

## GROUP EMBEDDING OF GENERALIZED BOL LOOPS

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**ABSTRACT.** In this paper, special embedding of generalized Bol loops in groups is obtained. It is found that if the generalized Bol loop is specially embeddable in a group, each loop isotopic to it is also specially embeddable in the group. A generalized Bol loop obtained from a special subset of a group is specially embeddable in the homomorphic image of the group. The universal embedding of generalized Bol loops into free groups on  $n$  generators is obtained.

### 1. INTRODUCTION AND PRELIMINARIES

In 1937 Gerrit Bol [?] introduced Bol loops in the context of web geometry wherein the loops that satisfy Bol and Moufang identities are shown to be related. Robinson [?] in his Ph.D. thesis investigated the algebraic properties of loops that satisfy; Bol, Moufang and Bruck identities. The isotopic and holomorphic theory of Bol loops were also developed and other notions on Bol loops among which is the special embedding of Bol loops in groups are also investigated. The theory of Bol loops is believed to have evolved through the thesis. Some later results on Bol loops can be found in [?, ?], [?, ?, ?, ?], [?, ?].

In the 1980s, many researchers among whom are Burn [?, ?, ?], Solarin and Sharma [?, ?, ?] became preoccupied by the construction of finite Bol loops and in the current era, such construction has caught the attention of Chein and Goodaire [?, ?, ?], Foguel et. al. [?], Kinyon and Phillips [?, ?]. One of the most important results in the theory of Bol loops is the solution of the open problem on the existence of a simple Bol loop which was finally laid to rest by Nagy [?, ?, ?].

In 1978, the algebraic properties of half-Bol loops was introduced and studied by Sharma and Sabinin [?, ?]. Also Ajimal (see [?, ?]) introduced and studied the algebraic properties of a dual notion of half-Bol loop called generalized Bol loops. Since the introduction of generalised Bol loops, several studies have been undertaken to reveal the structure and properties of generalized Bol loops. Some interesting results on generalized Bol loops can be found in [?, ?, ?, ?].

In the present study, we extend the notion of special embedding of Bol loops in groups [?] to generalized Bol loops. Thus, through this study, we are able

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to demonstrates how generalized Bol loops can be realized by introducing appropriate binary operation on a selected subset (special subset) of a group. It must however be noted that there is nothing unique about the group in which the loop is embedded. This realization is referred to as special embedding of generalized Bol loops in groups. The embedding is then examined relative to isotopes, subloops and homomorphic images. The study is concluded by looking at universal embedding of generalized Bol loops into free groups on  $n$  generators.

This section provides necessary background and some useful results needed throughout the study.

In what follows, all mappings are assumed to be single valued. If  $f$  is a mapping of set  $G$  into itself or some other sets and  $x \in G$ , then  $xf$  denotes the unique image of  $x$  under  $f$ .

**Definition 1.1.** A mapping  $f$  of a set  $G$  is a permutation of  $G$  if and only if  $f$  is one-to-one mapping of  $G$  onto  $G$ . If  $(G, \cdot)$  is a groupoid and  $x \in G$ , then mappings  $R_x$  and  $L_x$  are defined by  $yR_x = yx$  and  $yL_x = xy$  for all  $y \in G$ . The groupoid  $(G, \cdot)$  is a quasigroup if and only if  $(G, \cdot)$  is closed and for each  $x \in G$ ,  $R_x$  and  $L_x$  are permutation of  $G$ .

**Definition 1.2.** A loop  $(Q, \cdot)$  satisfying the identical relation

$$(xy \cdot z)y = x(yz \cdot y) \quad (1.1)$$

$\forall x, y, z \in Q$  is called a right Bol loop while a loop  $(Q, \cdot)$  satisfying the identical relation

$$y(z \cdot yx) = (y \cdot zy)x \quad (1.2)$$

$\forall x, y, z \in Q$  is called a left Bol loop.

Loops satisfying (1.1) and (1.2) are named after Gerit Bol [?], who showed that the identities (1.1) and (1.2) corresponds to certain configurations in 3-webs. Bol also showed that there is a duality between the identities (1.1) and (1.2).

