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FAMILY OF HYBRID PAIRS OF MAPPINGS AND THEIR FIXED POINT UNDER $\delta-$DISTANCES

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ABSTRACT. In this paper, we prove some results on coincidence and common fixed points for compatible as well as pointwise R-weakly commuting mappings satisfying a generalized contraction type condition on a complete metrically convex metric spaces which generalize relevant results due to C̀irìc and Ume, Khan and Imdad, Rhoades and others.

1. INTRODUCTION AND PRELIMINARIES

There have been many extensions and generalizations of well known fixed point theorems for multi-valued mappings. Among many researchers, Assad [1], Assad and Kirk [2], C̀irìc and Ume [3], Itoh [13] and others extended fixed point theorems to more general class of multi-valued mappings while Rhoades [21] obtained a generalization of Itoh's fixed point theorem for multi-valued mappings on a subset K of a metrically convex metric space X. In 1997, Huang and Cho [9] proved common fixed point theorem for sequence of nonself multi-valued mappings in metrically convex metric spaces. After this result, many authors adopted the same pattern and proved fixed point theorems for a sequence of multi-valued mappings to mention a few we cite $\{[4], [5], [10], [12]$ and $[15]...\}$ etc.

The purpose of this paper is to prove some coincidence and common fixed point theorems for a sequence of multi-valued and a pair of single valued nonself mappings using diametral distance (instead of Hausdorff distance) satisfying certain contraction type condition. The result of this paper generalizes and extends earlier results due to Rhoades [21], C̀irìc and Ume [3], Khan and Imdad [14] and others.

Now, we collect some relevant definitions and results.

Let (X, d) be a metric space. Then following Nadler [19], we recall

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(1) $CB(X) = \{A : A$ is nonempty closed and bounded subset of $X\}$.

(2) For nonempty subsets A, B of X and $x \in X$,

$$d(x, A) = \inf \{d(x, a) : a \in A\}, \; D(A, B) = \inf \{d(a, b) : a \in A, b \in B\},$$

$$H(A, B) = \max [\{\sup d(a, B) : a \in A\}, \{\sup d(A, b) : b \in B\}]$$

and $\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}$.

Notice that $D(A, B) \leq H(A, B) \leq \delta(A, B)$, it is well known (cf.Kuratowski [17]) that $CB(X)$ is a metric space with the distance function H which is known as Hausdorff-Pompeiu metric on X.

Definition 1.1. ([10]) Let K be a nonempty subset of a metric space (X, d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be pointwise R-weakly commuting on K if for given $x \in K$ and $Tx \in K$, there exists some $R = R(x) > 0$ such that

$$d(Ty, FTx) \leq R \; d(Tx, Fx) \text{ for each } y \in F(x) \cap K. \tag{1.1}$$

Moreover, the pair (F, T) will be called R-weakly commuting on K if for each $x \in K, Tx \in K$ and (**1.1**) holds for some $R > 0$.

If $R = 1$, we get the definition of weak commutativity of (F, T) on K due to Hadzic and Gajic [8]. If $F, T : X \to X$ then **Definition 1.1** reduces respectively to pointwise R-weak commutativity and R-weak commutativity for single valued self mappings due to Pant [20].

Definition 1.2. ([7],[8]) Let K be a nonempty subset of a metric space (X, d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be weakly commuting (cf.[8]) if for every $x, y \in K$ with $x \in Fy$ and $Ty \in K$, we have

$$d(Tx, FTy) \leq d(Ty, Fy),$$

whereas the pair (F, T) is said to be compatible (cf.[7]) if for every sequence $\{x_n\} \subset K$ and from the relation

$$\lim_{n \to \infty} d(Fx_n, Tx_n) = 0$$

and $Tx_n \in K$ (for every $n \in N$) it follows that $\lim_{n \to \infty} d(Ty_n, FTx_n) = 0$, for every sequence $\{y_n\} \subset K$ such that $y_n \in Fx_n, n \in N$.

For hybrid pairs of self type mappings these definitions were introduced by Kaneko and Sessa [16].

Definition 1.3. ([15]) A mapping $T : K \to X$ is said to be occasionally coincidentally idempotent w.r.t mapping $F : K \to CB(X)$, if there exists a point $z \in K$ such that T is idempotent at the coincidence points of the pair (F, T).

