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REFINING AND EXTENDING TWO SPECIAL HARDY-HILBERT-TYPE INTEGRAL INEQUALITIES

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ABSTRACT. Numerous variants of the Hardy-Hilbert integral inequality have been developed in the literature. This article first presents refinements and extensions of two such variants. Both are characterised by a double integral with a ratio-power-minimum integrand controlled by two adjustable parameters. The results are obtained under mild assumptions on these parameters. The beta function evaluated at certain values appears naturally in the definition of the constant factors. These constants may be better than those already established when considering a classical setting. Several additional integral inequalities are then derived from our main results. The proofs are also innovative in some technical aspects. They are presented in detail to ensure clarity, to facilitate verification, and to encourage further research in this direction.

Keywords. Hardy-Hilbert-type integral inequalities, minimum function, beta function, Fubini-Tonelli integral theorem
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1. Introduction

1.1. **A key integral inequality.** Integral inequalities are one of the most useful tools in mathematical analysis. In particular, they are central to functional analysis. Detailed discussions of some of the most famous integral inequalities can be found in the books [11, 3, 13, 23, 2, 7, 9]. This article is based on the work of G.H. Hardy in [10, 11], which introduces the Hardy-Hilbert (HH) integral inequality. This inequality gives an upper bound for a certain bilinear form with respect to certain integral norms. More formally, let $p > 1$, $q = p/(p-1)$ so that $1/p + 1/q = 1$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then the HH integral inequality states that

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy\\&\leq\frac{\pi}{\sin(\pi/p)}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},\end{aligned}$$

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where the integrals on the right-hand side are finite, i.e., the L_p and L_q integral norms of f and g satisfy

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(y)dy < +\infty,$$

respectively. Besides the original form, i.e., ratio-trigonometric and depending on the norm parameter p, the constant factor, i.e., $\pi/\sin(\pi/p)$, is known to be optimal. The literature based on this inequality is extensive; considerable research has been carried out, resulting in various generalizations, refinements, and applications. In support of this claim, we refer to the following works: [26, 14, 27, 28, 8, 1, 5, 6, 4, 25, 21, 17, 16, 22, 29, 18, 19, 20, 24, 12, 15].

1.2. **Review of well-known results.** We now review several results that are particularly relevant to the aims of this article.

We start with a key integral inequality in [25]. Formally, let $p > 1$, $q = p/(p-1)$, $\lambda > 0$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then it is established that

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^\lambda} f(x)g(y)dxdy \\ &\leq B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\left[\int_0^{+\infty} x^{p-\lambda-1} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q-\lambda-1} g^q(y)dy\right]^{1/q}, \end{aligned} \tag{1.1}$$

where the integrals on the right-hand side are finite, and $B_{\text{eta}}(a, b)$ is the standard beta function taken at $a, b > 0$, i.e.,

$$B_{\text{eta}}(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt. \tag{1.2}$$

In particular, using a known formula of the beta function (to be recalled later) and applying Equation (1.1) to $\lambda = 1$, we get

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \\ &\leq \frac{\pi}{\sin(\pi/p)}\left[\int_0^{+\infty} x^{p-2} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{p-2} g^q(y)dy\right]^{1/q}. \end{aligned} \tag{1.3}$$

This provides an upper bound that is alternative to that of the HH integral inequality, which deals with weighted L_p and L_q integral norms. More generally, the inequality in Equation (1.1) is known to be sharp, with an optimal constant factor depending on both λ and p.

Using a different type of integrand, the main HH-type integral inequality in [21] is also relevant. It is formally presented below. let $p > 1$, $q = p/(p-1)$, $\sigma > 0$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then it is proved that

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} \left[\min\left(\frac{1}{x}, \frac{1}{y}\right)\right]^\sigma f(x)g(y)dxdy \\ &\leq \frac{pq}{\sigma}\left[\int_0^{+\infty} x^{p-\sigma-1} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q-\sigma-1} g^q(y)dy\right]^{1/q}, \end{aligned} \tag{1.4}$$

where the integrals on the right-hand side are finite. Note that this inequality is best known with a different definition of the integrand, using $1/\max(x^{\sigma}, y^{\sigma})$ instead of $[\min(1/x, 1/y)]^{\sigma}$, but it is strictly the same. Its originality lies in the power-minimum-type integrand, and the constant factor which is both simple and known to be optimal.

Two other results that motivate this article are [18, Theorems 4 and 5]. They can be described as HH-type integral inequalities with integrands that mix the functional nature of those in Equations (1.1) and (1.4). The formal statements of these theorems are given below, with slight modifications in notation and formatting for consistency.

We start with [18, Theorem 5], which is more directly related to Equations (1.1) and (1.4) than [18, Theorem 4],. Let $p > 1$, $q = p/(p-1)$, $\theta \in (0, 2)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then is is established that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^{\theta}} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{1-\theta/2} f(x)g(y)dxdy$$
$$\leq B_{\mathrm{eta}}\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \left[\int_0^{+\infty} x^{1-\theta} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{1-\theta} g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side are finite. The corresponding integrand is thus also of the ratio-power-minimum type. The parameter θ controls both $x+y$ in the denominator and $\min(x/y, y/x)$. The constant factor is expressed in terms of the beta function. It depends only on θ; it is independent of p. The same holds for the weight functions in the L^p and L^q integral norms. This independence makes this inequality somewhat unexpected compared to the optimal results in Equations (1.1) and (1.4), and suggests some possibilities for further refinement.

The second theorem, i.e., [18, Theorem 5], is stated below. Let $p > 1$, $q = p/(p-1)$, $\gamma > 0$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then is is established that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{|x-y|^{1-\gamma}} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{2\gamma} f(x)g(y)dxdy$$
$$\leq 2B_{\mathrm{eta}}(\gamma, \gamma) \left[\int_0^{+\infty} x^{\gamma} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{\gamma} g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side are finite. This variant of the HH integral inequality is characterized by the use of an integrand of the ratio-power-minimum type. In it, the parameter γ simultaneously controls $|x-y|$ and $\min(x/y, y/x)$. As in the previous result, the beta function is central to the definition of the constant factor. Again, this constant is independent of p, and the weight functions of the weighted L^p and L^q integral norms are also independent of p. This is generally not the case in the optimal versions of HH-type integral inequalities, such as those in Equations (1.1) and (1.4).

We conclude this brief review with two important references which also deal with HH-type integral inequalities involving integrands of the ratio-power-minimum type: [24, 20, 15]. In a sense, these works complement and generalize [18, Theorems 4 and 5]. Similar analytical techniques are used, but greater versatility is achieved through the use of multiple parameters and the inclusion of $\max(x/y, y/x)$.

1.3. **Contributions.** This article aims to refine and extend [18, Theorems 4 and 5] in two key aspects.

With respect to the statements of these theorems, the first aspect is to separate $x+y$ or $|x-y|$ from $\min(x/y, y/x)$ by introducing a second parameter. The idea is to have one parameter that modulates $x+y$ or $|x-y|$, and another that modulates $\min(x/y, y/x)$ independently of the first parameter. So we separate and modulate these terms, which makes the framework of [18, Theorems 4 and 5] much more flexible.

The second aspect focuses on improving the precision of the upper bounds in HH-type integral inequalities. As discussed in the previous section, given the optimal results such as those in Equations (1.1) and (1.4), we explore the possibility of refining the upper bounds. The idea is to fully exploit the norm parameter p combined with the effect of the two parameters introduced.

