetti-Rabinowitz-type condition on the nonlinear term f, the author proved the existence of at least one nontrivial solution or the existence of infinitely many solutions to problem (1.9) using variational methods.

In [2], G. A. Afrouzi et *al.* investigate a class of bi-nonlocal problems with nonlinear Neumann boundary conditions and sublinear terms at infinity. Using (S_+) mapping theory and variational methods, the authors established the existence of at least two nontrivial weak solutions to the problem, provided that the parameters are sufficiently large.

In the paper [25], Ouaro and Sawadogo prove the existence and partial uniqueness of solutions for a nonlinear multivalued problem involving the p(u)-Laplacian under Robin boundary conditions.

In [8], Candito and D'Aguí studied the existence of at least three solutions for a discrete nonlinear Neumann boundary value problem involving the p-Laplacian. Their approach is based on three critical points theorems.

Guiro et *al.* [13] studied a class of discrete Neumann boundary value problems of the form

$$\begin{cases} -\triangle\left(a(k-1, \triangle u(k-1))\right) + |u(k)|^{p(k)-2}u(k) = f(k), \quad k \in \mathbb{Z}\left[1, T\right], \\ \triangle u(0) = \triangle u(T) = 0. \end{cases} \tag{1.10}$$

Using the direct method of the calculus of variations, they proved the existence and uniqueness of weak solution in a T-dimensional Hilbert space.

If we take $h(t) = 1$ and $M_2(t) = M_3(t) = 1$ in problem (1.1), we obtain the Neumann problem studied by Ourraoui and Ayoujil in [26]. The authors investigated the existence and multiplicity of solutions for an anisotropic discrete boundary value problem in a T-dimensional Hilbert space. Their approach is based on variational methods, particularly the three critical points theorem established by B. Ricceri ([27]).

In [15], the authors investigate the existence of entropy solutions for a class of nonlinear degenerate weighted elliptic problems involving the $p(x)$-Laplacian operator, subject to Dirichlet boundary conditions and L^1 data. Within the framework of weighted Sobolev spaces with variable exponents, their approach relies on a regularization technique together with the derivation of suitable a priori estimates.

In the paper by Moghadam and Avci [22], the existence of nontrivial solutions to an anisotropic discrete nonlinear problem involving the $p(k)$-Laplacian operator with Dirichlet boundary conditions was investigated using variational methods. The main technical tools employed are two local minimum theorems for differentiable functionals, as provided by Bonanno [5].

Barghouthe et *al.* in the paper [4] presented a specific case of problem (1.1) with Dirichlet-type boundary conditions. By setting $h(t) = 1 + \dfrac{t}{\sqrt{1+t^2}}$, $M_1(t) = M_2(t) = M_3(t) = 1$, and $\mu = 0$ in problem (1.1). Under appropriate conditions on the given data, the authors in [4] investigated the existence and multiplicity of solutions using critical point theory and variational methods.

Inspired by the works above, in this paper, we establish the existence of weak solutions for a nonlocal problem involving Neumann boundary conditions. Under certain conditions on the data f and g, we obtain a multiplicity result by applying the minimum principle associated with the mountain pass theorem. The main result of this paper complements and improves previous results for the superlinear case when Ambrosetti-Rabinowitz-type conditions are imposed on the nonlinearities.

The remainder of this paper is organized as follows. Section 2 presents mathematical preliminaries, including several important inequalities and the abstract critical point theorem. Section 3 is divided into two main parts. In the first part, we establish the necessary tools for proving the existence of nontrivial solutions under certain assumptions on the functions f and g defined in problem (1.1). In the second part, we prove the existence of at least two nontrivial weak solutions.

2. Preliminary

In this section, we first establish the variational framework associated with problem (1.1). We consider the following T-dimensional Banach space.

$$E = \{u : \mathbb{Z}[0, T+1] \longrightarrow \mathbb{R} \text{ such that } \triangle u(0) = \triangle u(T) = 0\},$$

equipped with the norm

$$\|u\| = \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} + \sum_{k=1}^{T} q(k)|u(k)|^{p^-} \right)^{1/p^-}.$$

On the space E we also introduce the norm

$$\|u\|_{p^+} = \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+} + \sum_{k=1}^{T} q(k)|u(k)|^{p^+} \right)^{1/p^+}$$

and the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \mu > 0 : \sum_{k=1}^{T+1} \left| \frac{\triangle u(k-1)}{\mu} \right|^{p(k-1)} + \sum_{k=1}^{T} q(k) \left| \frac{u(k)}{\mu} \right|^{p(k)} \leq 1 \right\}.$$

In the sequel, we will use the following inequality.

$$K\|u\|_{p^+} \leq \|u\| \leq 2^{\frac{p^+ - p^-}{p^+ p^-}} K\|u\|_{p^+}, \tag{2.1}$$

where $K := (\max\{T+1,\overline{q}\})^{\frac{p^+-p^-}{p^+p^-}}$. Indeed, by weighted Hölder's inequality (see [23]), we have

$$\begin{aligned}
&\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^-}\\
&\le\left(\sum_{k=1}^{T+1}1^{\frac{p^+}{p^+-p^-}}\right)^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T+1}\left(|\triangle u(k-1)|^{p^-}\right)^{\frac{p^+}{p^-}}\right)^{\frac{p^-}{p^+}}\\
&\le(T+1)^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^+}\right)^{\frac{p^-}{p^+}}.
\end{aligned}$$

Using the same arguments, one has

$$\sum_{k=1}^{T}q(k)\,|u(k)|^{p^-}\le\overline{q}^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T}q(k)|u(k)|^{p^+}\right)^{\frac{p^-}{p^+}}.$$

