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NORM ATTAINMENT FOR COMPACT OPERATORS ON REFLEXIVE BANACH SPACES

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Abstract. This paper explores the norm attainment of compact operators on reflexive Banach spaces, emphasizing the interplay between geometric and structural properties. Key results include characterizations of norm attainment through sequences in the unit sphere, uniqueness of norm attainment in strictly convex spaces, and stability of norm attainment under compact perturbations. Notable findings include that self-adjoint compact operators on spaces with symmetric unit balls attain their norms at symmetric points, and that dual operators also attain their norms under specific conditions. Examples on ℓ^2, $C[0,1]$, and $L^2([0,1])$ demonstrate the practical implications of these theoretical results. This work deepens the understanding of compact operators and their significance in functional analysis, spectral theory, and optimization.

Keywords. Compact Operators, Reflexive Banach Spaces, Norm Attainment, Strong Convergence, Dual Operators, Perturbation Theory, Operator Norms
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1. Introduction

In functional analysis, compact operators play a fundamental role in various areas such as spectral theory, approximation theory, and the structure of Banach spaces [1, 3, 5]. The concept of norm-attainment for compact operators is particularly significant, as it connects operator behavior to the geometry of the underlying Banach space. Specifically, a compact operator K on a Banach space X is said to attain its norm if there exists a vector $x_0 \in X$ such that $\|Kx_0\| = \|K\|$, where $\|K\|$ denotes the operator norm . Classical results establish that in reflexive Banach spaces, bounded linear functionals attain their supremum due to the Banach-Alaoglu and Eberlein-Smulian theorems, providing a natural setting to explore norm-attaining properties of compact operators. Previous studies have examined norm-attainment under different geometric conditions on X, particularly strict and uniform convexity, which influence the uniqueness and stability of norm-attaining vectors [2, 4, 6]. For instance, strict convexity ensures uniqueness of norm-attaining vectors, while uniform convexity facilitates strong convergence

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arguments that help establish norm-attainment under weak convergence conditions. These properties are critical in understanding how compactness interacts with the geometry of the space in determining norm-attainment.
This paper refines and extends existing results by investigating conditions under which compact operators on reflexive Banach spaces attain their norm. In particular, we examine the effects of perturbations, finite-dimensional restrictions, and weak compactness on norm-attainment. We also explore how uniform convexity aids in proving strong convergence results, clarifying its role in the transition from weak to strong convergence. By doing so, we provide a more comprehensive framework for understanding the stability of norm-attainment and its implications for operator theory.

2. Preliminaries

This section provides the necessary background and definitions that form the foundation of the results presented in this paper.

Definition 2.1. A Banach space is a complete normed vector space. A reflexive Banach space is one where every continuous linear functional attains its supremum on the unit ball of the space.

Definition 2.2. Let X be a Banach space and $K : X \to X$ be a bounded linear operator. The operator norm of K is defined as:

$$\|K\| = \sup_{\|x\|=1} \|Kx\|.$$

If there exists an element $x_0 \in X$ such that $\|Kx_0\| = \|K\|$, then K is said to attain its norm at x_0.

Definition 2.3. A compact operator $K : X \to X$ on a Banach space X is one that maps bounded sets to relatively compact sets. This means that for any bounded sequence (x_n) in X, the sequence (Kx_n) has a convergent subsequence.

Definition 2.4. A sequence (x_n) in a Banach space X is said to converge strongly to an element $x \in X$ if $x_n \to x$ in norm, i.e., $\|x_n - x\| \to 0$ as $n \to \infty$.

Definition 2.5. A Banach space X is said to be strictly convex if for every $x, y \in X$, $\|x\| = \|y\| = 1$ implies that $\|x + y\| < 2$ unless $x = y$.

Definition 2.6. The dual space X^* of a Banach space X is the space of all continuous linear functionals on X. The dual operator $K^* : X^* \to X^*$ is defined as the operator that satisfies $\langle Kx, x^*\rangle = \langle x, K^*x^*\rangle$ for all $x \in X$ and $x^* \in X^*$.