**Definition 1.3.** A loop is a generalized Bol loop if it satisfies the relation

$$(xy \cdot z)y^\alpha = x(yz \cdot y^\alpha)$$

where  $y^\alpha$  is the image of  $y$  under some mapping  $\alpha$  of the loop onto itself.

The dual of the relation in Definition 1.3 is given by

$$(y^\alpha \cdot zy)x = y^\alpha(z \cdot yx)$$

and it is called half-Bol loops.

**Definition 1.4.** Let  $(L, \circ)$  and  $(G, \cdot)$  be two distinct loops and let  $U, V, W$  be a bijection  $(U, V, W) : (L, \circ) \longrightarrow (G, \cdot)$ . For all  $x, y \in G$ ,  $(U, V, W)$  is called a loop isotopism if and only if  $(x \circ y)W = xV \cdot yU$ . Then,  $(L, \circ)$  and  $(G, \cdot)$  are called loop isotope.

**Definition 1.5.** A group is an ordered pair of a set  $G$  and one binary operation on that set  $G$  such that

- (1) the operation is associative;
- (2) there is an identity element;

- (3) (in the multiplicative notation) every element  $x$  of  $G$  has an inverse (there is an element  $y$  of  $G$  such that  $xy = yx = 1$ ).

**Definition 1.6.** A subgroup of a group  $G$  (written multiplicatively) is a subset  $H$  of  $G$  such that

- (1)  $1 \in H$  ;
- (2)  $x \in H$  implies  $x^{-1} \in H$ ;
- (3)  $x, y \in H$  implies  $xy \in H$

**Definition 1.7.** A subgroup  $H$  of  $G$  is normal if  $ghg^{-1} \in H$  for each  $g \in G$  and  $h \in H$ . Every subgroup of an abelian group is normal. If  $f$  is a homomorphism from  $G$  to  $H$ , then the kernel of  $f$ ,  $\ker f = \{x \in G : (x)f = 1\} = f^{-1}(1)$  is easily seen to be a normal subgroup of  $G$ .

**Definition 1.8.** A homomorphism of a groupoid  $A$  into a groupoid  $B$  (written multiplicatively) is a mapping  $\phi$  of  $A$  into  $B$  such that  $(xy)\phi = (x)\phi(y)\phi$  for all  $x, y \in A$ .

In [?], it was proved that every free groups of a finite rank can be obtained as a multiplication group of a loop. This fact stated in the results below becomes our motivating factor to consider universal embedding of generalized Bol loops into free groups.

**Theorem 1.9** ([?], Theorem 1). *For every free group  $F$  of a finite rank  $n \geq 1$  there exists a commutative loop  $Q$  with multiplication group isomorphic to  $F$ .*

**Corollary 1.10** ([?], Corollary 2). *For every free group  $F$  there exists a loop  $Q$  with multiplication group isomorphic to  $F$ .*

## 2. SPECIAL EMBEDDING OF GENERALIZED BOL LOOPS

**Theorem 2.1.** *Let  $G$  be a multiplicative group with 1 as the identity element of  $G$ . Let  $H$  be a non empty subset of  $G$  such that  $\alpha : H \rightarrow H$  preserves inverse. If*

- i.  $H$  generates  $G$
- ii. *for each  $y^\alpha \in H$  there exists an automorphism  $\theta_{y^\alpha}$  of  $G$  such that  $((y^\alpha)^{-1}z)\theta_{y^\alpha}^{-1} \in H$  whenever  $z \in H$  (in particular  $1 \in H$ )*
- iii.  $\theta_1 = I$

*We define*

$$x \circ y = y \cdot x\theta_y \text{ for all } x, y \in H. \quad (2.1)$$

*Under the above assumptions,  $(H, \circ)$  is a generalized Bol loop if and only if*

- iv.  $x \circ y \in H$  whenever  $x, y \in H$
- v.  $\theta_{(y \circ z) \circ y^\alpha} = \theta_y \theta_z \theta_{y^\alpha}$   $y, z \in H$ .

*Proof.* From (??), it is evident that

$$[(x \circ y) \circ z] \circ y^\alpha = x \circ [(y \circ z) \circ y^\alpha] \text{ for all } x, y, z \in H, \quad (2.2)$$

if and only if  $x\theta_y \theta_z \theta_{y^\alpha} = x\theta_{(y \circ z) \circ y^\alpha}$  for all  $x, y, z \in H$ .