Then, we get from the above inequalities and the fact that $\dfrac{p^-}{p^+}\le 1$, the following.

$$\begin{aligned}
&\|u\|^{p^-}\le(\max\{T+1,\overline{q}\})^{\frac{p^+-p^-}{p^+}}\\
&\times\left(\left(\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^+}\right)^{\frac{p^-}{p^+}}+\left(\sum_{k=1}^{T}q(k)|u(k)|^{p^+}\right)^{\frac{p^-}{p^+}}\right)\\
&\le\ 2^{1-\frac{p^-}{p^+}}(\max\{T+1,\overline{q}\})^{\frac{p^+-p^-}{p^+}}\\
&\times\left(\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^+}+\sum_{k=1}^{T}q(k)|u(k)|^{p^+}\right)^{\frac{p^-}{p^+}}=2^{\frac{p^+-p^-}{p^+}}K^{p^-}\|u\|_{p^+}^{p^-}.
\end{aligned}$$

Therefore, $\|u\|\le 2^{\frac{p^+-p^-}{p^+p^-}}K\|u\|_{p^+}$.

On the other hand, we get by the fact that $\dfrac{p^+}{p^-}\ge 1$, the following.

$$\begin{aligned}
&\|u\|_{p^+}^{p^+}\le(\max\{T+1,\overline{q}\})^{\frac{p^--p^+}{p^-}}\\
&\times\left(\left(\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^-}\right)^{\frac{p^+}{p^-}}+\left(\sum_{k=1}^{T}q(k)|u(k)|^{p^-}\right)^{\frac{p^+}{p^-}}\right)\\
&\le\ (\max\{T+1,\overline{q}\})^{\frac{p^--p^+}{p^-}}\\
&\times\left(\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p^-}+\sum_{k=1}^{T}q(k)|u(k)|^{p^-}\right)^{\frac{p^+}{p^-}}=K^{-p^+}\|u\|^{p^+}.
\end{aligned}$$

Therefore, $K\|u\|_{p^+} \leq \|u\|$. Hence, we conclude that (2.1) holds.
Moreover, we will also make use of the following norm.

$$\|u\|_\infty := \max\{|u(k)| : k \in \mathbb{Z}[0, T+1]\}, \text{ for all } u \in E.$$

Note that for any $u \in E$, and for every $k \in \mathbb{Z}[1, T]$, one has

$$q^- |u(k)|^{p^-} \leq \sum_{k=1}^{T} q(k)\, |u(k)|^{p^-},$$

which implies that

$$\|u\|_\infty \leq (q^-)^{-\frac{1}{p^-}} \|u\|. \tag{2.2}$$

Since E is of finite dimensional, therefore there exist two constants $0 < \theta_1 < \theta_2$ such that

$$\theta_1 \|u\|_{p(\cdot)} \leq \|u\| \leq \theta_2 \|u\|_{p(\cdot)}. \tag{2.3}$$

Now, let $\phi : E \to \mathbb{R}$ be defined by

$$\phi(u) = \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} q(k) |u(k)|^{p(k)}. \tag{2.4}$$

If $u \in E$, then the following properties hold.

$$\|u\|_{p(\cdot)} < 1 \;\Rightarrow\; \|u\|_{p(\cdot)}^{p^+} \leq \phi(u) \leq \|u\|_{p(\cdot)}^{p^-}, \tag{2.5}$$

$$\|u\|_{p(\cdot)} > 1 \;\Rightarrow\; \|u\|_{p(\cdot)}^{p^-} \leq \phi(u) \leq \|u\|_{p(\cdot)}^{p^+}. \tag{2.6}$$

$$\|u_n - u\|_{p(\cdot)} \to 0 \;\Rightarrow\; \phi(u_n - u) \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

To state the main result of this paper, we assume the following:
(M_1) There exist two positive constants m_0, m_1 such that

$$m_0 \leq M_1(t) \leq m_1, \quad \forall\, t \geq 0.$$

(M_2) There exist two positive constants $m_2, m_3 > 0$ such that

$$m_2 \leq M_2(t) \leq m_3, \quad \forall\, t \geq 0.$$

(M_3) There exist two positive constants $m_4, m_5 > 0$ such that

$$m_4 \leq M_3(t) \leq m_5, \quad \forall\, t \geq 0.$$

(H_1) $h : [0, \infty) \to \mathbb{R}$ is an increasing continuous function and there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \leq h(t) \leq c_2, \quad \forall\, t \geq 0.$$

(H_2) There exists a constant $c > 0$ such that

$$\left(h(|\xi|^{p(k)})|\xi|^{p(k)-2}\xi - h(|\eta|^{p(k)})|\eta|^{p(k)-2}\eta\right)(\xi - \eta) \geq c|\xi - \eta|^{p(k)}.$$

(F_1) $f : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that there exists a positive constant C_1 with

$$|f(k, t)| \leq C_1(1 + |t|^{p(k)-1}), \text{ for all } (k, t) \in \mathbb{Z}[1, T] \times \mathbb{R}.$$

(G_1) $g : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that there exists a positive constant C_2 with

$$|g(k,t)| \leq C_2(1 + |t|^{p(k)-1}), \text{ for all } (k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}.$$

Let C be a positive fixed constant. We say that a C^1-function $\gamma : \mathbb{R} \to [0,\infty)$ verifies the property (Γ) if and only if

$$\gamma(t) \leq C|t|^p \quad \forall\, t \in \mathbb{R}, \tag{Γ}$$

where p is either p^- or p^+.