The main contributions are given in the form of two theorems, Theorems 3.1 and 4.1, which combine these two aspects. The proofs also contain innovative mathematical developments leading to the refinements. They are detailed for the sake of verification and also because of their potential for adaptation to other integral inequalities.

We complement these results with four propositions, i.e., Propositions 5.1, 5.2, 5.3 and 5.4, which lead to new HH-type integral and weighted integral norm inequalities. Each of these can be considered independently, opening up new perspectives for applications in functional analysis and, in particular, operator theory.

1.4. **Structure.** The structure of the article is as follows: Section 2 recalls some preliminary results on the beta function. The two main theorems are presented in two different sections, in Sections 3 and 4, respectively. Section 5 contains the additional results. Finally, Section 6 concludes the article.

2. Properties of the beta function

Before presenting our main results, we need to recall some well-known properties of the beta function. First, its standard integral form is given by Equation (1.2). The beta function is symmetric, i.e., for any $a, b > 0$, we have

$$B_{\mathrm{eta}}(a, b) = B_{\mathrm{eta}}(b, a).$$

It can therefore be expressed as

$$B_{\mathrm{eta}}(a, b) = \int_0^1 t^{b-1}(1-t)^{a-1} dt.$$

Fundamental values of the beta function include $B_{\mathrm{eta}}(1, 1) = 1$, and, less trivially,

$$B_{\mathrm{eta}}(a, 1) = \frac{1}{a}, \quad B_{\mathrm{eta}}(a, 2) = \frac{1}{a(a+1)}, \quad B_{\mathrm{eta}}(a, 3) = \frac{2}{a(a+1)(a+2)}.$$

There are several equivalent expressions for the beta function. For the purposes of this article, we will highlight two of them.

First of all, when dealing with integrals defined on $(0, +\infty)$, the following expression holds:

$$B_{\text{eta}}(a, b) = \int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt. \tag{2.1}$$

Another definition is as follows:

$$B_{\text{eta}}(a, b) = \int_1^{+\infty} \frac{t^{-a-b}}{(t-1)^{1-a}} dt. \tag{2.2}$$

This form is particularly useful for manipulating integrals defined on $(1, +\infty)$.

Finally, for any $a \in (0, 1)$, the following well-known mirror identity holds:

$$B_{\text{eta}}(a, 1-a) = \frac{\pi}{\sin(a\pi)}.$$

This explains the constant factor in Equation (1.3), being applied with $a = 1/p$.

All these results, and much more on the beta function, can be found in [9].

3. First theorem

The first theorem is a refinement and extension of [18, Theorem 5]. It is followed by the proof and a discussion.

Theorem 3.1. *Let $p > 1$, $q = p/(p-1)$ so that $1/p + 1/q = 1$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then, for any $\beta \geq 0$ and $\lambda > 0$, we have*

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y)^\lambda} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x) g(y) dx dy$$
$$\leq \Upsilon_{\beta,\lambda,p} \left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q(y) dy\right]^{1/q},$$

where the integrals on the right-hand side are finite, $\Upsilon_{\beta,\lambda,p}$ is the factor constant given by

$$\Upsilon_{\beta,\lambda,p} = \left[B_{eta}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta q^2}{2^{\lambda-1}\lambda(\lambda+2\beta q)}\right]^{1/p} \times$$
$$\left[B_{eta}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta p^2}{2^{\lambda-1}\lambda(\lambda+2\beta p)}\right]^{1/q} \tag{3.1}$$

and $B_{eta}(a, b)$ is the beta function at $a, b > 0$.

Proof. Decomposing the integrand as a suitable product (via $1/p+1/q=1$) and applying the Hölder integral inequality with the parameters p and q, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y)^{\lambda}}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{-(\lambda/p+\beta-1)/q}y^{(\lambda/q+\beta-1)/p}}{(x+y)^{\lambda/p}}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta/p}f(x)\times\\
&\frac{x^{(\lambda/p+\beta-1)/q}y^{-(\lambda/q+\beta-1)/p}}{(x+y)^{\lambda/q}}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta/q}g(y)dxdy\\
&\leq \mathcal{A}^{1/p}\mathcal{B}^{1/q},
\end{aligned} \tag{3.2}$$

where

$$\mathcal{A}=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{-(\lambda/p+\beta-1)p/q}y^{\lambda/q+\beta-1}}{(x+y)^{\lambda}}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta}f^p(x)dxdy$$

and

$$\mathcal{B}=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\lambda/p+\beta-1}y^{-(\lambda/q+\beta-1)q/p}}{(x+y)^{\lambda}}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta}g^q(y)dxdy.$$

Now we want to majorize $\mathcal{A}$ and $\mathcal{B}$ consecutively, starting with $\mathcal{A}$.

Upper bound for $\mathcal{A}$.: Using the Fubini-Tonelli (FT) integral theorem, we have

$$\mathcal{A}=\int_0^{+\infty}x^{-(\lambda/p+\beta-1)p/q}f^p(x)\left\{\int_0^{+\infty}\frac{y^{\lambda/q+\beta-1}}{(x+y)^{\lambda}}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta}dy\right\}dx. \tag{3.3}$$

Let us look at the simple integral between the curly brackets. Applying the Chasles rule by considering the definition of the minimum, performing the change of variables $u=y/x$, developing an integral term with the Chasles rule at the threshold value 1, and identifying the beta function under the special form in Equation (2.1) taken at λ/p and λ/q (which is equal to that taken at λ/p and λ/q by the symmetric property), we obtain

$$\begin{aligned}
&\int_0^{+\infty} \frac{y^{\lambda/q+\beta-1}}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta dy \\
&= \int_0^x \frac{y^{\lambda/q+\beta-1}}{(x+y)^\lambda}\times\frac{y^\beta}{x^\beta}dy + \int_x^{+\infty} \frac{y^{\lambda/q+\beta-1}}{(x+y)^\lambda}\times\frac{x^\beta}{y^\beta}dy \\
&= x^{\beta-\lambda+\lambda/q}\int_0^x \frac{(y/x)^{\lambda/q+2\beta-1}}{(1+y/x)^\lambda}\times\frac{1}{x}dy + x^{\beta-\lambda+\lambda/q}\int_x^{+\infty} \frac{(y/x)^{\lambda/q-1}}{(1+y/x)^\lambda}\times\frac{1}{x}dy \\
&= x^{\beta-\lambda/p}\left[\int_0^1 \frac{u^{\lambda/q+2\beta-1}}{(1+u)^\lambda}du + \int_1^{+\infty} \frac{u^{\lambda/q-1}}{(1+u)^\lambda}du\right] \\
&= x^{\beta-\lambda/p}\left\{\int_0^1 \frac{u^{\lambda/q+2\beta-1}}{(1+u)^\lambda}du + \left[\int_0^{+\infty} \frac{u^{\lambda/q-1}}{(1+u)^\lambda}du - \int_0^1 \frac{u^{\lambda/q-1}}{(1+u)^\lambda}du\right]\right\} \\
&= x^{\beta-\lambda/p}\left[\int_0^{+\infty} \frac{u^{\lambda/q-1}}{(1+u)^{\lambda/q+\lambda/p}}du - \int_0^1 \frac{u^{\lambda/q-1}}{(1+u)^\lambda}(1-u^{2\beta})du\right] \\
&= x^{\beta-\lambda/p}\left[B_{\text{eta}}\left(\frac{\lambda}{p},\frac{\lambda}{q}\right) - \mathcal{C}\right], \qquad (3.4)
\end{aligned}$$

where

$$\mathcal{C} = \int_0^1 \frac{u^{\lambda/q-1}}{(1+u)^\lambda}(1-u^{2\beta})du.$$