Since  $H$  generate  $G$  and each  $\theta_u, u \in H$ , is an automorphism of  $G$ , it follows that

$$x\theta_y\theta_z\theta y^\alpha = x\theta_{(y\circ z)\circ y^\alpha} \text{ holds for all } x, y, z \in H$$

if and only if

$$\theta_y\theta_z\theta y^\alpha = \theta_{(y\circ z)\circ y^\alpha} \text{ for all } x, y, z \in H.$$

is satisfied. Conversely, assume that i., ii., iii., iv. and v. hold. By ii.,  $1 \in H$ . For all  $x, y \in H$   $x \circ 1 = 1 \cdot x\theta_1 = x$  and  $1 \circ y = y \cdot 1\theta_y = y$ . so 1 is also identity element for  $(H, \circ)$ . For  $y, z \in H$   $((y^\alpha)^{-1}z)\theta_{y^\alpha}^{-1} \in H$  and  $((y^\alpha)^{-1}z)\theta_{y^\alpha}^{-1} \circ y^\alpha = y^\alpha \cdot ((y^\alpha)^{-1}z)\theta_{y^\alpha}^{-1} = z$ .

Moreover, for  $x, y, z \in H$ ,  $x \circ y^\alpha = z$  implies that

$$y^\alpha \cdot x\theta_{y^\alpha} = z \text{ and } x = ((y^\alpha)^{-1}z)\theta_{y^\alpha}^{-1}.$$

Thus, for  $y, z \in H$ , there exists a unique  $x \in H$  such that  $x \circ y^\alpha = z$ . Hence,  $(H, \circ)$  satisfies the right cancellation law. For  $x \in H$ , let  $x'$  be that unique element in  $H$  such that  $x' \circ x = 1$ . For  $x, y, z \in H$ , assume that  $x \circ y = x \circ z$ . Then

$$x' \circ [(x \circ y) \circ x^\alpha] = x' \circ [(x \circ z) \circ x^\alpha]$$

By (??)

$$[(x' \circ x) \circ y] \circ x^\alpha = [(x' \circ x) \circ z] \circ x^\alpha$$

So,  $y \circ x^\alpha = z \circ x^\alpha$  and by right cancellation law  $y = z$ . Thus, we have been able to show that  $(H, \circ)$  satisfy the left cancellation law.

Now, if  $x, z \in H$ , then  $x' \circ (z \circ x^\alpha) \in H$ , by iv., so there is a unique  $y \in H$  so that

$$y \circ x^\alpha = x' \circ (z \circ x^\alpha)$$

Then

$$x' \circ [(x \circ y) \circ x^\alpha] = x' \circ (z \circ x^\alpha).$$

and by (??),

$$x' \circ [(x \circ y) \circ x^\alpha] = x' \circ (z \circ x^\alpha)$$

By cancellation laws  $x \circ y = z$ . So for  $x, z \in H$  there is unique  $y \in H$  such that  $x \circ y = z$ .

Consequently,  $(H, \circ)$  is a generalized Bol loop.  $\square$

**Definition 2.2.** A generalized Bol loop  $(K, \cdot)$  is specially embeddable in a group  $G$  if and only if there exists a special subset  $H$  of  $G$  so that  $(K, \cdot)$  and  $(H, \circ)$  are isomorphic.

**Lemma 2.3.** Let  $(K, \cdot)$  be a generalized Bol loop and let  $G_\lambda$  be the group generated by all  $L_x$  for all  $x \in K$ . If  $G_{\lambda, \rho}$  is the group generated by all  $L_x, R_x$  for all  $x \in K$ , then  $G_\lambda$  is a normal subgroup of  $G_{\lambda, \rho}$ .

*Proof.* Since  $(K, \cdot)$  is a generalized Bol loop,

$$R_{y^\alpha} L_x R_{(y^\alpha)^{-1}} = L_{(y)^{-1}} L_{xy}$$

for all  $x, y \in K$ . That is  $R_{y^\alpha} L_x R_{(y^\alpha)^{-1}} \in G_\lambda$  for all  $x, y \in K$ .