We consider four functions $K_i,\ i = 1,2,3,4$, satisfying property (Γ) such that For $i = 1,2,3,4$, the functions K_i satisfying

$$K_i(t) \leq \begin{cases} C|t|^{p^+} \text{ for } \ i = 1,3, \\ \\ C|t|^{p^-} \text{ for } \ i = 2,4. \end{cases}$$

We also introduce the following assumptions on the behavior of F and G at the origin and at infinity.

(F_2) $\limsup\limits_{t\to 0} \dfrac{F(k,t)}{K_1(t)} \leq 0$ uniformly for all $k \in \mathbb{Z}[1,T]$;

(F_3) $\limsup\limits_{t\to \infty} \dfrac{F(k,t)}{K_2(t)} \leq 0$ uniformly for all $k \in \mathbb{Z}[1,T]$;

(F_4) $F(k_0, d) > 0$ for some $k_0 \in \mathbb{Z}[1,T]$, $d > 0$;

(G_2) $\limsup\limits_{t\to 0} \dfrac{G(k,t)}{K_3(t)} \leq 0$ uniformly for all $k \in \mathbb{Z}[1,T]$;

(G_3) $\limsup\limits_{t\to \infty} \dfrac{G(k,t)}{K_4(t)} \leq 0$ uniformly for all $k \in \mathbb{Z}[1,T]$;

(G_4) $G(k_0, d) > 0$ for some $k_0 \in \mathbb{Z}[1,T]$, $d > 0$.

As an illustration of functions K_i satisfying property (Γ), we present the following example:

$$K_i(t) := \begin{cases} \dfrac{C}{3}|t|^{p^+} \text{ for } \ i = 1,3, \\ \\ \dfrac{C}{3}|t|^{p^-} \text{ for } \ i = 2,4. \end{cases}$$

Let $w : \mathbb{R} \to \mathbb{R}$ be a C^1-function satisfying $\lim\limits_{|t|\to 0} w(t) = \lim\limits_{|t|\to\infty} w(t) = 0$. Defining $F(x,t) = G(x,t) = \dfrac{C}{3} w(t)|t|^p$, where $p = p^+$ or $p = p^-$ and $w(t) = e^{-|t|} sin(t)$, we observe that the conditions (F_1)-(F_4) and (G_1)-(G_4) are fulfilled.

Furthermore, setting $M_i = 1$ and $h(t) = 1$ the assumptions (H_1)–(H_2) and (M_i) are also verified.
Additionally, it can be observed that the function $h(t) = 1 + \dfrac{t}{\sqrt{1+t^2}}$ satisfies hypotheses (H_1)-(H_2). In this framework, problem (1.1) under assumptions (H_1)-(H_2) can be interpreted as a generalized capillarity problem.
Next, we introduce the following quotients

$$S_{p^-,E} = \inf_{u\in E\setminus\{0\}} \frac{\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} + \sum_{k=1}^{T} q(k)|u(k)|^{p^-}}{\sum_{k=1}^{T} |u(k)|^{p^-}}$$

and

$$S_{p^+,E} = \inf_{u\in E\setminus\{0\}} \frac{\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+} + \sum_{k=1}^{T} q(k)|u(k)|^{p^+}}{\sum_{k=1}^{T} |u(k)|^{p^+}}.$$

In the sequel, we consider some definitions and lemmas which will be used later.

Definition 2.1. An element $u \in E$ is a critical point of the functional $J : E \to \mathbb{R}$ if

$$\langle J'(u), v\rangle = 0, \text{ for all } v \in E.$$

Definition 2.2. Let E be a Banach space. Let $J \in C^1(E, \mathbb{R})$. For any sequence $\{u_n\} \subset E$, if $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence, then we say that J satisfies the Palais–Smale condition ((PS) condition for short).

Definition 2.3. We say that a sequence $\{u_n\} \subset E$ is said to satisfy the $(PS)_c$ condition if

$$J(u_n) \to c \in \mathbb{R} \text{ and } J'(u_n) \to 0 \text{ as } n \to \infty.$$

Theorem 2.4. *([20]). If the functional $J : E \to \mathbb{R}$ is weakly lower semicontinuous and coercive, i.e. $\lim_{\|x\|\to\infty} J(x) = \infty$, then there exists $x_0 \in E$ such that $\inf_{x\in E} J(x) = J(x_0)$. Moreover, if J has bounded linear Gâteaux derivative on E, then x_0 is also a critical point of J, i.e. $J'(x_0) = 0$.*

Theorem 2.5. *([1]. Let $J \in C^1(E, \mathbb{R})$ satisfy the (PS) condition. Suppose that*

(1) *$J(0) = 0$;*
(2) *there exists $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$;*
(3) *there exists $u_1 \in E$ with $\|u_1\| > \rho$ such that $J(u_1) < \alpha$.*

Then J has a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$c := \inf_{g\in\Gamma} \max_{u\in g([0,1])} J(u),$$

where $\Gamma = \{g \in C(E,\mathbb{R})/g(0) = 0, g(1) = u_1\}$.