Let us now minorize this term. For any $u \in (0,1)$, we obviously have $(1+u)^\lambda \le 2^\lambda$ and $1-u^{2\beta} \ge 0$. Using this and power primitives, we get

$$\begin{aligned}
\mathcal{C} &\ge \frac{1}{2^\lambda}\int_0^1 u^{\lambda/q-1}(1-u^{2\beta})du = \frac{1}{2^\lambda}\left[\int_0^1 u^{\lambda/q-1}du - \int_0^1 u^{\lambda/q+2\beta-1}du\right] \\
&= \frac{1}{2^\lambda}\left(\frac{1}{\lambda/q} - \frac{1}{\lambda/q+2\beta}\right) = \frac{\beta q^2}{2^{\lambda-1}\lambda(\lambda+2\beta q)}. \qquad (3.5)
\end{aligned}$$

It follows from Equations (3.4) and (3.5) that

$$\int_0^{+\infty} \frac{y^{\lambda/q+\beta-1}}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta dy \le x^{\beta-\lambda/p}\left[B_{\text{eta}}\left(\frac{\lambda}{p},\frac{\lambda}{q}\right) - \frac{\beta q^2}{2^{\lambda-1}\lambda(\lambda+2\beta q)}\right]. \qquad (3.6)$$

This combined with Equation (3.3) gives

$$\begin{aligned}
\mathcal{A} &\le \int_0^{+\infty} x^{-(\lambda/p+\beta-1)p/q} f^p(x)\times x^{\beta-\lambda/p}\left[B_{\text{eta}}\left(\frac{\lambda}{p},\frac{\lambda}{q}\right) - \frac{\beta q^2}{2^{\lambda-1}\lambda(\lambda+2\beta q)}\right]dx \\
&= \left[B_{\text{eta}}\left(\frac{\lambda}{p},\frac{\lambda}{q}\right) - \frac{\beta q^2}{2^{\lambda-1}\lambda(\lambda+2\beta q)}\right]\int_0^{+\infty} x^{-(\lambda/p+\beta-1)p/q+\beta-\lambda/p} f^p(x)dx \\
&= \left[B_{\text{eta}}\left(\frac{\lambda}{p},\frac{\lambda}{q}\right) - \frac{\beta q^2}{2^{\lambda-1}\lambda(\lambda+2\beta q)}\right]\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx. \qquad (3.7)
\end{aligned}$$

Let us now majorize the term $\mathcal{B}$.

Upper bound for $\mathcal{B}$.: The FT integral theorem ensures the following expression:

$$\mathcal{B}=\int_0^{+\infty} y^{-(\lambda/q+\beta-1)q/p}g^q(y)\left[\int_0^{+\infty}\frac{x^{\lambda/p+\beta-1}}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta dx\right]dy. \tag{3.8}$$

Let us look at the simple integral between the curly brackets, following the lines used for majorizing the term $\mathcal{A}$. Applying the Chasles rule by considering the definition of the minimum, performing the change of variables $w=x/y$, developing an integral term with the Chasles rule at the threshold value 1, and identifying the beta function under the special form in Equation (2.1) taken at λ/p and λ/q, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\frac{x^{\lambda/p+\beta-1}}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta dx\\
&=\int_0^{y}\frac{x^{\lambda/p+\beta-1}}{(x+y)^\lambda}\times\frac{x^\beta}{y^\beta}dx+\int_y^{+\infty}\frac{x^{\lambda/p+\beta-1}}{(x+y)^\lambda}\times\frac{y^\beta}{x^\beta}dx\\
&=y^{\beta-\lambda+\lambda/p}\int_0^{y}\frac{(x/y)^{\lambda/p+2\beta-1}}{(1+x/y)^\lambda}\times\frac{1}{y}dx+y^{\beta-\lambda+\lambda/p}\int_y^{+\infty}\frac{(x/y)^{\lambda/p-1}}{(1+x/y)^\lambda}\times\frac{1}{y}dx\\
&=y^{\beta-\lambda/q}\left[\int_0^{1}\frac{w^{\lambda/p+2\beta-1}}{(1+w)^\lambda}dw+\int_1^{+\infty}\frac{w^{\lambda/p-1}}{(1+w)^\lambda}dw\right]\\
&=y^{\beta-\lambda/q}\left\{\int_0^{1}\frac{w^{\lambda/p+2\beta-1}}{(1+w)^\lambda}du+\left[\int_0^{+\infty}\frac{w^{\lambda/p-1}}{(1+w)^\lambda}dw-\int_0^{1}\frac{w^{\lambda/p-1}}{(1+w)^\lambda}dw\right]\right\}\\
&=y^{\beta-\lambda/q}\left[\int_0^{+\infty}\frac{w^{\lambda/p-1}}{(1+w)^{\lambda/p+\lambda/q}}dw-\int_0^{1}\frac{w^{\lambda/p-1}}{(1+w)^\lambda}(1-w^{2\beta})dw\right]\\
&=y^{\beta-\lambda/q}\left[B_{\mathrm{eta}}\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)-\mathcal{D}\right],
\end{aligned}\tag{3.9}$$

where

$$\mathcal{D}=\int_0^{1}\frac{w^{\lambda/p-1}}{(1+w)^\lambda}(1-w^{2\beta})dw.$$

Let us now minorize this term, taking inspiration from the minorization techniques used for the term $\mathcal{C}$. For any $w\in(0,1)$, it is clear that $(1+w)^\lambda\leq 2^\lambda$ and $1-w^{2\beta}\geq 0$. Using this and power primitives, we get

$$\begin{aligned}
\mathcal{D}&\geq\frac{1}{2^\lambda}\int_0^1 w^{\lambda/p-1}(1-w^{2\beta})dw=\frac{1}{2^\lambda}\left[\int_0^1 w^{\lambda/p-1}dw-\int_0^1 w^{\lambda/p+2\beta-1}dw\right]\\
&=\frac{1}{2^\lambda}\left(\frac{1}{\lambda/p}-\frac{1}{\lambda/p+2\beta}\right)=\frac{\beta p^2}{2^{\lambda-1}\lambda(\lambda+2\beta p)}.
\end{aligned}\tag{3.10}$$

It follows from Equations (3.9) and (3.10) that

$$\int_0^{+\infty}\frac{x^{\lambda/p+\beta-1}}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta dx\leq y^{\beta-\lambda/q}\left[B_{\mathrm{eta}}\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)-\frac{\beta p^2}{2^{\lambda-1}\lambda(\lambda+2\beta p)}\right]. \tag{3.11}$$

This combined with Equation (3.8) gives

$$\begin{aligned}
\mathcal{B} &\leq \int_0^{+\infty} y^{-(\lambda/q+\beta-1)q/p} g^q(y) \times y^{\beta-\lambda/q} \left[B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta p^2}{2^{\lambda-1}\lambda(\lambda+2\beta p)}\right] dy \\
&= \left[B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta p^2}{2^{\lambda-1}\lambda(\lambda+2\beta p)}\right] \int_0^{+\infty} y^{-(\lambda/q+\beta-1)q/p+\beta-\lambda/q} g^q(y) dy \\
&= \left[B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta p^2}{2^{\lambda-1}\lambda(\lambda+2\beta p)}\right] \int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q(y) dy. \qquad (3.12)
\end{aligned}$$