Replacing  $y$  by  $y^{-1}$ , we have

$$R_{(y^\alpha)^{-1}} L_x R_{y^\alpha} = L_{(y^{-1})^{-1}} L_{(xy^{-1})} \in G_\lambda \forall x, y \in K \text{ since } R_{(y^{-1})} = R_{(y)^{-1}}$$

Taking inverses,

$$R_{y^\alpha} L_{(x)^{-1}} R_{(y^\alpha)^{-1}} \in G_\lambda$$

and

$$R_{(y^\alpha)^{-1}} L_x R_{y^\alpha} \in G_\lambda \quad \forall x, y \in K$$

Clearly,

$$L_{(y^\alpha)^{-1}} L_x^{\pm 1} L_{y^\alpha} \in G_\lambda$$

and

$$L_{y^\alpha} L_x^{\pm 1} L_{(y^\alpha)^{-1}} \in G_\lambda$$

for all  $x, y \in K$ . Hence  $G_\lambda$  is normal in  $G_{\lambda, \rho}$ .  $\square$

**Theorem 2.4.** *If  $(K, \cdot)$  is a generalized Bol loop then  $(K, \cdot)$  is specially embeddable in a group.*

*Proof.* Define  $G_\lambda$  and  $G_{\lambda, \rho}$  as in the preceding lemma. Let  $H = \{L_x | x \in K\}$  and let  $G = G_\lambda$ . For each  $L_x \in H$ , define  $\bar{\theta}_{L_x}$  by

$$Y \bar{\theta}_{L_x} = R_x Y R_{(x)^{-1}}$$

for all  $Y \in G_{\lambda, \rho}$ . Each  $\bar{\theta}_{L_x}$  is an inner automorphism of  $G_{\lambda, \rho}$ .

For each  $L_x \in H$ , let  $\theta_{L_x}$  be the restriction of  $\bar{\theta}_{L_x}$  to  $G$ . Clearly  $H$  generates  $G$  and in view of (??), each  $\theta_{L_x}$  is an automorphism of  $G$ . Since  $(K, \cdot)$  is a generalized Bol loop

$$L_{xy} = L_y R_{y^\alpha} L_x R_{(y^\alpha)^{-1}} \text{ for all } x, y \in K. \quad (2.3)$$

So,

$$[L_{(y)^{-1}} L_z] \theta_{L_y}^{-1} = R_{(y^\alpha)^{-1}} L_{(y)^{-1}} L_z R_{y^\alpha}$$

from (??) with  $x = y^{-1}$  we obtain,

$$L_{(y)^{-1}} = R_{y^\alpha} L_{(y^{-1})} R_{(y^\alpha)^{-1}} \text{ for all } y \in K.$$

Hence,

$$\begin{aligned} [L_{(y)^{-1}} L_z] \theta_{L_y}^{-1} &= L_{(y)^{-1}} R_{(y^\alpha)^{-1}} L_z R_{y^\alpha} \\ &= L_{(zy^{-1})} \in H \end{aligned}$$

Also,  $Y \theta_{L_1} = Y$  for all  $Y \in G$ . Define  $\phi$  by  $\phi(x) = L_x$  for all  $x \in K$ . Then  $\phi$  is a one-to-one mapping of  $K$  onto  $H$ . For  $x, y \in K$ ,

$$\begin{aligned} \phi(x) \circ \phi(y) &= L_x \circ L_y \\ &= L_y \cdot L_x \theta_{L_y} \\ &= L_y R_{y^\alpha} L_x R_{(y^\alpha)^{-1}} \\ &= L_{xy} = \phi(xy) \end{aligned}$$

So,  $\phi$  is an isomorphism of  $(K, \cdot)$  onto  $(H, \circ)$ . Thus  $(H, \circ)$  is a generalized Bol loop.  $\square$

**2.1. On isotope, subloop and homomorphic image of the special embedding.** In this subsection we treated the special embedding relative to isotope, subloop and homomorphic image of the generalized Bol loops.