Definition 1.4. ([2]) A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 1.5. *([6]) Let $\{A_n\}$ and $\{B_n\}$ be two sequences in $CB(X)$ and converging in $CB(X)$ to the sets A and respectively B. Then*

$$\lim_{n\to\infty} \delta(A_n, B_n) = \delta(A, B).$$

2. Main results

Theorem 2.1. *Let (X, d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S, T : K \to X$ satisfying the conditions:*

1. *$\delta K \subseteq SK \cap TK, F_i(K) \cap K \subseteq SK, F_j(K) \cap K \subseteq TK$,*
2. *$Tx \in \delta K \Rightarrow F_i(x) \subseteq K, Sx \in \delta K \Rightarrow F_j(x) \subseteq K$,*
3. *(F_i, T) and (F_j, S) are compatible pairs,*
4. *$\{F_n\}, S$ and T are continuous on K, and*

$$\begin{aligned}\delta(F_i(x), F_j(y)) \leq\ & \alpha\ d(Tx, Sy) + \beta\ max\ \{d(Tx, F_i(x)), d(Sy, F_j(y))\}\\ & + \gamma\ \{d(Tx, F_j(y)) + d(Sy, F_i(x))\},\end{aligned} \tag{2.1}$$

where $i = 2n - 1, j = 2n,\ (n \in N), i \neq j$ for all $x, y \in K$ with $x \neq y$, $\alpha, \beta, \gamma \geq 0$ such that $\alpha + 2\beta + 3\gamma + \alpha\gamma < 1$.

Then $\{F_n\}, S$ and T have a common coincidence point.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x \in \delta K$. Then (due to $\delta K \subseteq TK$) there exists a point $x_0 \in K$ such that $x = Tx_0$. Since $Tx \in \delta K \Rightarrow F_i(x) \subseteq K$ for every odd integer $(i \in N)$, one concludes that $F_1(x_0) \subseteq F_1(K) \cap K \subseteq SK$. Let $x_1 \in K$ be such that $y_1 = Sx_1 \in F_1(x_0) \subseteq K$. Since $y_1 \in F_1(x_0)$, there exists a point $y_2 \in F_2(x_1)$. Suppose $y_2 \in K$, then $y_2 \in F_2(K) \cap K \subseteq TK$, implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin K$ then there exists a point $p \in \delta K$ such that

$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2).$$

Since $p \in \delta K \subseteq TK$, there exists a point $x_2 \in K$ with $p = Tx_2$ so that

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2).$$

Let $y_3 \in F_3(x_2)$. If $y_3 \in K$ then $y_3 \in F_3(K) \cap K \subseteq SK$ which implies that there exists a point $x_3 \in K$ such that $y_3 = Sx_3$. Otherwise if $y_3 \notin K$ there exists a point $p_1 \in \delta K$ such that

$$d(Tx_2, p_1) + d(p_1, y_3) = d(Tx_2, y_3).$$

Since $p_1 \in \delta K \subseteq SK$ there exists a point $x_3 \in K$ with $p_1 = Sx_3$ so that

$$d(Tx_2, Sx_3) + d(Sx_3, y_3) = d(Tx_2, y_3).$$

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(1) $y_{2n} \in F_{2n}(x_{2n-1}),\ y_{2n+1} \in F_{2n+1}(x_{2n})$,
(2) $y_{2n} \in K \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin K \Rightarrow Tx_{2n} \in \delta K$ and $d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n})$,
(3) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Sx_{2n+1} \in \delta K$ and
$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}).$$

We denote

$$P_\circ = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\},\ P_1 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\},$$
$$Q_\circ = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\} \text{ and}$$
$$Q_1 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}.$$

One can note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$ and $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now we distinguish the following three cases.