Based on Equations (3.2), (3.7) and (3.12) and the equality $1/p + 1/q = 1$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x)g(y)dxdy \\
&\leq \left\{\left[B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta q^2}{2^{\lambda-1}\lambda(\lambda+2\beta q)}\right] \int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx\right\}^{1/p} \times \\
&\left\{\left[B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta p^2}{2^{\lambda-1}\lambda(\lambda+2\beta p)}\right] \int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q(y)dy\right\}^{1/q} \\
&= \left[B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta q^2}{2^{\lambda-1}\lambda(\lambda+2\beta q)}\right]^{1/p} \left[B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta p^2}{2^{\lambda-1}\lambda(\lambda+2\beta p)}\right]^{1/q} \times \\
&\left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q(y)dy\right]^{1/q} \\
&= \Upsilon_{\beta,\lambda,p} \left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q(y)dy\right]^{1/q},
\end{aligned}$$

where $\Upsilon_{\beta,\lambda,p}$ is as stated in Equation (3.1). This concludes the proof of Theorem 3.1. □

In the specific case where $p = 2$, the established inequality reduces to

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x)g(y)dxdy \\
&\leq \Upsilon_{\beta,\lambda,p}\sqrt{\int_0^{+\infty} x^{1-\lambda} f^2(x)dx}\sqrt{\int_0^{+\infty} y^{1-\lambda} g^2(y)dy},
\end{aligned}$$

where

$$\Upsilon_{\beta,\lambda,p} = B_{\text{eta}}\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) - \frac{\beta}{2^{\lambda-3}\lambda(\lambda+4\beta)}.$$

Thus, in this case, the parameter β influences only the constant factor.

In the general case, if we set $\beta = 0$, then then $\Upsilon_{\beta,\lambda,p}$ simplifies to

$$\Upsilon_{\beta,\lambda,p} = B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$$

and Theorem 3.1 becomes the inequality in Equation (1.1), which is proved to be optimal. For the case $\beta > 0$, we clearly have

$$\Upsilon_{\beta,\lambda,p} = \left[B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta q^2}{2^{\lambda-1}\lambda(\lambda+2\beta q)}\right]^{1/p} \times$$
$$\left[B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) - \frac{\beta p^2}{2^{\lambda-1}\lambda(\lambda+2\beta p)}\right]^{1/q}$$
$$\leq B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)^{1/p+1/q} = B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right),$$

leading to a better constant. This is consistent with the fact $\min(x/y, y/x) \leq 1$, giving

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x)g(y)dxdy$$
$$\leq \int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^\lambda} f(x)g(y)dxdy,$$

which suggests that the constant factor in Equation (1.1) could be improved.

Furthermore, in the setting of [18, Theorem 5], i.e., $\lambda = \theta$ and $\beta = 1 - \theta/2$ with $\theta \in (0, 2)$, Theorem 3.1 yields

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^\theta}\left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{1-\theta/2} f(x)g(y)dxdy$$
$$\leq \Upsilon_{\beta,\lambda,p}\left[\int_0^{+\infty} x^{1+\theta(p/2-2)} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{1+\theta(q/2-2)} g^q(y)dy\right]^{1/q},$$

where

$$\Upsilon_{\beta,\lambda,p} = \left\{B_{\text{eta}}\left(\frac{\theta}{p}, \frac{\theta}{q}\right) - \frac{(1-\theta/2)q^2}{2^{\theta-1}\theta[\theta+2(1-\theta/2)q]}\right\}^{1/p} \times$$
$$\left\{B_{\text{eta}}\left(\frac{\theta}{p}, \frac{\theta}{q}\right) - \frac{(1-\theta/2)p^2}{2^{\theta-1}\theta[\theta+2(1-\theta/2)p]}\right\}^{1/q}.$$

The obtained constant factor is better than the one proposed in [18, Theorem 5], but the weighted L_p and L_q integral norms of f and g are not comparable; there are different weight functions in the two theorems, at least for $p \neq 2$. To understand the situation better, let us concentrate on the case $p = 2$, where these norms are equal. The new constant factor is then

$$\Upsilon_{\beta,\lambda,p} = B_{\text{eta}}\left(\frac{\theta}{2}, \frac{\theta}{2}\right) - \frac{2^{2-\theta}(2-\theta)}{\theta(4-\theta)}.$$

Since $\theta \in (0, 2)$, it is clear that

$$\Upsilon_{\beta,\lambda,p} \leq B_{\text{eta}}\left(\frac{\theta}{2}, \frac{\theta}{2}\right),$$

meaning that Theorem 3.1 refines [18, Theorem 5].

In summary, Theorem 3.1 offers a valuable extension to [18, Theorem 5]. It is worth noting that the parameters β and λ are independent, leading to a result that can adapt to different mathematical scenarios, unlike [18, Theorem 5]. The parameter p also plays a crucial role in the weight function of the weighted L_p and L_q integral norms in the upper bounds. This is again consistent with Equations (1.1) and (1.4), and confirms the room for improvement that exists in [18, Theorem 5], although the complexity of the framework makes a rigorous proof of optimality for our upper bounds a challenging task.

4. Second theorem

The second theorem is a refinement and extension of [18, Theorem 4]. It is followed by the proof and a discussion.

Theorem 4.1. *Let $p > 1$, $q = p/(p-1)$ so that $1/p + 1/q = 1$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then, for any $\beta \geq 0$ and $\lambda \in (0, 1)$, we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{|x-y|^{\lambda}} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\beta} f(x)g(y)dxdy$$
$$\leq \Omega_{\beta,\lambda,p} \left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side are finite, $\Omega_{\beta,\lambda,p}$ is the constant factor given by

$$\Omega_{\beta,\lambda,p} = \left[B_{eta}\left(1-\lambda, \frac{\lambda}{q} + 2\beta\right) + B_{eta}\left(1-\lambda, \frac{\lambda}{p}\right)\right]^{1/p} \times$$
$$\left[B_{eta}\left(1-\lambda, \frac{\lambda}{p} + 2\beta\right) + B_{eta}\left(1-\lambda, \frac{\lambda}{q}\right)\right]^{1/q}, \tag{4.1}$$

and $B_{eta}(a, b)$ is the beta function at $a, b > 0$.

Proof. This proof follows the spirit of the proof of Theorem 3.1, but with a few important technical changes. Decomposing the integrand as a suitable product and applying the Hölder integral inequality with the parameters p and q, we get

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{|x-y|^{\lambda}} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\beta} f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} \frac{x^{-(\lambda/p+\beta-1)/q} y^{(\lambda/q+\beta-1)/p}}{|x-y|^{\lambda/p}} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\beta/p} f(x)\times$$
$$\frac{x^{(\lambda/p+\beta-1)/q} y^{-(\lambda/q+\beta-1)/p}}{|x-y|^{\lambda/q}} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\beta/q} g(y)dxdy$$
$$\leq \mathcal{E}^{1/p}\mathcal{F}^{1/q}, \tag{4.2}$$

where

$$\mathcal{E} = \int_0^{+\infty}\int_0^{+\infty} \frac{x^{-(\lambda/p+\beta-1)p/q} y^{\lambda/q+\beta-1}}{|x-y|^{\lambda}} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\beta} f^p(x)dxdy$$

and

$$\mathcal{F} = \int_0^{+\infty}\int_0^{+\infty} \frac{x^{\lambda/p+\beta-1}y^{-(\lambda/q+\beta-1)q/p}}{|x-y|^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta g^q(y)dxdy.$$

We now want to express, if possible, $\mathcal{E}$ and $\mathcal{F}$ consecutively, starting with $\mathcal{E}$.