**Theorem 2.5.** *If the generalized Bol loop  $(K, \cdot)$  is specially embeddable in a group  $G$ , then each loop isotopic to  $(K, \cdot)$  is also specially embeddable in  $G$ .*

*Proof.* Let  $H$  be that special subset of  $G$  such that  $(H, \circ)$  and  $(K, \cdot)$  are isomorphic. Let  $\alpha$  be a self map on  $H$  such that  $f^\alpha \in H$  and let  $L = H\theta_{f^\alpha}$ . For each  $x \in L$ , define  $\bar{\theta}_x$  by

$$\bar{\theta}_x = \theta_{f^\alpha}^{-1} \theta_{x\theta_{f^\alpha}^{-1} \circ f^\alpha}$$

Since for each  $x \in L$ ,  $x\theta_{f^\alpha}^{-1} \circ f^\alpha \in H$ ,  $\bar{\theta}_x$  is an automorphism of  $G$ . In a similar manner to the definition of “ $\circ$ ” in  $H$  define “ $*$ ” in  $L$  by

$$x * y = y \cdot x\bar{\theta}_y$$

for all  $x, y \in L$ . Now let  $(H, \oplus)$  be that principal isotope of  $(H, \circ)$  defined by

$$x \oplus y = xR_\circ(f^\alpha) \circ yL_\circ(f^\alpha)^{-1} \text{ for all } x, y \in H.$$

We shall prove that  $\theta_{f^\alpha}$  is an isomorphism of  $(H, \oplus)$  onto  $(L, *)$ . Let  $f^{\alpha(-1)}$  be the inverse of  $f^\alpha$  in the generalized Bol loop  $(H, \circ)$ . Since 1 is the identity for  $(H, \circ)$  as well as for the group  $G$ .

$$1 = f^\alpha \cdot f^{\alpha(-1)} \theta_{f^\alpha}$$

So,

$$f^{\alpha(-1)} = (f^\alpha)^{-1} \theta_{f^\alpha}^{-1}.$$

Also by (v),

$$\theta_{(f \circ f^{(-1)}) \circ f^\alpha} = \theta_f \theta_{f^{(-1)}} \theta_{f^\alpha}.$$

So,  $\theta_{f^{(-1)}} = \theta_f^{-1}$ . Thus, we have

$$f^{\alpha(-1)} = f^{-1} \theta_f^{-1}, \quad \theta_{f^{(-1)}} = \theta_f^{-1}$$

and using the fact that  $L_{(y)^{-1}} = R_{(y^\alpha)} L_{(y^{-1})} R_{(y^\alpha)^{-1}}$ , we have

$$(L_\circ)_{(f)^{-1}} = (R_\circ)_{(f^\alpha)} (L_\circ)_{(f^{(-1)})} (R_\circ)_{(f^\alpha)^{-1}}$$

Now for  $x, y \in H$ ,

$$\begin{aligned}
(x \oplus y)\theta_f &= [x(R_\circ)_{(f)} \circ y(L_\circ)_{(f)^{-1}}]\theta_f \\
&= [(x \circ f) \circ y(R_\circ)_{(f^\alpha)}(L_\circ)_{(f^{(-1)})}(R_\circ)_{(f^\alpha)^{(-1)}}]\theta_f \\
&= [(x \circ f) \circ \{[f^{(-1)} \circ (y \circ f^\alpha)] \circ (f^\alpha)^{(-1)}\}]\theta_f \\
&= [\{[(x \circ f) \circ f^{(-1)}] \circ (y \circ f^\alpha)\} \circ (f^\alpha)^{(-1)}]\theta_f \\
&= \{[x \circ (y \circ f^\alpha)] \circ (f^\alpha)^{(-1)}\}\theta_f \\
&= \{f^{\alpha(-1)} \cdot (f^\alpha \cdot y\theta_{f^\alpha} \cdot x\theta_{y \circ f^\alpha})\theta_{(f^\alpha)^{(-1)}}\}\theta_f \\
&= \{((f^\alpha)^{-1}\theta_{f^\alpha}^{-1})(f^\alpha \cdot y\theta_{f^\alpha} \cdot x\theta_{y \circ f^\alpha})\theta_{(f^\alpha)^{(-1)}}\}\theta_f \\
&= (f^\alpha)^{-1} \cdot f^\alpha \cdot y\theta_{f^\alpha} \cdot x\theta_{y \circ f^\alpha} \\
&= y\theta_{f^\alpha} \cdot x\theta_{y \circ f^\alpha}
\end{aligned}$$