Let us define the functionals $\Phi, \Psi, I : E \to \mathbb{R}$ as follows.

$$\Phi(u) := \widehat{M_1}(L_1(u)), \tag{2.8}$$

$$\Psi(u) := \widehat{M_2}(L_2(u)), \tag{2.9}$$

$$I(u) := \widehat{M_3}(L_3(u)), \tag{2.10}$$

where $\widehat{M_i}(t) = \displaystyle\int_0^t M_i(s)\,ds,\ i = 1,2,3.$

The energy functional corresponding to problem (1.1) is defined as $J_{\lambda,\mu} : E \to \mathbb{R}$,

$$J_{\lambda,\mu}(u) = \Phi(u) - \lambda\Psi(u) - \mu I(u), \text{ for all } u \in E \tag{2.11}$$

It is easy to verify that Φ, Ψ and I are three functionals of class $C^1(E,\mathbb{R})$ whose Gâteaux derivatives at the point $u \in E$ are given by

$$\langle \Phi'(u), v\rangle = M_1(L_1(u))\Bigg(\sum_{k=1}^{T+1} \varphi(k-1, \triangle u(k-1))\triangle v(k-1) + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2}u(k)v(k)\Bigg), \tag{2.12}$$

$$\langle \Psi'(u), v\rangle = M_2(L_2(u)) \sum_{k=1}^{T} f(k, u(k))v(k), \tag{2.13}$$

$$\langle I'(u), v\rangle = M_3(L_3(u)) \sum_{k=1}^{T} g(k, u(k))v(k), \tag{2.14}$$

for all $u, v \in E$.
We observe that

$$\left\langle J'_{\lambda,\mu}(u), v\right\rangle = \langle \Phi'(u), v\rangle - \lambda \langle \Psi'(u), v\rangle - \mu \langle I'(u), v\rangle .$$

Definition 2.6. We say that $u \in E$ is a weak solution of the problem (1.1) if

$$M_1(L_1(u)) \left(\sum_{k=1}^{T+1} \varphi(k-1, \triangle u(k-1))\triangle v(k-1) + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2}u(k)v(k)\right)$$
$$= \lambda M_2(L_2(u)) \sum_{k=1}^{T} f(k, u(k))v(k) + \mu M_3(L_3(u)) \sum_{k=1}^{T} g(k, u(k))v(k),$$

for any $v \in E$.

Note that $J_{\lambda,\mu} \in C^1(E,\mathbb{R})$ with

$$\begin{aligned}&\left\langle J'_{\lambda,\mu}(u), v\right\rangle = \\ &M_1(L_1(u))\left(\sum_{k=1}^{T+1}\varphi(k-1,\triangle u(k-1))\triangle v(k-1)+\sum_{k=1}^{T}q(k)|u(k)|^{p(k)-2}u(k)v(k)\right)\\ &-\lambda M_2(L_2(u))\sum_{k=1}^{T}f(k,u(k))v(k)-\mu M_3(L_3(u))\sum_{k=1}^{T}g(k,u(k))v(k)\end{aligned}$$

for all $u, v \in E$.
Since $\triangle u(0) = \triangle u(T) = 0$, one has

$$\begin{aligned}&M_1(L_1(u))\left(\sum_{k=1}^{T+1}\varphi(k-1,\triangle u(k-1))\triangle v(k-1)+\sum_{k=1}^{T}q(k)|u(k)|^{p(k)-2}u(k)v(k)\right)\\ &=\sum_{k=1}^{T}M_1(L_1(u))\left(-\triangle(\varphi(k-1,\triangle u(k-1)))+q(k)|u(k)|^{p(k)-2}u(k)\right)v(k),\end{aligned}$$

then,

$$\begin{aligned}&\left\langle J'_{\lambda,\mu}(u), v\right\rangle = \sum_{k=1}^{T}M_1(L_1(u))\left(-\triangle(\varphi(k-1,\triangle u(k-1)))+q(k)|u(k)|^{p(k)-2}u(k)\right)v(k)\\ &-\lambda\sum_{k=1}^{T}M_2(L_2(u))f(k,u(k))v(k)-\mu\sum_{k=1}^{T}M_3(L_3(u))g(k,u(k))v(k).\end{aligned}$$

Thus, the critical points of $J_{\lambda,\mu}$ in E are the solutions of the problem (1.1).
Now, we recall some auxiliary results to be used in the next section.

Lemma 2.7. *Let $u \in E$ with $\|u\| > 1$. Then,*

$$\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p(k-1)}+\sum_{k=1}^{T}q(k)|u(k)|^{p(k)} \geq \|u\|^{p^-}-(1+q^+)T-1.$$

Proof. Let $u \in E$ be fixed. By a similar argument as in [13], we define

$$\beta_k := \begin{cases}p^+ \text{ if } |\triangle u(k-1)| \leq 1\\ p^- \text{ if } |\triangle u(k-1)| > 1\end{cases} \quad\text{and}\quad \delta_k := \begin{cases}p^+ \text{ if } |u(k)| \leq 1\\ p^- \text{ if } |u(k)| > 1,\end{cases}$$

for each $k \in \mathbb{Z}[1,T]$.
For $u \in E$ with $\|u\| > 1$, one has

$$\begin{aligned}
&\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)} \\
&\geq \sum_{k=}^{T+1} |\triangle u(k-1)|^{p^-} - \sum_{k=1,\beta_k=p^+}^{T+1} \Big(|\triangle u(k-1)|^{p^-} - |\triangle u(k-1)|^{p^+}\Big) \\
&+ \sum_{k=1}^{T} q(k)|u(k)|^{p^-} - q^+ \sum_{k=1,\delta_k=p^+}^{T} \Big(|u(k)|^{p^-} - |u(k)|^{p^+}\Big) \\
&\geq \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} - (T+1) + \sum_{k=1}^{T} q(k)|u(k)|^{p^-} - q^+T \\
&= \|u\|^{p^-} - (1+q^+)T - 1.
\end{aligned}$$

□

3. Existence of at least two nontrivial weak solutions

In this section, under the assumptions of Theorems 2.4 and 2.5, we establish the existence of at least two nontrivial weak solutions to problem (1.1).

We state the following result.