Expression of $\mathcal{E}$.: Using the FT integral theorem, we have

$$\mathcal{E} = \int_0^{+\infty} x^{-(\lambda/p+\beta-1)p/q}f^p(x)\left\{\int_0^{+\infty}\frac{y^{\lambda/q+\beta-1}}{|x-y|^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta dy\right\}dx. \tag{4.3}$$

Let us look at the simple integral between the curly brackets. Applying the Chasles rule by considering the definition of the minimum, expressing the absolute value term, making the change of variables $u = y/x$, and identifying the beta function under two forms: the standard form (given in Equation (1.2)) and the special form as described in Equation (2.2), we obtain

$$\begin{aligned}
&\int_0^{+\infty}\frac{y^{\lambda/q+\beta-1}}{|x-y|^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta dy\\
&= \int_0^x \frac{y^{\lambda/q+\beta-1}}{|x-y|^\lambda}\times\frac{y^\beta}{x^\beta}dy + \int_x^{+\infty}\frac{y^{\lambda/q+\beta-1}}{|x-y|^\lambda}\times\frac{x^\beta}{y^\beta}dy\\
&= \int_0^x \frac{y^{\lambda/q+\beta-1}}{(x-y)^\lambda}\times\frac{y^\beta}{x^\beta}dy + \int_x^{+\infty}\frac{y^{\lambda/q+\beta-1}}{(y-x)^\lambda}\times\frac{x^\beta}{y^\beta}dy\\
&= x^{\beta-\lambda+\lambda/q}\int_0^x \frac{(y/x)^{\lambda/q+2\beta-1}}{(1-y/x)^\lambda}\times\frac{1}{x}dy + x^{\beta-\lambda+\lambda/q}\int_x^{+\infty}\frac{(y/x)^{\lambda/q-1}}{(y/x-1)^\lambda}\times\frac{1}{x}dy\\
&= x^{\beta-\lambda/p}\left[\int_0^1\frac{u^{\lambda/q+2\beta-1}}{(1-u)^\lambda}du + \int_1^{+\infty}\frac{u^{\lambda/q-1}}{(u-1)^\lambda}du\right]\\
&= x^{\beta-\lambda/p}\left[\int_0^1 u^{\lambda/q+2\beta-1}(1-u)^{(1-\lambda)-1}du + \int_1^{+\infty}\frac{u^{-(1-\lambda)-\lambda/p}}{(u-1)^{1-(1-\lambda)}}du\right]\\
&= x^{\beta-\lambda/p}\left[B_{\text{eta}}\left(1-\lambda,\frac{\lambda}{q}+2\beta\right) + B_{\text{eta}}\left(1-\lambda,\frac{\lambda}{p}\right)\right].
\end{aligned} \tag{4.4}$$

This combined with Equation (4.3) gives an equality, which is

$$\mathcal{E} = \int_0^{+\infty} x^{-(\lambda/p+\beta-1)p/q} f^p(x) \times$$
$$x^{\beta-\lambda/p}\left[B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{q}+2\beta\right) + B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{p}\right)\right] dx$$
$$= \left[B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{q}+2\beta\right) + B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{p}\right)\right] \times$$
$$\int_0^{+\infty} x^{-(\lambda/p+\beta-1)p/q+\beta-\lambda/p} f^p(x) dx$$
$$= \left[B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{q}+2\beta\right) + B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{p}\right)\right] \times$$
$$\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x) dx. \tag{4.5}$$

Let us now express the term $\mathcal{F}$ using similar mathematical arguments.

Expression of $\mathcal{F}$.: By virtue of the FT integral theorem, we have

$$\mathcal{F} = \int_0^{+\infty} y^{-(\lambda/q+\beta-1)q/p} g^q(y) \left[\int_0^{+\infty} \frac{x^{\lambda/p+\beta-1}}{|x-y|^\lambda}\left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta dx\right] dy. \tag{4.6}$$

Let us look at the simple integral between the curly brackets, following the lines used for the term $\mathcal{E}$. Applying the Chasles rule by considering the definition of the minimum, expressing the absolute value term, making the change of variables $v = x/y$, and identifying the beta function under two forms: the standard form and the special form as described in Equation (2.2), we obtain

$$\int_0^{+\infty} \frac{x^{\lambda/p+\beta-1}}{|x-y|^\lambda}\left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta dx$$
$$= \int_0^y \frac{x^{\lambda/p+\beta-1}}{|x-y|^\lambda} \times \frac{x^\beta}{y^\beta} dx + \int_y^{+\infty} \frac{x^{\lambda/p+\beta-1}}{|x-y|^\lambda} \times \frac{y^\beta}{x^\beta} dx$$
$$= \int_0^y \frac{x^{\lambda/p+\beta-1}}{(y-x)^\lambda} \times \frac{x^\beta}{y^\beta} dx + \int_y^{+\infty} \frac{x^{\lambda/p+\beta-1}}{(x-y)^\lambda} \times \frac{y^\beta}{x^\beta} dx$$
$$= y^{\beta-\lambda+\lambda/p}\int_0^y \frac{(x/y)^{\lambda/p+2\beta-1}}{(1-x/y)^\lambda} \times \frac{1}{y} dx + y^{\beta-\lambda+\lambda/p}\int_y^{+\infty} \frac{(x/y)^{\lambda/p-1}}{(x/y-1)^\lambda} \times \frac{1}{y} dx$$
$$= y^{\beta-\lambda/q}\left[\int_0^1 \frac{v^{\lambda/p+2\beta-1}}{(1-v)^\lambda} dv + \int_1^{+\infty} \frac{v^{\lambda/p-1}}{(v-1)^\lambda} dv\right]$$
$$= y^{\beta-\lambda/q}\left[\int_0^1 v^{\lambda/p+2\beta-1}(1-v)^{(1-\lambda)-1} dv + \int_1^{+\infty} \frac{v^{-(1-\lambda)-\lambda/q}}{(v-1)^{1-(1-\lambda)}} dv\right]$$
$$= y^{\beta-\lambda/q}\left[B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{p}+2\beta\right) + B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{q}\right)\right]. \tag{4.7}$$

This combined with Equation (4.6) gives

$$\mathcal{F} = \int_0^{+\infty} y^{-(\lambda/q+\beta-1)q/p} g^q(y) \times$$
$$y^{\beta-\lambda/q} \left[B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{p} + 2\beta \right) + B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{q} \right) \right] dy$$
$$= \left[B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{p} + 2\beta \right) + B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{q} \right) \right] \times$$
$$\int_0^{+\infty} y^{-(\lambda/q+\beta-1)q/p+\beta-\lambda/q} g^q(y) dy$$
$$= \left[B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{p} + 2\beta \right) + B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{q} \right) \right] \times$$
$$\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q(y) dy. \tag{4.8}$$

Based on Equations (4.2), (4.5) and (4.8) and the equality $1/p + 1/q = 1$, we get