But also, for  $x, y \in H$ ,

$$x\theta_{f^\alpha} * y\theta_{f^\alpha} = y\theta_{f^\alpha} \cdot (x\theta_{f^\alpha})\theta_y\theta_{f^\alpha} = y\theta_{f^\alpha} \cdot x\theta_{f^\alpha}\theta_{f^\alpha}^{-1}\theta_{y \circ f^\alpha} = y\theta_{f^\alpha} \cdot x\theta_{y \circ f^\alpha}$$

So,

$$(x \oplus y)\theta_{f^\alpha} = x\theta_{f^\alpha} * y\theta_{f^\alpha} \text{ for all } x, y \in H.$$

Since  $(L, *)$  is isomorphic to the isotope  $(H, \oplus)$  of  $(H, \circ)$ , it follows that  $(L, *)$  is also a generalized Bol loop and since  $L$  generates  $G$ ,  $L$  is a special subset of  $G$ .  $\square$

**Theorem 2.6.** *If a generalized Bol loop  $(K, \cdot)$  is specially embeddable in a group  $G$ , then each subloop of  $(K, \cdot)$  is specially embeddable in a subgroup of  $G$*

*Proof.* Let  $H$  be that special subset of  $G$  such that  $(H, \circ)$  and  $(K, \cdot)$  are isomorphic. Let  $\bar{H}$  be any subloop of  $(H, \circ)$  and let  $\bar{G}$  be the subgroup of  $G$  generated by  $\bar{H}$ . For each  $x^\alpha \in H$  and consequently, for each  $x^\alpha \in \bar{H}$ , there corresponds an automorphism  $\theta_{x^\alpha}$  of  $G$  so that (i), (ii), (iii), (iv) & (v) are satisfied. For  $y, z \in \bar{H}$ ,  $((y^\alpha)^{-1}z)\theta_{y^\alpha}^{-1} \circ y^\alpha = z$  and thus  $((y^\alpha)^{-1}z)\theta_{y^\alpha}^{-1} \in \bar{H}$ . Hence, to prove that  $\bar{H}$  is a special subset of  $\bar{G}$  it suffices to show that  $\bar{G}\theta_{x^\alpha} = \bar{H}$  for each  $x^\alpha \in \bar{H}$ .

For  $x, y \in \bar{H}$ , let  $\alpha : \bar{H} \longrightarrow \bar{H}$ , we have  $y\theta_{x^\alpha} = (x^\alpha)^{-1}(y \circ x^\alpha)$ . Since,  $\bar{H}$  is a subloop of  $(H, \circ)$ ,  $y \circ x^\alpha \in H$ . So,  $y\theta_{x^\alpha} \in \bar{G}$  for all  $x, y \in H$ . Also, for  $x, y \in \bar{H}$ ,  $y^{-1}\theta_{x^\alpha} = (y\theta_{x^\alpha})^{-1} \in \bar{G}$ . Now let  $z \in \bar{G}$ . The  $z = y_1^{\epsilon_1}y_2^{\epsilon_2}\cdots y_n^{\epsilon_n}$  where  $y_i \in \bar{H}$  and  $\epsilon_i = 1$  or  $-1$ . Let  $x^\alpha \in \bar{H}$ , then

$$z\theta_{x^\alpha} = y_1^{\epsilon_1}\theta_{x^\alpha}, y_2^{\epsilon_2}\theta_{x^\alpha}, \dots, y_n^{\epsilon_n}\theta_{x^\alpha}$$

But, each  $y_i^{\epsilon_i}\theta_{x^\alpha} \in \bar{G}$ . So,  $\bar{G}\theta_{x^\alpha} \subseteq \bar{G}$  for all  $x^\alpha \in \bar{H}$ . For  $x^\alpha \in \bar{H}$ , let  $x^{\alpha(-1)}$  be the inverse of  $x^\alpha$  in the loop  $(\bar{H}, \circ)$ .