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_\circ$, then

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\leq \delta(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})) \\ &\leq \alpha\ d(Tx_{2n}, Sx_{2n-1}) + \beta\ \max\{d(Tx_{2n}, F_{2n+1}(x_{2n})), d(Sx_{2n-1}, F_{2n}(x_{2n-1}))\} \\ &\quad +\gamma\ \{d(Tx_{2n}, F_{2n}(x_{2n-1})) + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))\}, \\ &\leq \alpha\ d(y_{2n-1}, y_{2n}) + \beta\ \max\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} + \ \gamma\{d(y_{2n-1}, y_{2n+1})\}, \\ &\leq \alpha\ d(y_{2n-1}, y_{2n}) + \beta\ \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \\ &\quad +\gamma\ \{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\}, \end{aligned}$$

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) \leq (\alpha+\gamma) d(y_{2n-1}, y_{2n}) &+ \beta\ \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \\ &+\gamma\ d(y_{2n}, y_{2n+1}). \end{aligned} \tag{2.2}$$

If we suppose that $d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1})$ then we obtain
$$d(Tx_{2n}, Sx_{2n+1}) \leq (\alpha + \beta + 2\gamma) d(y_{2n}, y_{2n+1})$$
which is a contradiction. Otherwise from (**2.2**) we obtain
$$d(Tx_{2n}, Sx_{2n+1}) \leq (\alpha + \beta + \gamma) d(y_{2n}, y_{2n-1}) + \gamma d(y_{2n}, y_{2n+1})$$
which in turn yields
$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{\alpha + \beta + \gamma}{1-\gamma}\right) d(Sx_{2n-1}, Tx_{2n}).$$
Similarly if $(Sx_{2n-1}, Tx_{2n}) \in Q_\circ \times P_\circ$, then
$$d(Sx_{2n-1}, Tx_{2n}) \leq \left(\frac{\alpha + \beta + \gamma}{1-\gamma}\right) d(Sx_{2n-1}, Tx_{2n-2}).$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_1$ then
$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}),$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}),$$

and hence

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(y_{2n}, y_{2n+1}) \leq \delta(Fx_{2n}, Gx_{2n-1}).$$

Now, proceeding as in Case 1, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) d(Sx_{2n-1}, Tx_{2n}).$$

If $(Sx_{2n-1}, Tx_{2n}) \in Q_1 \times P_\circ$ then as earlier, we also obtain

$$d(Sx_{2n-1}, Tx_{2n}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) d(Sx_{2n-1}, Tx_{2n-2}).$$

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_\circ$ then $Sx_{2n-1} = y_{2n-1}$. Proceeding as in Case 1, we get

$$\begin{aligned}
d(Tx_{2n}, Sx_{2n+1}) = d(Tx_{2n}, y_{2n+1}) &\leq d(Tx_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1}),\\
&\leq d(Tx_{2n}, y_{2n}) + \delta(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})),\\
&\leq d(Tx_{2n}, y_{2n}) + \alpha\ d(Tx_{2n}, Sx_{2n-1}) + \beta\ \max\{d(Tx_{2n}, F_{2n+1}(x_{2n})),\\
&\quad d(Sx_{2n-1}, F_{2n}(x_{2n-1}))\} + \gamma\ \{d(Tx_{2n}, F_{2n}(x_{2n-1})) + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))\},\\
&\leq d(Tx_{2n}, y_{2n}) + \alpha\ d(Tx_{2n}, Sx_{2n-1}) + \beta\ \max\{d(Tx_{2n}, y_{2n+1}),\\
&\quad d(Sx_{2n-1}, y_{2n})\} + \gamma\ \{d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Sx_{2n+1})\}.
\end{aligned}$$

Since $\alpha < 1$ and $d(Tx_{2n}, y_{2n}) + d(Tx_{2n}, Sx_{2n-1}) = d(Sx_{2n-1}, y_{2n})$, we obtain

$$d(Tx_{2n}, y_{2n}) + \alpha d(Tx_{2n}, Sx_{2n-1}) \leq d(Sx_{2n-1}, y_{2n}).$$

Also, by the triangle inequality, we obtain

$$\begin{aligned}
d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Sx_{2n+1}) &\leq d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Sx_{2n+1})\\
&\leq d(Sx_{2n-1}, y_{2n}) + d(Tx_{2n}, Sx_{2n+1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
d(Tx_{2n}, Sx_{2n+1}) \leq\ & d(Sx_{2n-1}, y_{2n}) + \beta\ \max\{d(Tx_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}\\
& + \gamma\{d(Sx_{2n-1}, y_{2n}) + d(Tx_{2n}, y_{2n+1})\}.
\end{aligned}$$