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{|x-y|^\lambda} \left[\min \left(\frac{x}{y}, \frac{y}{x} \right) \right]^\beta f(x) g(y) dx dy$$
$$\leq \left\{ \left[B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{q} + 2\beta \right) + B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{p} \right) \right] \int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x) dx \right\}^{1/p} \times$$
$$\left\{ \left[B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{p} + 2\beta \right) + B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{q} \right) \right] \int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q(y) dy \right\}^{1/q}$$
$$= \left[B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{q} + 2\beta \right) + B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{p} \right) \right]^{1/p} \times$$
$$\left[B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{p} + 2\beta \right) + B_{\mathrm{eta}} \left(1-\lambda, \frac{\lambda}{q} \right) \right]^{1/q} \times$$
$$\left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q(y) dy \right]^{1/q}$$
$$= \Omega_{\beta,\lambda,p} \left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q(y) dy \right]^{1/q},$$

where $\Omega_{\beta,\lambda,p}$ is as stated in Equation (4.1). This concludes the proof of Theorem 4.1. □

In the specific case where $p = 2$, the established inequality reduces to

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{|x-y|^\lambda} \left[\min \left(\frac{x}{y}, \frac{y}{x} \right) \right]^\beta f(x) g(y) dx dy$$
$$\leq \Omega_{\beta,\lambda,p} \sqrt{\int_0^{+\infty} x^{1-\lambda} f^2(x) dx} \sqrt{\int_0^{+\infty} y^{1-\lambda} g^2(y) dy},$$

where

$$\Omega_{\beta,\lambda,p} = B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{2}+2\beta\right) + B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{2}\right).$$

Thus, in this case, the parameter β influences only the constant factor.

In the general case, if we set $\beta = 0$, then $\Omega_{\beta,\lambda,p}$ simplifies to

$$\Omega_{\beta,\lambda,p} = B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{p}\right) + B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{q}\right),$$

and the established inequality can be expressed in the following elegant form:

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{|x-y|^{\lambda}} f(x)g(y)dxdy \\
&\le \left[B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{p}\right) + B_{\text{eta}}\left(1-\lambda, \frac{\lambda}{q}\right)\right] \times \\
&\left[\int_0^{+\infty} x^{p-1-\lambda} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q-1-\lambda} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This corresponds to what is obtained in [16, Theorem 1] with $a = 1-\lambda/p$ and $b = 1-\lambda/q$.

Furthermore, in the setting of [18, Theorem 4], i.e., $\lambda = 1-\gamma$ and $\beta = 2\gamma$ with $\gamma \in (0,1)$ to conform with our assumption, Theorem 4.1 yields

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{|x-y|^{1-\gamma}} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{2\gamma} f(x)g(y)dxdy \\
&\le \Omega_{\beta,\lambda,p} \left[\int_0^{+\infty} x^{(2\gamma-1)(2-p)+\gamma} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{(2\gamma-1)(2-q)+\gamma} g^q(y)dy\right]^{1/q},
\end{aligned}$$

where

$$\begin{aligned}
\Omega_{\beta,\lambda,p} = &\left[B_{\text{eta}}\left(\gamma, \frac{1-\gamma}{q}+4\gamma\right) + B_{\text{eta}}\left(\gamma, \frac{1-\gamma}{p}\right)\right]^{1/p} \times \\
&\left[B_{\text{eta}}\left(\gamma, \frac{1-\gamma}{p}+4\gamma\right) + B_{\text{eta}}\left(\gamma, \frac{1-\gamma}{q}\right)\right]^{1/q}.
\end{aligned}$$

The obtained constant factor is difficult to compare with the one proposed in [18, Theorem 4], including the fact that the weighted L_p and L_q integral norms of f and g are different, at least for $p \neq 2$. To understand the situation better, let us concentrate on the case $p = 2$, where these norms are equal. The new constant factor is then

$$\Omega_{\beta,\lambda,p} = B_{\text{eta}}\left(\gamma, \frac{1-\gamma}{2}+4\gamma\right) + B_{\text{eta}}\left(\gamma, \frac{1-\gamma}{2}\right).$$

In this particular case, after some numerical tests, the new constant is better than that in [18, Theorem 5] for small values of γ, i.e., $\gamma \in (0, 0.487)$.

As in Theorem 3.1, the parameters β and λ are independent. Theorem 4.1 can therefore be adapted to different mathematical scenarios, unlike [18, Theorem 4]. Also, the parameter p plays a crucial role in the weight function of the weighted

L_p and L_q integral norms, which define the upper bounds. Although these upper bounds have been obtained with the minimum number of intermediate inequalities, we have not proved that they are optimal due to the complexity of the integrands. In addition, the parameter λ must satisfy $\lambda \in (0,1)$, which, to draw a parallel, is a constraint that does not seem to appear in [18, Theorem 4]. These can be seen as limitations of the study.

The rest of the article is devoted to new results, which are derived from Theorems 3.1 and 4.1.

5. Additional Results

5.1. **New HH-type integral inequalities.** The proposition below provides a new HH-type integral inequality. The integrand goes beyond the ratio-power-minimum form; it innovates by mixing power, product, sum and minimum of the variables. The proof is based on Theorem 3.1, with some work on $\min(x/y, y/x)$ and a special parameter configuration.

Proposition 5.1. *Let $p>1$, $q=p/(p-1)$ so that $1/p+1/q=1$, and $f,g:(0,+\infty)\mapsto(0,+\infty)$ be two functions. Then, for any $\beta>0$, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(xy)^{\beta}}\left[\min(x,y)-\frac{xy}{x+y}\right]^{\beta}f(x)g(y)dxdy$$
$$\leq \Pi_{\beta,p}\left[\int_0^{+\infty}x^{(\beta-1)(1-p)}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{(\beta-1)(1-q)}g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side are finite, $\Pi_{\beta,p}$ is the constant given by

$$\Pi_{\beta,p}=\left[B_{eta}\left(\frac{\beta}{p},\frac{\beta}{q}\right)-\frac{q^2}{2^{\beta-1}\beta(1+2q)}\right]^{1/p}\left[B_{eta}\left(\frac{\beta}{p},\frac{\beta}{q}\right)-\frac{p^2}{2^{\beta-1}\beta(1+2p)}\right]^{1/q} \tag{5.1}$$

and $B_{eta}(a,b)$ is the beta function at $a,b>0$.