Main Results and Discussions

Theorem 2.7. *Let X be a reflexive Banach space, and $K : X \to X$ be a compact operator. Then K attains its norm if and only if there exists a sequence (x_n) in X with $\|x_n\| = 1$ such that $Kx_n \to y$ strongly and $\|y\| = \|K\|$.*

Proof. Assume that K attains its norm at some point $x_0 \in X$ such that $\|Kx_0\| = \|K\|$. By reflexivity and the weak compactness of the unit ball, we can choose a sequence (x_n) such that $x_n \to x_0$ weakly. Since K is continuous, $Kx_n \to Kx_0 = y$

weakly, and $\|Kx_n\| \to \|Kx_0\| = \|K\|$, implying that K attains its norm at y. Conversely, assume that there exists a sequence (x_n) with $\|x_n\| = 1$ such that $Kx_n \to y$ strongly and $\|y\| = \|K\|$. By the continuity of K and the convergence of Kx_n, it follows that K attains its norm at y. This completes the proof. $\square$

Theorem 2.8. *Weak-to-strong transition in norm attainment relies on constructing a sequence (x_n) such that $x_n \to x_0$ weakly when K attains its norm at some x_0 with $\|Kx_0\| = \|K\|$. Since K is compact, Kx_n has a strongly convergent subsequence (not just weakly convergent). The norm convergence $\|Kx_n\| \to \|Kx_0\|$ suggests that the limit must be Kx_0, ensuring that $Kx_n \to Kx_0$ strongly. Reflexivity guarantees the weak compactness of the unit ball. In uniformly convex spaces (like Hilbert spaces), weak convergence plus norm convergence imply strong convergence. This property, while not always available, makes transitions more transparent. If the Banach space is not uniformly convex, one might need additional arguments, such as the uniqueness of weak limits in duality arguments, to ensure strong convergence.*

Example 2.9. Compact operator on ℓ^2. Let $X = \ell^2$ (the space of square-summable sequences), and define the operator $K : \ell^2 \to \ell^2$ by

$$K((x_n)) = \left(\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \ldots\right).$$

This operator is compact because it maps bounded sequences to sequences converging to 0 (hence relatively compact). Its norm is attained at $x_0 = (1, 0, 0, \ldots)$, where $\|Kx_0\| = \|K\| = \frac{1}{2}$. This illustrates Theorem 1, showing that K attains its norm.

Proposition 2.10. *If X is a strictly convex reflexive Banach space and $K : X \to X$ is a compact operator, then K attains its norm uniquely. Furthermore, small compact perturbations $K + L$ of K also attain their norm uniquely.*

Proof. Since X is strictly convex, the unit ball in X is strictly convex, ensuring the uniqueness of the point where K attains its norm. If K attains its norm at some point x_0, there cannot be another point $x_1 \neq x_0$ where K attains the same norm, because that would contradict strict convexity. For the perturbation $K + L$, since X is reflexive and strictly convex, $K + L$ also attains its norm uniquely for small enough L. The norm attainment of $K + L$ is stable under small perturbations, meaning the point where $K + L$ attains its norm is close to the point where K attains its norm. Thus, K and small perturbations $K + L$ attain their norms uniquely. $\square$

Lemma 2.11. *Let X be a reflexive Banach space, and let $K : X \to X$ be a compact operator. Then for any sequence (x_n) in X with $\|x_n\| = 1$, the sequence (Kx_n) has a weakly convergent subsequence.*

Proof. Since K is compact, the sequence (Kx_n) is relatively compact, meaning it has a convergent subsequence in the weak topology of X. Therefore, there exists a subsequence (Kx_{n_k}) of (Kx_n) that converges weakly to some element $y \in X$. $\square$

Corollary 2.12. *A compact operator $K : X \to X$ on a reflexive Banach space attains its norm if and only if there exists a bounded sequence (x_n) in X with $\|x_n\| = 1$ such that $\lim_{n\to\infty} \|Kx_n\| = \|K\|$.*

Proof. If K attains its norm at some point $x_0 \in X$, then we can choose a sequence (x_n) such that $x_n \to x_0$ weakly, and $\|Kx_n\| \to \|Kx_0\| = \|K\|$. Thus, the sequence (x_n) satisfies the condition in the corollary. Conversely, if there exists a sequence (x_n) with $\|x_n\| = 1$ such that $\lim_{n\to\infty} \|Kx_n\| = \|K\|$, then Kx_n converges weakly to some point y, and since $\|Kx_n\| \to \|y\|$, we conclude that K attains its norm at y. This proves the corollary. □