Recall that  $\theta_{x^{\alpha(-1)}} = \theta_{x^\alpha}^{-1}$ . So,  $\bar{G}\theta_{x^{\alpha(-1)}} \subseteq \bar{G}$  or  $\bar{G} \subseteq \bar{G}\theta_{x^\alpha}$ . Consequently,  $\bar{G}\theta_{x^\alpha} = \bar{G}$  for all  $x^\alpha \in \bar{H}$  and the proof is complete.  $\square$

**Theorem 2.7.** *Let  $H$  be a special subset of a group  $(G, \cdot)$ . If  $\theta$  is a homomorphism of the group  $(G, \cdot)$  onto the group  $G\theta$  with kernel  $K$  such that*

$$(1) \ x^\alpha = y^\alpha \text{ whenever } x^\alpha, y^\alpha \in H \text{ and } x^\alpha\beta = y^\alpha\theta$$

(2)  $k\theta_{y^\alpha} \subseteq K$  for all  $y^\alpha \in H$ ,

then the generalized Bol loop  $(H, \circ)$  is specially embeddable in  $G\theta$

*Proof.* Define a one-to-one map,  $\theta$ , of  $H$  onto  $H\theta$  to be restriction of  $\theta$  to  $H$ . Also, define  $\bar{\theta}_{y^\alpha\theta}$  for  $y^\alpha \in H$  by

$$(x^\alpha\theta)\bar{\theta}_{y^\alpha\theta} = (x^\alpha\theta_{y^\alpha})\theta \text{ for all } x^\alpha \in G.$$

Before we proceed, we shall establish that each  $\bar{\theta}_{y^\alpha\theta}$ ,  $y^\alpha \in H$  is a well defined map of  $G\theta$  into  $G\theta$ . Now suppose that  $x^\alpha\theta = z^\alpha\theta$  for  $x, z \in G$ . Then set  $z^\alpha = x^\alpha \cdot k$  for some  $k \in K$  and

$$\begin{aligned} (z^\alpha\theta)\bar{\theta}_{y^\alpha\theta} &= (z^\alpha\theta_{y^\alpha})\theta = ((x^\alpha \cdot k)\theta_{y^\alpha})\theta = (x^\alpha\theta_{y^\alpha} \cdot k\theta_{y^\alpha})\theta \\ &= (x^\alpha\theta_{y^\alpha})\theta \cdot (k\theta_{y^\alpha})\theta \\ &= (x^\alpha\theta_{y^\alpha})\theta \text{ (Since } k\theta_{y^\alpha} \in K \text{ by 2 of Theorem ??)} \\ &= (x^\alpha\theta)\bar{\theta}_{y^\alpha\theta} \end{aligned}$$

Hence,  $\bar{\theta}_{y^\alpha\theta}$  is well-defined for each  $y^\alpha \in H$ .

Now, let  $x^\alpha, z^\alpha \in G$  and  $y^\alpha \in H$ ,

$$\begin{aligned} (x^\alpha\theta \cdot z^\alpha\theta)\bar{\theta}_{y^\alpha\theta} &= ((x^\alpha \cdot z^\alpha)\theta)\bar{\theta}_{y^\alpha\theta} = ((x^\alpha \cdot z^\alpha)\theta_{y^\alpha})\theta \\ &= (x^\alpha\theta_{y^\alpha} \cdot z^\alpha\theta_{y^\alpha})\theta = (x^\alpha\theta_{y^\alpha})\theta \cdot (z^\alpha\theta_{y^\alpha})\theta \\ &= (x^\alpha\theta)\bar{\theta}_{y^\alpha\theta} \cdot (z^\alpha\theta)\bar{\theta}_{y^\alpha\theta}. \end{aligned}$$

Thus, for each  $y^\alpha \in H$ , it follows that  $\bar{\theta}_{y^\alpha\theta}$  is a self homomorphism of the group  $G\theta$ .