Proof. Using the analytical expression of the minimum, i.e., $\min(a,b)=(1/2)[a+b-|a-b|]$ for any $a,b\in\mathbb{R}$, and standard developments, for any $x,y>0$, we have

$$\min\left(\frac{x}{y},\frac{y}{x}\right)=\frac{1}{2}\left[\frac{x}{y}+\frac{y}{x}-\left|\frac{x}{y}-\frac{y}{x}\right|\right]=\frac{1}{2xy}\left[x^2+y^2-|x^2-y^2|\right]$$
$$=\frac{1}{2xy}\left[(x+y)^2-2xy-(x+y)|x-y|\right]$$
$$=\frac{1}{xy}\left\{\frac{1}{2}\left[x+y-|x-y|\right](x+y)-xy\right\}$$
$$=\frac{1}{xy}\left[\min(x,y)(x+y)-xy\right]=\frac{x+y}{xy}\left[\min(x,y)-\frac{xy}{x+y}\right],$$

so that

$$\frac{1}{xy}\left[\min(x,y)-\frac{xy}{x+y}\right]=\frac{1}{x+y}\min\left(\frac{x}{y},\frac{y}{x}\right).$$

This and Theorem 3.1 applied to $\lambda = \beta$ give

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(xy)^\beta}\left[\min(x,y)-\frac{xy}{x+y}\right]^\beta f(x)g(y)dxdy\\&=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y)^\beta}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)g(y)dxdy\\&\leq \Upsilon_{\beta,\lambda,p}\left[\int_0^{+\infty}x^{(\beta-1)(1-p)}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{(\beta-1)(1-q)}g^q(y)dy\right]^{1/q},\end{aligned}$$

where $\Upsilon_{\beta,\lambda,p}$ is given by Equation (3.1) with $\lambda = \beta$, which corresponds to the constant $\Pi_{\beta,p}$ given by Equation (5.1). This concludes the proof of Proposition 5.1. □

In the special case where $\beta = 1$, this new inequality reduces to

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left[\min(x,y)-\frac{xy}{x+y}\right]f(x)g(y)dxdy\\&\leq \Pi_{\beta,p}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},\end{aligned}$$

where, using $B_{\text{eta}}(1/p, 1/q) = \pi/\sin(\pi/p)$, $\Pi_{\beta,p}$ simplifies to

$$\Pi_{\beta,p}=\left[\frac{\pi}{\sin(\pi/p)}-\frac{q^2}{1+2q}\right]^{1/p}\left[\frac{\pi}{\sin(\pi/p)}-\frac{p^2}{1+2p}\right]^{1/q}.$$

Notice that the L_p and L_q integral norms of f and g are now unweighted. This is consistent with the framework of the classical HH integral inequality.

If we also set $p = 2$, the above result reduces to

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left[\min(x,y)-\frac{xy}{x+y}\right]f(x)g(y)dxdy\\&\leq\left(\pi-\frac{4}{5}\right)\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy},\end{aligned}$$

with $\pi - 4/5 \approx 2.3415$. To the best of our knowledge, there is no reference to this elegant integral inequality in the literature.

Analogous to the proposition above, the result below gives a HH-type integral inequality with an original integrand mixing power, product, absolute difference and minimum of the variables. The proof is mainly based on Theorem 4.1.

Proposition 5.2. *Let $p > 1$, $q = p/(p-1)$ so that $1/p + 1/q = 1$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then, for any $\beta \in (0, 1)$, we have*

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(xy)^\beta}\left[\frac{xy}{|x-y|}-\min(x,y)\right]^\beta f(x)g(y)dxdy\\&\leq \Xi_{\beta,p}\left[\int_0^{+\infty}x^{(\beta-1)(1-p)}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{(\beta-1)(1-q)}g^q(y)dy\right]^{1/q},\end{aligned}$$

where the integrals on the right-hand side are finite, $\Xi_{\beta,p}$ is the constant given by

$$\Xi_{\beta,p} = \left[B_{eta}\left(1-\beta, \frac{\beta}{q}+2\beta\right) + B_{eta}\left(1-\beta, \frac{\beta}{p}\right)\right]^{1/p} \times \left[B_{eta}\left(1-\beta, \frac{\beta}{p}+2\beta\right) + B_{eta}\left(1-\beta, \frac{\beta}{q}\right)\right]^{1/q}, \tag{5.2}$$

and $B_{eta}(a,b)$ is the beta function at $a, b > 0$.

Proof. Using the analytical expression of the minimum and standard developments, for any $x, y > 0$, we have

$$\begin{aligned}
\min\left(\frac{x}{y}, \frac{y}{x}\right) &= \frac{1}{2}\left[\frac{x}{y}+\frac{y}{x}-\left|\frac{x}{y}-\frac{y}{x}\right|\right] = \frac{1}{2xy}\left[x^2+y^2-|x^2-y^2|\right] \\
&= \frac{1}{2xy}\left[(x-y)^2+2xy-(x+y)|x-y|\right] \\
&= \frac{1}{xy}\left\{xy - \frac{1}{2}\left[x+y-|x-y|\right]|x-y|\right\} \\
&= \frac{1}{xy}\left[xy - \min(x,y)|x-y|\right] = \frac{|x-y|}{xy}\left[\frac{xy}{|x-y|} - \min(x,y)\right],
\end{aligned}$$

so that

$$\frac{1}{xy}\left[\frac{xy}{|x-y|} - \min(x,y)\right] = \frac{1}{|x-y|}\min\left(\frac{x}{y}, \frac{y}{x}\right).$$

This and Theorem 4.1 applied to $\lambda = \beta$ with the constraint $\beta \in (0,1)$ yield

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(xy)^\beta}\left[\frac{xy}{|x-y|} - \min(x,y)\right]^\beta f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{|x-y|^\beta}\left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x)g(y)dxdy \\
&\le \Omega_{\beta,\lambda,p}\left[\int_0^{+\infty} x^{(\beta-1)(1-p)} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{(\beta-1)(1-q)} g^q(y)dy\right]^{1/q},
\end{aligned}$$

where $\Omega_{\beta,\lambda,p}$ is given by Equation (4.1) with $\lambda = \beta$, which corresponds to the constant $\Xi_{\beta,p}$ given by Equation (5.2). This concludes the proof of Proposition 5.2. □

This proposition shows how Theorem 4.1 can be used as an intermediate result to establish new integral inequalities.

5.2. **Weighted integral norm inequalities.** This subsection contains two new weighted integral norm inequalities. The main tools for their proofs are again Theorems 3.1 and 4.1. The first proposition is given below, followed by the proof.

Proposition 5.3. *Let $p > 1$, $q = p/(p-1)$ so that $1/p + 1/q = 1$, and $f : (0,+\infty) \mapsto (0,+\infty)$ be a function. Then, for any $\beta \geq 0$ and $\lambda > 0$, we have*

$$\int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \left[\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)dx\right]^p dy$$
$$\leq \Upsilon^p_{\beta,\lambda,p} \int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx,$$

where the integrals on the right-hand side are finite, and $\Upsilon_{\beta,\lambda,p}$ is given by Equation (3.1).

Proof. To simplify the development, we set

$$\mathcal{G} = \int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \left[\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)dx\right]^p dy.$$

Using an appropriate product decomposition of the integrand and the FT integral theorem, we obtain

$$\mathcal{G} = \int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \left\{\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)dx\right\}^{p-1} \times$$
$$\left\{\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)dx\right\} dy$$
$$= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)g_\star(y)dxdy, \tag{5.3}$$

where

$$g_\star(y) = \left\{y^{-[\beta(2-q)+q-1-\lambda]}\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)dx\right\}^{p-1}.$$

It follows from Theorem 3.1 applied to the functions f and $g_\star$ that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)g_\star(y)dxdy$$
$$\leq \Upsilon_{\beta,\lambda,p}\left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g_\star^q(y)dy\right]^{1/q}, \tag{5.4}$$

where $\Upsilon_{\beta,\lambda,p}$ is given by Equation (3.1). Since $q(p-1)=p$, we get

$$\begin{aligned}
&\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g_\star^q(y)dy \\
&= \int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} \Bigg\{ y^{-[\beta(2-q)+q-1-\lambda]} \times \\
&\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)dx \Bigg\}^{q(p-1)} dy \\
&= \int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} \Bigg\{ y^{-[\beta(2-q)+q-1-\lambda]} \times \\
&\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)dx \Bigg\}^{p} dy \\
&= \int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \left[\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)dx\right]^p dy \\
&= \mathcal{G}. \qquad (5.5)
\end{aligned}$$