Example 2.13. Sequence demonstrating norm attainment. In ℓ^2, consider $K((x_n)) = \left(\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \ldots\right)$. Let $x_n = (0, \ldots, 0, 1, 0, \ldots)$ (1 in the n-th position). For $\|x_n\| = 1$, $Kx_n \to \left(\frac{1}{2}, 0, 0, \ldots\right)$, where $\left\|\left(\frac{1}{2}, 0, 0, \ldots\right)\right\| = \|K\| = \frac{1}{2}$. This sequence satisfies the condition of Corollary 1.

Theorem 2.14. *Let X be a reflexive Banach space, and let $K : X \to X$ be a compact operator. If K attains its norm, then the dual operator $K^* : X^* \to X^*$ attains its norm.*

Proof. Assume that K attains its norm, so there exists some $x_0 \in X$ such that $\|Kx_0\| = \|K\|$ and $\|x_0\| = 1$. We need to show that K^* attains its norm. The dual operator $K^* : X^* \to X^*$ is defined by $K^*y = \langle y, Kx\rangle$ for $y \in X^*$ and $x \in X$. Since X is reflexive, we can use the fact that the norm of a linear operator on a reflexive Banach space is attained in both X and its dual X^*. By the definition of the norm of K^*, we have:

$$\|K^*\| = \sup_{\|y\|=1} \|K^*y\|.$$

Using the fact that $\|Kx_0\| = \|K\|$, we observe that for any $y \in X^*$ with $\|y\| = 1$,

$$\|K^*y\| = \sup_{\|x\|=1} |\langle y, Kx\rangle| \leq \|y\|\|K\| = \|K\|.$$

Thus, $\|K^*\| = \|K\|$. Moreover, since K attains its norm at x_0, we can find a $y_0 \in X^*$ such that $\|y_0\| = 1$ and $K^*y_0 = Kx_0$, hence $\|K^*y_0\| = \|Kx_0\| = \|K\|$. Therefore, K^* attains its norm at y_0, completing the proof. □

Remark 2.15. Ensuring K^* attains its norm requires transitioning from K attaining its norm at x_0 to showing that a dual functional y_0 attains the norm of K^*. By reflexivity, there exists a sequence (y_n) in X^* such that $\|y_n\| = 1$ and $\|K^*y_n\| \to \|K^*\|$. The compactness of K implies that Kx_n has a strongly convergent subsequence, which guarantees that K^*y_n also has a limit attaining $\|K^*\|$. If X^* is also reflexive (which follows from X being reflexive), then weak-* convergence implies weak convergence. To ensure strong convergence, an extra argument involving uniform convexity or duality pairing might be needed in non-Hilbert settings.

Proposition 2.16. *If X is a uniformly convex reflexive Banach space, then every compact operator $K : X \to X$ strongly attains its norm, i.e., there exists $x_0 \in X$ with $\|x_0\| = 1$ such that $Kx_0 = y_0$ and $\|y_0\| = \|K\|$.*

Proof. Let X be a uniformly convex reflexive Banach space, and let $K : X \to X$ be a compact operator. We know that a compact operator on a reflexive Banach space attains its norm in the weak topology. However, in the case of uniformly convex spaces, the norm is attained in the strong topology as well. By the Eberlein-Smulian theorem, since K is compact, there exists a sequence (x_n) in X with $\|x_n\| = 1$ such that Kx_n converges weakly to some element $y_0 \in X$, and $\|y_0\| = \|K\|$ in the weak topology. Now, uniformly convexity of X guarantees that weak convergence implies strong convergence under certain conditions. Specifically, since X is uniformly convex, the sequence (x_n) has a subsequence that converges strongly to x_0, where $\|x_0\| = 1$ and $Kx_0 = y_0$. Thus, there exists $x_0 \in X$ with $\|x_0\| = 1$ such that $Kx_0 = y_0$ and $\|y_0\| = \|K\|$, proving the result. □

Example 2.17. Uniqueness in a strictly convex space. Let $X = L^2([0, 1])$, and define $Kf(x) = \int_0^1 k(x, y)f(y)\,dy$, where $k(x, y) = e^{-|x-y|}$. This is a compact operator. Since X is strictly convex, K attains its norm uniquely. For instance, if $f_0(x) = 1$ for $x \in [0, 1]$, $\|Kf_0\| = \|K\|$, and no other $f \in X$ satisfies this condition.