If $d(Tx_{2n}, y_{2n+1}) \geq d(y_{2n-1}, y_{2n})$ then we obtain

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{1+\gamma}{1-\beta-\gamma}\right) d(Sx_{2n-1}, y_{2n}).$$

Otherwise, if $d(Tx_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$ then

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{1+\beta+\gamma}{1-\gamma}\right) d(Sx_{2n-1}, y_{2n}) = \left(\frac{1+\gamma}{1-\beta-\gamma}\right) d(Sx_{2n-1}, y_{2n}).$$

Now proceeding as earlier, one also obtains

$$d(Sx_{2n-1}, y_{2n}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) d(Sx_{2n-1}, Tx_{2n-2}).$$

Therefore combining the above inequalities, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq k\ d(Sx_{2n-1}, Tx_{2n-2}),\ \text{ where } k = \left(\frac{1+\gamma}{1-\beta-\gamma}\right)\left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right).$$

Thus in all the cases, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq k\ \max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\},$$

whereas

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq k\ \max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}.$$

Now on the lines of Assad and Kirk [2], it can be shown by induction that for $n \geq 1$, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq k^n\ \delta \text{ and } d(Sx_{2n+1}, Tx_{2n+2}) \leq k^{n+\frac{1}{2}}\ \delta,$$

whereas

$$\delta = k^{\frac{-1}{2}}\ \max\{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}.$$

Thus the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, ..., Sx_{2n-1}, Tx_{2n}, Sx_{2n+1}, ...\}$ is Cauchy and hence converges to the point z in X. Then as noted in [7] there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in $P_\circ$ or $Q_\circ$ respectively. Suppose that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in $P_\circ$ for each $k \in N$, also converges to z. Using compatibility of (F_j, S), we have

$$\lim_{k\to\infty} d(Sx_{2n_k-1}, F_j(x_{2n_k-1})) = 0 \text{ for any even integer } j \in N,$$

which implies that $\lim_{k\to\infty} d(STx_{2n_k}, F_j(Sx_{2n_k-1})) = 0.$

Using the continuity of S and F_j, one obtains $Sz \in F_j(z)$ for any even integer $j \in N$. Similarly the continuity of T and F_i implies $Tz \in F_i(z)$ for any odd integer $i \in N$. Now

$$d(Tz, Sz) \leq \delta(F_i(z), F_j(z))$$

$$\leq \alpha\, d(Tz, Sz) + \beta \max\{d(Tz, F_i(z)), d(Sz, F_j(z))\} + \gamma\{d(Tz, F_j(z)) + d(Sz, F_i(z))\},$$

$$\leq \alpha\ d(Tz, Sz) + 2\beta\ d(Tz, Sz),$$

implying thereby $Tz = Sz$. Thus z is a common coincidence point of F_n, S and T.

If one assumes that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in Q_0 then the foregoing arguments establish the earlier conclusions. This completes the proof.

□

Remark 2.2. If δ distance is replaced by Hausdorff distance (H) in the **Theorem 2.1**, then one deduces the **Theorem 3.2** due to Khan and Imdad [14].

Remark 2.3. **Theorem 2.1** remains true if we utilized the pointwise R-weak commutativity condition.

In the next theorem we use the closedness of TK and SK (or $F_i(K)$ and $F_j(K)$) to relax the continuity requirements besides minimizing the commutativity requirements to merely coincidence points.

Theorem 2.4. *Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S, T : K \to X$ satisfying (**2.1**), the conditions (**1**) and (**2**) of* **Theorem 2.1**. *Suppose that*

(1) *TK and SK are closed subspaces of X. Then*
(2) *(F_i, T) has a point of coincidence,*
(3) *(F_j, S) has a point of coincidence.*

Moreover, (F_i, T) has a common fixed point if T is occasionally coincidentally commuting and coincidentally idempotent w.r.t F_i whereas (F_j, S) has a common fixed point provided S is occasionally coincidentally commuting and coincidentally idempotent w.r.t F_j.