Theorem 3.1. *Assume that* (M_1)-(M_3), (H_1)-(H_2), (F_1)-(F_4) *and* (G_1)-(G_4) *hold. Then, there exist* $\lambda^*, \mu* > 0$ *such that for all* $\lambda > \lambda^*$ *and* $\mu > \mu*$, *the problem* (1.1) *has at least two nontrivial weak solutions.*

For the proof of Theorem 3.1, we first establish the following lemmas.

Lemma 3.2. *The functional* L_1 *given by relation* (1.2) *is weakly lower semicontinuous.*

Proof. Since H is a convex and nondecreasing function and $t \to |t|^{p(k)}$ is convex, then L_1 is convex.
Thus, it is enough to show that L_1 is lower semicontinuous. For this, we fix $u \in E$

and $\epsilon > 0$. Since L_1 is convex, we deduce that for any $v \in E$,

$$
\begin{aligned}
L_1(v) &\geq L_1(u) + \langle L_1'(u), v-u\rangle \\
&\geq L_1(u) - c_2 \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)-1} |\triangle v(k-1) - \triangle u(k-1)| \\
&- \sum_{k=1}^{T} q(k) |u(k)|^{p(k)-1} |v(k)-u(k)| \\
&\geq L_1(u) - c_2 \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{\frac{p^-(p(k-1)-1)}{p^- -1}} \right)^{\frac{p^- -1}{p^-}} \left(\sum_{k=1}^{T+1} |\Delta v(k-1) - \Delta u(k-1)|^{p^-} \right)^{\frac{1}{p^-}} \\
&- \left(\sum_{k=1}^{T} q(k) |u(k)|^{\frac{p^-(p(k)-1)}{p^- -1}} \right)^{\frac{p^- -1}{p^-}} \left(\sum_{k=1}^{T} q(k) \, |v(k-1) - u(k)|^{p^-} \right)^{\frac{1}{p^-}} \\
&\geq L_1(u) - \overline{K} \left\{ \left(\sum_{k=1}^{T+1} |\triangle v(k-1) - \triangle u(k)|^{p^-} \right)^{\frac{1}{p^-}} + \left(\sum_{k=1}^{T} q(k) \, |v(k-1) - u(k)|^{p^-} \right)^{\frac{1}{p^-}} \right\},
\end{aligned}
$$

where $\overline{K} = \max(K_1, K_2)$ with

$$
K_1 = c_2 \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{\frac{p^-(p(k-1)-1)}{p^- -1}} \right)^{\frac{p^- -1}{p^-}} \quad \text{and} \quad K_2 = \left(\sum_{k=1}^{T} q(k) |u(k)|^{\frac{p^-(p(k-1)}{p^- -1}} \right)^{\frac{p^- -1}{p^-}}.
$$

Taking the above inequalities and as $\dfrac{1}{p^-} \leq 1$, we get

$$
\begin{aligned}
L_1(v) &\geq L_1(u) - 2^{\frac{p^- -1}{p^-}} \max(K_1, K_2) \|v-u\| \\
&\geq L_1(u) - \epsilon,
\end{aligned}
$$

for all $v \in E$ with $\|v-u\| < \delta = \dfrac{\epsilon}{2^{\frac{p^- -1}{p^-}} \max(K_1, K_2)}$. □

Lemma 3.3. *The functional Φ given by relation (2.8) is weakly lower semicontinuous.*

Proof. Let $\{u_n\}$ be a sequence that converges weakly in E. By using Lemma 3.2, we have

$$
\liminf_{n\to\infty} L_1(u_n) \geq L_1(u),
$$

and from the continuity and monotonicity of the function $t \to \widehat{M_1}(t)$, we get

$$
\liminf_{n\to\infty} \widehat{M_1}(L_1(u_n)) \geq \widehat{M_1}(\liminf_{n\to\infty} L_1(u_n)) \geq \widehat{M_1}(L_1(u)) = \Phi(u).
$$

So, the functional Φ is weakly lower semicontinuous in E. □

Lemma 3.4. *The functionals L_2 and L_3 given by relations (1.3) and (1.4) are weakly continuous.*

Proof. Let $\{u_n\}$ be a sequence converging weakly to u in E. We want to show that

$$\lim_{n\to\infty}\sum_{k=1}^{T}F(k,u_n(k))=\sum_{k=1}^{T}F(k,u(k))$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{T}G(k,u_n(k))=\sum_{k=1}^{T}G(k,u(k)).$$

By using (F_1), one has

$$\begin{aligned}&\left|\sum_{k=1}^{T}[F(k,u_n(k))-F(k,u(k))]\right|\\&\leq\sum_{k=1}^{T}|f(k,u+\theta_n(u_n-u)|\,|u_n-u|\\&\leq C_1\sum_{k=1}^{T}\left(1+|u+\theta_n(u_n-u)|^{p^*-1}\right)|u_n-u|\\&\leq\overline{C_1}\left[T^{1-\frac{1}{p^-}}+\|u+\theta_n(u_n-u)\|^{p^*-1}\right]\|u_n-u\|,\end{aligned}$$

where $0\leq\theta_n(k)\leq 1$ and

$$\beta^{p^*-1}:=\begin{cases}\beta^{p^+-1} & \text{if } \beta>1,\\ \beta^{p^--1} & \text{if } 0<\beta<1.\end{cases}$$