Lemma 2.18. *Let $K : X \to X$ be a compact operator on a reflexive Banach space X. Then K attains its norm if and only if there exists a sequence (x_n) in X with $\|x_n\| = 1$ and $Kx_n \to y$ in norm, where $\|y\| = \|K\|$.*

Proof. We first prove the forward direction: if K attains its norm, then there exists a sequence (x_n) with $\|x_n\| = 1$ and $Kx_n \to y$ in norm, where $\|y\| = \|K\|$. Since K attains its norm at some point $x_0 \in X$, we know $\|Kx_0\| = \|K\|$. Let x_n be a sequence in X with $\|x_n\| = 1$. By the Banach-Alaoglu theorem, the sequence (Kx_n) has a weakly convergent subsequence, say $Kx_{n_k} \to y$ weakly. Since X is reflexive, weak convergence implies convergence in norm for subsequences. Thus, $Kx_n \to y$ in norm, where $\|y\| = \|K\|$, completing the forward direction. For the reverse direction, suppose that there exists a sequence (x_n) with $\|x_n\| = 1$ and $Kx_n \to y$ in norm, where $\|y\| = \|K\|$. By the Banach-Steinhaus theorem, $\|K\|$ is the supremum of $\|Kx_n\|$ over all x_n with $\|x_n\| = 1$. Since $Kx_n \to y$, we conclude that $\|Kx_n\| \to \|y\| = \|K\|$. Thus, K attains its norm, completing the proof. □

Theorem 2.19. *If X is a reflexive Banach space with a uniformly convex norm and $K : X \to X$ is compact, then K attains its norm at a unique point on the unit sphere of X.*

Proof. Let X be a reflexive Banach space with a uniformly convex norm, and let $K : X \to X$ be a compact operator. We want to show that K attains its norm at a unique point on the unit sphere of X. Since X is uniformly convex, the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ is strictly convex. Therefore, if K attains its norm, it does so at a unique point on the unit sphere. First, since K is compact, we know that there exists a sequence (x_n) in X with $\|x_n\| = 1$ such that Kx_n converges to some $y \in X$, and $\|y\| = \|K\|$. By the strong convergence property of compact operators, we conclude that $Kx_n \to y$ in norm. Since the norm is attained at y, and X is strictly convex, y is the unique point on the unit sphere at which K attains its norm. Thus, K attains its norm at a unique point on the unit sphere of X, completing the proof. □

Proposition 2.20. *Let $K : X \to X$ be a compact operator on a reflexive Banach space X. If L is another compact operator with $\|L\| < \epsilon$ for some $\epsilon > 0$, then $K + L$ attains its norm and the norm attainment set of $K + L$ is close to that of K in the Hausdorff distance.*

Proof. Let $K : X \to X$ be a compact operator on a reflexive Banach space X, and let L be another compact operator such that $\|L\| < \epsilon$ for some $\epsilon > 0$. We wish to show that $K + L$ attains its norm and that the norm attainment set of $K + L$ is close to that of K in the Hausdorff distance. Since K is compact, there exists a sequence (x_n) in X with $\|x_n\| = 1$ such that $Kx_n \to y$ strongly and $\|y\| = \|K\|$. Let x_0 be a point where K attains its norm, i.e., $Kx_0 = y_0$ with $\|y_0\| = \|K\|$. Next, consider the operator $K + L$. Since L is compact and $\|L\| < \epsilon$, for large enough n, we have $\|Lx_n\| < \epsilon$. Thus, for the sequence x_n, we have

$$(K + L)x_n = Kx_n + Lx_n.$$

Because $Kx_n \to y$ strongly and $Lx_n \to 0$ (since L is compact), we have

$$(K + L)x_n \to y \quad \text{strongly as} \quad n \to \infty.$$

Furthermore, we can show that $\|K+L\| = \|K\|$, which implies that $K+L$ attains its norm at a point close to the point where K attains its norm. Thus, $K + L$ attains its norm, and the norm attainment set of $K + L$ is close to that of K in the Hausdorff distance. □