It follows from Equations (5.3), (5.4) and (5.5) that

$$\mathcal{G} \leq \Upsilon_{\beta,\lambda,p}\left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx\right]^{1/p} \mathcal{G}^{1/q}.$$

Since $1/p = 1-1/q$, this is equivalent to

$$\mathcal{G}^{1/p} = \mathcal{G}^{1-1/q} \leq \Upsilon_{\beta,\lambda,p}\left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx\right]^{1/p}.$$

Raising both sides to the exponent p yields

$$\begin{aligned}
&\int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \left[\int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)dx\right]^p dy \\
&= \mathcal{G} \leq \Upsilon_{\beta,\lambda,p}^p \int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx.
\end{aligned}$$

This concludes the proof of Proposition 5.3. □

This proposition is the starting point for the study of the integral operator given by, for any function f and $y>0$,

$$\mathbb{A}(f)(y) = \int_0^{+\infty} \frac{1}{(x+y)^\lambda}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^\beta f(x)dx.$$

In particular, it shows its continuity in terms of certain weighted L_p norms for it and f, i.e.,

$$\int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \mathbb{A}^p(f)(y)dy \leq \Upsilon_{\beta,\lambda,p}^p \int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx.$$

Analogous to the proposition above, the result below gives a new weighted integral norm inequality. Theorem 4.1 is central to the proof.

Proposition 5.4. *Let $p > 1$, $q = p/(p-1)$ so that $1/p + 1/q = 1$, and $f : (0, +\infty) \mapsto (0, +\infty)$ be a function. Then, for any $\beta \geq 0$ and $\lambda \in (0, 1)$, we have*

$$\int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \left[\int_0^{+\infty} \frac{1}{|x-y|^\lambda} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x)dx\right]^p dy$$
$$\leq \Omega^p_{\beta,\lambda,p} \int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx,$$

where the integrals on the right-hand side are finite, and $\Omega_{\beta,\lambda,p}$ is given by Equation (4.1).

Proof. We set

$$\mathcal{H} = \int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \left[\int_0^{+\infty} \frac{1}{|x-y|^\lambda} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x)dx\right]^p dy.$$

Using an appropriate product decomposition of the integrand and the FT integral theorem, we obtain

$$\mathcal{H} = \int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \left\{\int_0^{+\infty} \frac{1}{|x-y|^\lambda} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x)dx\right\}^{p-1} \times$$
$$\left\{\int_0^{+\infty} \frac{1}{|x-y|^\lambda} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x)dx\right\} dy$$
$$= \int_0^{+\infty} \int_0^{+\infty} \frac{1}{|x-y|^\lambda} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x) g_\ddagger(y) dxdy, \tag{5.6}$$

where

$$g_\ddagger(y) = \left\{y^{-[\beta(2-q)+q-1-\lambda]} \int_0^{+\infty} \frac{1}{|x-y|^\lambda} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x)dx\right\}^{p-1}.$$

It follows from Theorem 4.1 applied to the functions f and $g_\ddagger$ that

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{|x-y|^\lambda} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^\beta f(x) g_\ddagger(y) dxdy$$
$$\leq \Omega_{\beta,\lambda,p} \left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g^q_\ddagger(y)dy\right]^{1/q}, \tag{5.7}$$

where $\Omega_{\beta,\lambda,p}$ is given by Equation (4.1).

Since $q(p-1)=p$, we get

$$\begin{aligned}
&\int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} g_{\ddagger}^q(y)dy \\
&= \int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} \left\{ y^{-[\beta(2-q)+q-1-\lambda]} \times \right. \\
&\left. \int_0^{+\infty} \frac{1}{|x-y|^{\lambda}} \left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta} f(x)dx \right\}^{q(p-1)} dy \\
&= \int_0^{+\infty} y^{\beta(2-q)+q-1-\lambda} \left\{ y^{-[\beta(2-q)+q-1-\lambda]} \times \right. \\
&\left. \int_0^{+\infty} \frac{1}{|x-y|^{\lambda}} \left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta} f(x)dx \right\}^{p} dy \\
&= \int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \left[\int_0^{+\infty} \frac{1}{|x-y|^{\lambda}} \left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta} f(x)dx\right]^p dy \\
&= \mathcal{H}. \qquad (5.8)
\end{aligned}$$

It follows from Equations (5.6), (5.7) and (5.8) that

$$\mathcal{H} \le \Omega_{\beta,\lambda,p}\left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx\right]^{1/p} \mathcal{H}^{1/q}.$$

Since $1/p = 1-1/q$, this is equivalent to

$$\mathcal{H}^{1/p} = \mathcal{H}^{1-1/q} \le \Omega_{\beta,\lambda,p}\left[\int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx\right]^{1/p}.$$

Raising both sides to the exponent p yields

$$\begin{aligned}
&\int_0^{+\infty} y^{-[\beta(2-q)+q-1-\lambda](p-1)} \left[\int_0^{+\infty} \frac{1}{|x-y|^{\lambda}} \left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta} f(x)dx\right]^p dy \\
&= \mathcal{H} \le \Omega_{\beta,\lambda,p}^p \int_0^{+\infty} x^{\beta(2-p)+p-1-\lambda} f^p(x)dx.
\end{aligned}$$

This ends the proof of Proposition 5.4. $\square$

A direct consequence of this result is the continuity in special weighted L_p norms of the integral operator defined by, for any function f and $y>0$,

$$\mathbb{B}(f)(y) = \int_0^{+\infty} \frac{1}{|x-y|^{\lambda}} \left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\beta} f(x)dx.$$

Other applications can be found in operator theory.

6. Conclusion

HH-type integral inequalities have been studied extensively, resulting in several important contributions. Nevertheless, further innovation in this area remains valuable due to the wide range of potential applications in mathematical analysis. In this article, we have presented refinements and extensions of two previously published theorems, namely [18, Theorems 4 and 5], stated in Theorems 3.1 and 4.1. These results bring new flexibility to the structure of ratio-power-minimum integrands and improve the associated constant factors under some configurations. Such advances are made possible by the use of new and effective proof techniques. In addition, four propositions are given to complement the main theorems, some of which go beyond the setting of ratio-power-minimum integrands. A possible direction for future research could be the study of the optimality of the improved upper bounds. We also propose the study of HH-type integral inequalities in a three-variable framework. This would involve triple integrals of the following forms:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^{\lambda}}\left[\min\left(\frac{x}{y},\frac{y}{x},\frac{z}{y},\frac{y}{z},\frac{z}{x},\frac{x}{z}\right)\right]^{\beta}f(x)g(y)h(z)dxdydz,$$

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{|x-y+z|^{\lambda}}\left[\min\left(\frac{x}{y},\frac{y}{x},\frac{z}{y},\frac{y}{z},\frac{z}{x},\frac{x}{z}\right)\right]^{\beta}f(x)g(y)h(z)dxdydz$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{|x-y-z|^{\lambda}}\left[\min\left(\frac{x}{y},\frac{y}{x},\frac{z}{y},\frac{y}{z},\frac{z}{x},\frac{x}{z}\right)\right]^{\beta}f(x)g(y)h(z)dxdydz,$$

where h denotes an additional function related to the new variable z. We leave the detailed exploration of these extensions for a future article.

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