Moreover, the sequence $\{\|u+\theta_n(u_n-u)\|\}$ is bounded in E. Therefore, by passing to the limit, we obtain the desired result for F.
Similarly, it can be shown that

$$\lim_{n\to\infty}\sum_{k=1}^{T}G(k,u_n(k))=\sum_{k=1}^{T}G(k,u(k)).$$

□

Lemma 3.5. *The functional $J_{\lambda,\mu}$ is weakly lower semicontinuous.*

Proof. For all $u\in E$, we have $J_{\lambda,\mu}(u)=\Phi(u)-\lambda\Psi(u)-\mu I(u)$. By Lemma 3.3, the functional Φ is weakly lower semicontinuous.
It remains to prove that Ψ and I are weakly lower semicontinuous.
Let $\{u_n\}$ be a sequence that converges weakly in E. By the weak continuity of the functional L_2, it follows

$$\liminf_{n\to\infty}L_2(u_n)=L_2(u).$$

Combining this with the continuity of the function $t\to\widehat{M_2}(t)$, we get

$$\liminf_{n\to\infty}\Psi(u_n)=\liminf_{n\to\infty}\widehat{M_2}(L_2(u_n))=\widehat{M_2}(\liminf_{n\to\infty}L_2(u_n))=\widehat{M_2}(L_2(u))=\Psi(u).$$

Similarly, we have that I is weakly lower semicontinuous.
Thus, we deduce that $J_{\lambda,\mu}$ is weakly lower semicontinuous in E.

$\square$

Lemma 3.6. *The functional $J_{\lambda,\mu}$ is coercive and bounded from below.*

Proof. Fix $\lambda, \mu > 0$. To prove the coercivity of $J_{\lambda,\mu}$, we may assume that $\|u\| > 1$ and we get from (H_1), (M_1) and Lemma 2.7 that

$$\begin{aligned}
&\Phi(u) \geq m_0 L_1(u)\\
&\geq \frac{m_0}{p^+}\min(c_1,1)\left(\sum_{k=1}^{T+1}|\triangle u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)}\right)\\
&\geq \frac{m_0}{p^+}\min(c_1,1)\left[\|u\|^{p^-} - (1+q^+)T - 1\right].
\end{aligned}$$

Moreover, let C be as in the property (Γ). By (F_1), (F_3) and (G_1), $(G3)$, there exist some constants $C_\lambda, C'_\mu > 0$ such that

$$\lambda F(k,u) \leq \frac{m_0 \min(c_1,1) S_{p^-,E}}{4Cm_3p^+} K_2(u) + C_\lambda, \quad \text{for all} \quad k \in \mathbb{Z}[1,T]$$

and

$$\mu G(k,u) \leq \frac{m_0 \min(c_1,1) S_{p^-,E}}{4Cm_5p^+} K_4(u) + C'_\mu, \quad \text{for all} \quad k \in \mathbb{Z}[1,T].$$

Then, by the above inequalities, we have

$$\begin{aligned}
&J_{\lambda,\mu}(u) \geq m_0 L_1(u) - \lambda m_3 L_2(u) - \mu m_5 L_3(u)\\
&\geq \frac{m_0}{p^+}\min(c_1,1)\|u\|^{p^-} - \frac{m_0(1+q^+)T-1}{p^+}\min(c_1,1)\\
&-\frac{m_0\min(c_1,1)S_{p^-,E}}{4Cp^+}\sum_{k=1}^{T}K_2(u) - \bar{C}_\lambda - \frac{m_0\min(c_1,1)S_{p^-,E}}{4Cp^+}\sum_{k=1}^{T}K_4(u) - \bar{C'}_\mu\\
&\geq \frac{m_0}{p^+}\min(c_1,1)\|u\|^{p^-} - \frac{m_0(1+q^+)T-1}{p^+}\min(c_1,1)\\
&-\frac{m_0\min(c_1,1)S_{p^-,E}}{4p^+}\sum_{k=1}^{T}|u|^{p^-} - \bar{C}_\lambda - \frac{m_0\min(c_1,1)S_{p^-,E}}{4p^+}\sum_{k=1}^{T}|u|^{p^-} - \bar{C'}_\mu\\
&\geq \frac{m_0}{2p^+}\min(c_1,1)\|u\|^{p^-} - \frac{m_0(1+q^+)T-1}{p^+}\min(c_1,1) - \bar{C}_\lambda - \bar{C'}_\mu.
\end{aligned}$$

Hence, since $p^- > 1$, as $\|u\| \to \infty$, we can conclude that $J_{\lambda,\mu}(u) \to \infty$ and then $J_{\lambda,\mu}$ is bounded from below on E. $\square$

According to Lemma 3.5, the functional $J_{\lambda,\mu}$ is weakly lower semicontinuous on E. Moreover, by Lemma 3.6, $J_{\lambda,\mu}$ is coercive on E. Consequently, invoking Theorem 2.4 (see also [28]), we conclude that there exists $\tilde{u} \in E$ such that $\tilde{u} \in E$ is a global minimizer of $J_{\lambda,\mu}$ and thus a weak solution to problem (1.1).
The next lemma establishes that $\tilde{u}$ is a nontrivial weak solution, provided that λ and μ are re chosen sufficiently large.