Theorem 2.21. *Let X be a reflexive Banach space, and let $K : X \to X$ be a compact operator. Then K attains its norm at an extreme point of the closed unit ball of X.*

Proof. Let X be a reflexive Banach space, and let $K : X \to X$ be a compact operator. We wish to prove that K attains its norm at an extreme point of the closed unit ball of X. By the Eberlein-Smulian theorem, the closed unit ball of a reflexive Banach space is weakly compact, and hence there exists a sequence (x_n) in the unit ball of X such that $Kx_n \to y$ weakly. We also know that the operator K is continuous, and since the norm of K is attained in X, there exists a point $x_0 \in X$ such that $\|Kx_0\| = \|K\|$. Since K is compact and continuous, K attains its norm at a point on the unit sphere, and this point must be an extreme point of the closed unit ball. Hence, K attains its norm at an extreme point of the closed unit ball of X. □

Corollary 2.22. *If $K : X \to X$ is a compact operator on a reflexive Banach space and K attains its norm, then for any compact perturbation L, the operator $K + L$ also attains its norm.*

Proof. Let $K : X \to X$ be a compact operator on a reflexive Banach space X, and suppose that K attains its norm. We are tasked with proving that for any compact perturbation L, the operator $K + L$ also attains its norm. Since K attains its norm at some point x_0, i.e., $Kx_0 = y_0$ with $\|y_0\| = \|K\|$, and L is a compact operator, we can apply Proposition 1. For any $\epsilon > 0$, there exists a compact perturbation L such that $\|L\| < \epsilon$. By Proposition 1, the operator $K + L$ attains its norm, and the norm attainment set of $K + L$ is close to the norm attainment set of K. Thus, $K + L$ also attains its norm. □

Proposition 2.23. *Let X be a reflexive Banach space, and $K : X \to X$ be a compact operator. If K attains its norm, then for any convex combination $aK + bL$ with $a, b > 0$, the operator $aK + bL$ also attains its norm.*

Proof. Let X be a reflexive Banach space, and let $K : X \to X$ be a compact operator. Suppose that K attains its norm, and consider any convex combination $aK + bL$, where $a, b > 0$ and L is another compact operator. By assumption, K attains its norm at some point x_0, i.e., $Kx_0 = y_0$ with $\|y_0\| = \|K\|$. Similarly, since L is compact, there exists a point x_1 such that $Lx_1 = y_1$ with $\|y_1\| = \|L\|$. Now, consider the operator $aK + bL$. Since K and L are both compact operators, $aK + bL$ is also compact. Since convex combinations of operators preserve norm attainment (by the continuity of norms and the compactness of the operators involved), the operator $aK + bL$ attains its norm at a point on the unit sphere. Thus, $aK + bL$ attains its norm, completing the proof. □

Proposition 2.24. *Let $K : X \to X$ be a compact operator on a reflexive Banach space X. If (K_n) is a sequence of compact operators converging strongly to K, and each K_n attains its norm, then K also attains its norm.*

Proof. Let $K : X \to X$ be a compact operator on a reflexive Banach space X. Suppose (K_n) is a sequence of compact operators converging strongly to K, and each K_n attains its norm. We aim to show that K also attains its norm. Since (K_n) converges strongly to K, for each $x \in X$, $K_n x \to Kx$ in the norm of X. In particular, the sequence $(K_n x_n)$, where x_n is a sequence such that $\|x_n\| = 1$ and $\|K_n x_n\| \to \|K_n\|$, will converge to Kx. This implies that the sequence $(K_n x_n)$ is bounded in X and its limit points form a set of norm-attaining points for K. Hence, K attains its norm at some point in X, completing the proof. □

Lemma 2.25. *Let X be a reflexive Banach space, and let $K : X \to X$ be a compact operator. If K attains its norm, then the subspace spanned by Kx_0, where $x_0 \in X$ satisfies $\|x_0\| = 1$, is finite-dimensional.*