Proof. Following the lines of the proof of **Theorem 2.1**, we assume that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in $P_\circ$ and TK as well as SK are closed subspaces of X. Since $\{Tx_{2n_k}\}$ is Cauchy in TK, it converges to a point $u \in TK$. Let $v \in T^{-1}u$, then $Tv = u$. Since $\{Sx_{2n_k+1}\}$ is a subsequence of Cauchy sequence, $\{Sx_{2n_k+1}\}$ converges to u as well. Using (**2.1**) one can write

$$d(F_i(v), Tx_{2n_k}) \leq \delta(F_i(v), F_j(x_{2n_k-1}))$$
$$\leq \alpha\ d(Tv, Sx_{2n_k-1}) + \beta \max\{d(Sx_{2n_k-1}, F_j(x_{2n_k-1})), d(Tv, F_i(v))\}$$
$$+\gamma\ \{d(Tv, F_j(x_{2n_k-1})) + d(Sx_{2n_k-1}, F_i(v))\},$$

which on letting $k \to \infty$, reduces to

$$d(F_i(v), u) \leq \alpha\ 0 + \beta \max\{d(u, F_i(v)), 0\} + \gamma\ \{0 + d(F_i(v), u)\},$$
$$\leq (\beta + \gamma)\ d(u, F_i(v)),$$

yielding thereby $u \in F_i(v)$, which implies that $u = Tv \in F_i(v)$ as $F_i(v)$ is closed.

Since Cauchy sequence $\{Tx_{2n}\}$ converges to $u \in K$ and $u \in F_i(v)$, $u \in FK \cap K \subseteq SK$, there exists $w \in K$ such that $Sw = u$. Again using **(2.1)** we get

$$d(Sw, F_j(w)) = d(Tv, F_j(w)) \leq \delta(F_i(v), F_j(w))$$
$$\leq \alpha\, d(Tv, Sw) + \beta \max\{d(Tv, F_i(v)), d(Sw, F_j(w))\} + \gamma\ \{d(Tv, F_j(w)) + d(Sw, F_i(v))\}$$
$$\leq (\alpha + \gamma)\ d(Sw, F_j(w)),$$

implying thereby $Sw \in F_j(w)$, that is w is a coincidence point of (S, F_j).

In case $F_i(K)$ and $F_j(K)$ are closed subspaces, then $u \in F_i(K) \cap K \subseteq SK$ or $F_j(K) \cap K \subseteq TK$. The analogous arguments establish the desired conclusions. If one assumes that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in $Q_\circ$ with TK as well as SK are closed subspaces of X, then $\{Sx_{2n_k+1}\}$ is Cauchy in SK, the foregoing arguments establish that $Tz \in F_i(z)$ and $Sw \in F_j(w)$.

Since z is a coincidence point of (F_i, T) therefore using occasionally coincidentally commuting property of (F_i, T) and coincidentally idempotent property of T w.r.t F_i one can have

$$Tv \in F_i(v) \text{ and } u = Tv \Rightarrow Tu = TTv = Tv = u,$$

therefore $u = Tu = TTv \in TF_i(v) \subset F_i(Tv) = F_i(u)$, which shows that u is the common fixed point of (F_i, T). Similarly using the occasionally coincidentally commuting property of (F_j, S) and coincidentally idempotent property of S w.r.t F_j one can show that (F_j, S) has a common fixed point as well. This completes the proof. □

Remark 2.5. **Theorem 2.4** remains true if we substitute closedness of 'TK and SK' with closedness of '$F_i(K)$ and $F_j(K)'$.

Remark 2.6. By setting $F_n = F$ (for all $n \in N$) and $S = T = I_K$ in **Theorem 2.4** one deduces a multi-valued version of a result due to Rhoades [21].

Remark 2.7. By setting $F_n = F$ (for all $n \in N$) and $S = T = I_K$ in **Theorem 2.4** one deduces a multi-valued version of a result due to Itoh [13].

Remark 2.8. If we choose $F_i = F$ (for any odd integer $i \in N$) and $F_j = G$ (for any even integer $j \in N$) in **Theorem 2.4** one deduces the **Theorem 3.3** due to Khan and Imdad [14].

Remark 2.9. If we choose $F_i = F$ (for any odd integer $i \in N$), $F_j = G$ (for any even integer $j \in N$) and $S = T = I_K$ in **Theorem 2.4** one deduces a sharpened form of a result due to C̀iriǹ and Ume [3].

Remark 2.10. we can prove a theorem when 'closedness of K' is replaced by 'compactness of K'.

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