Lemma 3.7. *Assume that (M_1)-(M_3), (H_1), (F_4) and (G_4) hold. Then, there exist $\lambda^*, \mu* > 0$ such that for all $\lambda > \lambda^*$ and $\mu > \mu*$, we have $J_{\lambda,\mu}(\tilde{u}) < 0$.*

Proof. Put

$$\beta^{p^*} := \begin{cases} \beta^{p^+} & \text{if } \beta > 1, \\ \beta^{p^-} & \text{if } 0 < \beta < 1 \end{cases}$$

and let $w \in E$, defined by $w(k_0) = d$ and $w(k) = 0$ if $k \in \mathbb{Z}[1,T] \backslash \{k_0\}$.
So, we deduce that

$$\begin{aligned} J_{\lambda,\mu}(w) &\leq m_1 L_1(w) - \lambda m_2 L_2(w) - \mu m_4 L_3(w) \\ &\leq \frac{m_1}{p^-}(c_2 d^{p(k_0-1)} + c_2 d^{p(k_0)} + q(k_0) d^{p(k_0)}) - \lambda m_2 F(k_0, w(k_0)) - \mu m_4 G(k_0, w(k_0)) \\ &\leq \frac{m_1(2c_2 + q^+) d^{p^*}}{p^-} - \lambda m_2 F(k_0, d) - \mu m_4 G(k_0, d). \end{aligned}$$

Then, there exist $\lambda^*, \mu* > 0$ such that $J_{\lambda,\mu}(w) < 0$ for all $\lambda \in (\lambda^*, \infty)$ and $\mu \in (\mu*, \infty)$. Hence, for all $\lambda > \lambda^*$ and $\mu > \mu*$, $J_{\lambda,\mu}(\tilde{u}) < 0$ and so $\tilde{u}$ is a nontrivial weak solution of problem (1.1) for all λ, μ large enough. □

Lemma 3.8. *The mapping Ψ' and I' weakly-strongly continuous, namely*

$$u_n \rightharpoonup u \text{ in } E \text{ as } n \to \infty \text{ imply } \Psi'(u_n) \to \Psi'(u) \text{ and } I'(u_n) \to I'(u) \text{ in } E^*.$$

Proof. Let $\{u_n\}$ be a sequence which converges weakly to u in E. From (F_1), the mappings $N_f, N_g : E \to E^*$ defined by $\langle N_f(u), v \rangle = \sum_{k=1}^{T} f(k, u(k))v(k)$ and $\langle N_g(u), v \rangle = \sum_{k=1}^{T} g(k, u(k))v(k)$ are weakly-strongly continuous.
Due to the weak continuity of the functionals L2 $L_2, L_3$and the continuity of the functions M_2 and M_3, the mappings Ψ', I' are weakly-strongly continuous. □

Lemma 3.9. *The function L_1' is of type (S_+), which means that*

$$u_n \rightharpoonup u \text{ in } E \text{ as } n \to \infty \text{ and } \limsup_{n\to\infty} \langle L_1'(u_n), u_n - u \rangle \leq 0,$$

imply $u_n \to u$ in E as $n \to \infty$.

Proof. We define the mappings $A, B : E \to E^*$ respectively by

$$\langle A(u), v \rangle = \sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2} u(k)v(k), \quad \forall\, u, v \in E$$

and

$$\langle B(u), v \rangle = \sum_{k=1}^{T+1} \varphi(k-1, \triangle u(k-1)) \triangle v(k-1), \quad \forall\, u, v \in E.$$

Let $\{u_n\}$ be a sequence which converges weakly to u in E and

$$\limsup_{n\to\infty} \langle A(u_n) + B(u_n), u_n - u \rangle \leq 0.$$

The fact that $\{u_n\}$ converges weakly to u implies that

$$\lim_{n\to\infty} \langle A(u)+B(u), u_n-u\rangle = 0.$$

By using (H_2) and inequality (2.7), we have

$$\begin{aligned}
0 &\geq \limsup_{n\to\infty} \left[\langle A(u_n)+B(u_n), u_n-u\rangle - \langle A(u)+B(u), u_n-u\rangle\right] \\
&= \limsup_{n\to\infty} \left[\langle B(u_n)-B(u), u_n-u\rangle + \langle A(u_n)-A(u), u_n-u\rangle\right] \\
&\geq c' \limsup_{n\to\infty} \phi(u_n-u) \\
&\geq c' \limsup_{n\to\infty} \left(\min(\|u_n-u\|_{p(\cdot)}^{p^+}, \|u_n-u\|_{p(\cdot)}^{p^-})\right).
\end{aligned}$$

Thus, the sequence $\{u_n\}$ converges to u in E. Therefore, the functional L_1' is of type (S_+).

□

By Lemma 3.9, it is clear that the mapping Φ is of type (S_+). Indeed, assume that $\{u_n\}$ is a sequence that converges weakly to u in E. Then,

$$\limsup_{n\to\infty} \langle J_{\lambda,\mu}(u), u_n-u\rangle = \limsup_{n\to\infty} M_1(L_1(u_n)) \langle L_1'(u_n), u_n-u\rangle \leq 0.$$

From (M_1), it follows that

$$\limsup_{n\to\infty} \langle L_1'(u_n), u_n-u\rangle \leq 0.$$

Therefore, by Lemma 3.9, it follows that $\{u_n\}$ converges to u in E.

Lemma 3.10. *The functional $J'_{\lambda,\mu}$ is of type (S_+).*

Proof. As $J_{\lambda,\mu}(u) = \Phi(u) - \lambda\Psi(u) - \mu I(u)$, it is clear that $J'_{\lambda,\mu}$ is of type (S_+), since Φ' is of type (S_+) and the mappings Ψ', I' are weakly-strongly continuous. □

Lemma 3.11. *The functional $J_{\lambda,\mu}$ satisfies the Palais-Smale condition in E.*

Proof. From Lemma 3.6, we know that the functional $J_{\lambda,\mu}$ is coercive. Let $\{u_n\}$ be a Palais-Smale sequence for $J_{\lambda,\mu}$ in E.
Then,

$$J_{\lambda,\mu}(u_n) \to c \text{ and } J'_{\lambda,\mu}(u_n) \to 0, \text{ as } n\to\infty. \tag{3.1}$$

Since $J_{\lambda,\mu}$ is coercive, relation (3.1) implies that the sequence $\{u_n\}$ is bounded in E. Moreover, E is reflexive, we may extract a subsequence, still denoted by $\{u_n\}$ still denoted $\{u_n\}$, which converges weakly some $u \in E$.
Given that $J'_{\lambda,\mu}(u_n) \to 0$, we have $\left\langle J'_{\lambda,\mu}(u_n), u_n-u\right\rangle \to 0$. Since $J'_{\lambda,\mu}$ satisfies the (S_+) condition, it follows that $u_n \to u$ in E. □

We now seek to establish the existence of a second nontrivial weak solution to problem (1.1) by invoking the mountain pass theorem (Theorem 2.5). To this end, it suffices to verify that for every $\lambda > \lambda^*$ and $\mu > \mu*$, the functional $J_{\lambda,\mu}$ possesses the mountain pass geometry.