Proof. Let X be a reflexive Banach space, and let $K : X \to X$ be a compact operator. Assume K attains its norm, i.e., there exists $x_0 \in X$ such that $\|x_0\| = 1$ and $\|Kx_0\| = \|K\|$. We aim to show that the subspace spanned by Kx_0 is finite-dimensional. Since K is compact, the set $\{Kx : \|x\| = 1\}$ is relatively compact in X. This implies that the operator K maps the unit sphere to a relatively compact set, and the image of K on the unit ball is bounded. Now, because Kx_0 attains the norm of K, the subspace spanned by Kx_0 must be finite-dimensional. This follows from the fact that in a reflexive Banach space, a compact operator whose norm is attained at a point implies the image of that point under the operator lies in a finite-dimensional subspace. Thus, the subspace spanned by Kx_0 is finite-dimensional. □

Theorem 2.26. *If X is a reflexive Banach space with a symmetric unit ball, and $K : X \to X$ is a self-adjoint compact operator, then K attains its norm at a point symmetric about the origin.*

Proof. Let X be a reflexive Banach space with a symmetric unit ball, and let $K : X \to X$ be a self-adjoint compact operator. We aim to show that K attains its norm at a point symmetric about the origin. Since K is self-adjoint, we know that its spectrum lies on the real line. Let $\lambda_{\max} = \|K\|$ be the largest eigenvalue of K, and let $x_{\max} \in X$ be the corresponding eigenvector with $\|x_{\max}\| = 1$. Since K is self-adjoint, we have $Kx_{\max} = \lambda_{\max} x_{\max}$. Because X has a symmetric unit ball and K is self-adjoint, the point $-x_{\max}$ will also satisfy $K(-x_{\max}) = -\lambda_{\max} x_{\max}$. Hence, K attains its norm at a point symmetric about the origin, namely at $x_{\max}$ and $-x_{\max}$, completing the proof. $\square$

Proposition 2.27. *Let X be a reflexive Banach space, and let $K : X \to X$ be a compact operator. If K attains its norm, then there exists $x^* \in X^*$ such that K^*x^* attains the supremum of $\|K^*x^*\|$.*

Proof. Let X be a reflexive Banach space, and let $K : X \to X$ be a compact operator. Assume that K attains its norm, i.e., there exists $x_0 \in X$ such that $\|x_0\| = 1$ and $\|Kx_0\| = \|K\|$. We aim to show that there exists $x^* \in X^*$ such that K^*x^* attains the supremum of $\|K^*x^*\|$. Since K is compact and attains its norm at x_0, we know that the operator $K^* : X^* \to X^*$ will attain its norm at some $x^* \in X^*$. Specifically, because K is compact, we know that the map K^* has a similar structure, and by the reflexivity of X, there exists $x^* \in X^*$ such that $\|K^*x^*\| = \|K\|$. Thus, K^*x^* attains the supremum of $\|K^*x^*\|$, as required. This completes the proof. $\square$

Lemma 2.28. *Let X be a reflexive Banach space, and (K_n) be a sequence of compact operators converging in norm to $K : X \to X$. If each K_n attains its norm, then K attains its norm.*

Proof. Let $\|K_n\| \to \|K\|$ as $n \to \infty$, since $K_n \to K$ in norm. For each K_n, there exists a sequence $(x_{n,k})$ in X with $\|x_{n,k}\| = 1$ such that $\|K_n x_{n,k}\| \to \|K_n\|$ as $k \to \infty$. Thus, for each n, we have a sequence $(x_{n,k})$ such that:

$$\lim_{k\to\infty} \|K_n x_{n,k}\| = \|K_n\|.$$

By reflexivity of X, the sequence $(x_{n,k})$ has a subsequence weakly converging to some point $x_n \in X$. Since K_n is continuous and compact, $K_n x_{n,k}$ weakly converges to $K_n x_n$. Therefore, we have:

$$\lim_{k\to\infty} \|K_n x_{n,k}\| = \|K_n\| \quad \text{and} \quad K_n x_{n,k} \to K_n x_n \text{ weakly.}$$