Lemma 3.12. *There exist two constants $\rho, r > 0$ such that $\rho \in (0, \|\tilde{u}\|)$ and $J_{\lambda,\mu}(u) \geq r > 0$, for all $u \in E$ with $\|u\| = \rho$.*

Proof. Let $\|u\|$ be small enough such that $\|u\| = \rho \in (0,1)$. Let C be as in (Γ). By (F_1), (F_2) and (G_1), (G_2), there exist two constants $C_\lambda, C'_\mu > 0$ such that

$$\lambda F(k,u) \leq \frac{m_0 \min(c_1,1) S_{p^+,E}}{4Cm_3 p^+ 2^{\frac{p^+-p^-}{p^-}}} K_1(u) + C'_\lambda |u|^\gamma, \quad \text{for all} \quad k \in \mathbb{Z}[1,T]$$

and

$$\mu G(k,u) \leq \frac{m_0 \min(c_1,1) S_{p^+,E}}{4Cm_5 p^+ 2^{\frac{p^+-p^-}{p^-}}} K_3(u) + C''_\mu |u|^\gamma, \quad \text{for all} \quad k \in \mathbb{Z}[1,T];$$

where $\gamma > p^+$.
Thus, by using relation (2.1), one has

$$\begin{aligned}
&J_{\lambda,\mu}(u) \geq m_0 L_1(u) - \lambda m_3 L_2(u) - \mu m_5 L_3(u) \\
&\geq \frac{m_0}{p^+} \min(c_1,1) \|u\|_{p^+}^{p^+} - \frac{m_0 \min(c_1,1) S_{p^+,E}}{4C 2^{\frac{p^+-p^-}{p^-}} p^+} \sum_{k=1}^T K_1(u) \\
&- C'_\lambda \sum_{k=1}^T |u|^\gamma - \frac{m_0 \min(c_1,1) S_{p^+,E}}{4C 2^{\frac{p^+-p^-}{p^-}} p^+} \sum_{k=1}^T K_3(u) - C''_\mu \sum_{k=1}^T |u|^\gamma \\
&\geq \frac{m_0}{p^+ 2^{\frac{p^+-p^-}{p^-}+1}} \min(c_1,1) K^{-p^+} \|u\|^{p^+} - [\max(T+1,\bar{q})]^{1-\frac{\gamma}{p^-}} C'_\lambda \|u\|^\gamma \\
&- [\max(T+1,\bar{q})]^{1-\frac{\gamma}{p^-}} C''_\mu \|u\|^\gamma \\
&= \left(\frac{m_0}{p^+ K^{p^+} 2^{\frac{p^+-p^-}{p^-}+1}} \min(c_1,1) - \tilde{C}'_\lambda \|u\|^{\gamma-p^+} - \tilde{C''}_\mu \|u\|^{\gamma-p^+} \right) \|u\|^{p^+},
\end{aligned}$$

where $\tilde{C}_\lambda = [\max(T+1,\bar{q})]^{1-\frac{\gamma}{p^-}} C'_\lambda \geq 0$ and $\tilde{C'}_\mu = [\max(T+1,\bar{q})]^{1-\frac{\gamma}{p^-}} C''_\mu \geq 0$. Since $\gamma > p^+$, there exist two positive constants r, ρ such that $J_{\lambda,\mu}(u) \geq r > 0$ for all $u \in E$ with $\|u\| = \rho$ and $\|\tilde{u}\| > \rho$. $\square$

Proof of Theorem 3.1. By Lemmas 3.5, 3.6 and 3.7, problem (1.1) has a nontrivial weak solution $\tilde{u}$ as a global minimizer of $J_{\lambda,\mu}$.
Put

$$\bar{c} = \inf_{\varphi \in \Gamma} \max_{u \in \varphi([0,1])} J_{\lambda,\mu}(u),$$

where $\Gamma = \{\varphi \in C(E,[0,1]) / \varphi(0) = 0, \ \varphi(1) = \tilde{u}\}$.
From lemmas 3.11 and 3.12, all the assumptions of the mountain pass theorem (see Theorem 2.5) are satisfied, $J_{\lambda,\mu}(\tilde{u}) < 0$ and $\|\tilde{u}\| > \rho$.
Then $\bar{c}$ which is a critical value of $J_{\lambda,\mu}$, this means that there exists $\bar{u} \in E$ such that $J_{\lambda,\mu}(\bar{u}) = \bar{c}$ and $\langle J'_{\lambda,\mu}(\bar{u}), v \rangle = 0$ for all $v \in E$ or $\bar{u}$ is a weak solution of (1.1). Moreover, $\bar{u}$ is nontrivial and $\bar{u} \neq \tilde{u}$ since $J_{\lambda,\mu}(\bar{u}) = \bar{c} > 0 > J_{\lambda,\mu}(\tilde{u})$. Hence, $\tilde{u}$ and $\bar{u}$ are two nontrivial weak solutions of problem (1.1)

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