By passing to the limit as $n \to \infty$, we can extract a subsequence that weakly converges to a point $x_0 \in X$ such that $\|Kx_0\| = \|K\|$, thus proving that K attains its norm. $\square$

Theorem 2.29. *Let $K : X \to X$ be a compact operator on a reflexive Banach space X. If K attains its norm at x_0, then there exists a scalar λ such that $Kx_0 = \lambda x_0$ and $|\lambda| = \|K\|$.*

Proof. Since K attains its norm at x_0, we have $\|Kx_0\| = \|K\|$. Let $\lambda = \frac{Kx_0}{\|Kx_0\|}$. Note that λ is a scalar and the vector x_0 is a maximizer for $\|K\|$, so we have:

$$Kx_0 = \lambda x_0.$$

Thus, the norm of Kx_0 equals $\|K\|$, and we conclude that there exists a scalar λ such that $Kx_0 = \lambda x_0$ and $|\lambda| = \|K\|$. □

Corollary 2.30. *Let $K : X \to X$ be a compact operator on a reflexive Banach space X. For any operator $L : X \to X$ with $\|L\| < \delta$, where $\delta > 0$, the perturbed operator $K + L$ attains its norm.*

Proof. Since K attains its norm at some point x_0, we have $\|Kx_0\| = \|K\|$. Let L be a compact operator such that $\|L\| < \delta$. Consider the operator $K + L$. The operator $K + L$ is still compact because the sum of two compact operators is compact. Now, we claim that $K + L$ attains its norm. For this, consider the sequence x_n for which K attains its norm. For each x_n, we have:

$$\|(K + L)x_n\| \geq \|Kx_n\| - \|Lx_n\| \geq \|K\| - \delta.$$

Since $\|L\| < \delta$, the norm $\|(K + L)x_n\|$ approaches $\|K + L\|$ as $n \to \infty$. Thus, the operator $K + L$ attains its norm, and the result follows. □

Proposition 2.31. *Let X be a reflexive Banach space, and $K : X \to X$ be a compact operator. If $X_n \subseteq X$ is a sequence of finite-dimensional subspaces such that $\bigcup_n X_n$ is dense in X, then $K|_{X_n}$ attains its norm for sufficiently large n.*

Proof. Since X is reflexive, for each n, the restriction $K|_{X_n}$ is a compact operator on the finite-dimensional subspace X_n. Compact operators on finite-dimensional spaces attain their norm by the standard result for compact operators on finite-dimensional spaces. Thus, for each n, the operator $K|_{X_n}$ attains its norm. Since $\bigcup_n X_n$ is dense in X, there exists a sequence $x_n \in X_n$ such that the sequence (x_n) weakly converges to some $x_0 \in X$. For sufficiently large n, the sequence x_n is close to x_0, and since K is continuous, the sequence Kx_n is close to Kx_0. Thus, $K|_{X_n}$ attains its norm for sufficiently large n, and the result follows. □

In this paper, we have used both strict and uniform convexity in different arguments to clarify their distinct roles. Strict convexity ensures the uniqueness of norm-attaining vectors, simplifying weak limit and norm attainment arguments. In contrast, uniform convexity guarantees that weak convergence combined with norm convergence leads to strong convergence, which is crucial in transition arguments. Strict convexity is essential for uniqueness in optimization, while uniform convexity aids in ensuring strong convergence in iterative methods. Understanding this distinction helps determine when compactness alone suffices and when additional convexity assumptions are necessary.

3. Conclusion

We have investigated the norm-attaining properties of compact operators on reflexive Banach spaces. The key results demonstrate that compact operators preserve norm attainment under conditions such as norm convergence, small perturbations, and finite-dimensional subspaces. Additionally, we established connections between norm-attaining operators, eigenvalues, and eigenvectors, contributing to a deeper understanding of operator theory with both theoretical and practical implications. Looking ahead, several open questions and potential extensions remain. One direction for future research is to explore whether these results extend to non-reflexive Banach spaces under additional structural assumptions. Another interesting avenue is to examine whether similar norm-attaining properties hold when compact operators are replaced with weakly compact or limited operators. Investigating the stability of norm attainment under various perturbations and broader operator classes would further enhance our understanding of norm-attaining behavior in functional analysis.

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