Also for  $x^\alpha \in G$  and  $y^\alpha \in H$  observe that

$$((x^\alpha\theta_{(y^\alpha)^{-1}})\theta)\bar{\theta}_{y^\alpha\theta} = (x^\alpha\theta_{(y^\alpha)^{-1}}\theta_{y^\alpha})\theta = x^\alpha\theta.$$

Thus,  $\bar{\theta}_{y^\alpha\theta}$  maps  $G\theta$  onto  $G\theta$ . For  $x^\alpha, y^\alpha \in H$  define  $x^\alpha\theta \bar{\circ} y^\alpha\theta = y^\alpha\theta \cdot (x^\alpha\theta\bar{\theta}_{y^\alpha\theta})$  and observe that

$$\begin{aligned} (x^\alpha \circ y^\alpha) &= (y^\alpha \cdot x^\alpha\theta_{y^\alpha})\theta = y^\alpha\theta \cdot (x^\alpha\theta_{y^\alpha})\theta \\ &= y^\alpha\theta \cdot (x^\alpha\theta)\bar{\theta}_{y^\alpha\theta} = x^\alpha\theta \bar{\circ} y^\alpha\theta \end{aligned}$$

Thus  $(H, \circ)$  and  $(H\theta, \bar{\circ})$  are isomorphic generalized Bol loops. Since  $H\beta$  generates  $G\theta$  implies that  $H\theta$  is a special subset of  $G\theta$ ,  $(H, \circ)$  is specially embeddable in  $G\theta$ .  $\square$

## 2.2. Universal embedding of generalized Bol loops into free groups.

Motivated by Corollary 2 of [?] stated as Corollary 2.1 above, we are poised to investigate the universal embedding of generalized Bol loops into free groups.

**Theorem 2.8.** *A generalized Bol loops of order  $n+1$  is specially embeddable into free group on  $n$  generators.*

*Proof.* Let  $(K, \cdot)$  be a generalized Bol loop of order  $n+1$ . Let  $K = \{a_0, a_1, \dots, a_n\}$  with  $a_0$  denoting the identity element of  $(K, \cdot)$ . Now let  $G$  be the free group on  $n$  free generators  $x_1, x_2, \dots, x_n$  and let  $x_0$  be the identity element of  $G$ . Now let

$H = \{x_0, x_1, \dots, x_n\}$ , clearly  $H$  generates  $G$ . Define a binary operation “ $\circ$ ” on  $H$  by

$$x_i \circ x_j^\alpha = x_k \text{ if and only if } a_i \circ a_j^\alpha = a_k$$

for all  $i, j, k = 0, 1, \dots, n$ . Then  $(H, \circ)$  is a generalized Bol loop isomorphic to  $(K, \cdot)$ . For each fixed integer  $j$ ,  $0 \leq j \leq n$ , let us define a mapping  $\pi_j$  by

$$x_i \pi_j = x_i \circ x_j^\alpha$$

for  $i = 0, 1, \dots, n$ . Since  $(H, \circ)$  is a loop,  $\pi_j$  is a permutation of  $H$ .

We shall show that for each  $x_j^\alpha \in H$ , there exists an automorphism  $\theta_{x_j^\alpha}$  of  $G$  such that

$$x_i \theta_{x_j^\alpha} = (x_j^\alpha)^{-1} (x_i \pi_j) \text{ for all } x_i \in H \quad (2.4)$$

Then we shall have  $x_i \circ x_j^\alpha = x_j^\alpha \cdot x_i \theta_{x_j^\alpha}$  for all  $x_i, x_j^\alpha \in H$  and it follows that  $H$  is a special subset of  $G$ . Let  $\theta_{x_0} = I$  and note that (??) with  $j = 0$  is satisfied for all  $x_i \in H$ .

Now for  $p, q = 1, 2, \dots, n$ ,  $p \neq q$ , there exists automorphisms  $P_{pq}, V_p, W_{pq}$  of  $G$  such that

$$P_{pq} : x_p \longrightarrow x_q, x_q \longrightarrow x_p, x_k \longrightarrow x_k \text{ for } k \neq p, q$$

$$V_p : x_p \longrightarrow x_p^{-1}, x_k \longrightarrow x_k \text{ for } k \neq p$$

$$W_{pq} : x_q \longrightarrow x_p x_q, x_k \longrightarrow x_k \text{ for } k \neq q$$

For  $x_j^\alpha \in H, j \neq 0$  of order  $m$  in  $(H, \circ)$ , let

$$x_{j_1}^\alpha = x_{x_j^\alpha}^\alpha \pi_j, x_{j_2}^\alpha = x_{x_{j_1}^\alpha}^\alpha \pi_j \cdots, x_{j_{m-1}}^\alpha = x_0.$$

Since  $\pi_j$  is a permutation on  $H - \{x_j^\alpha, x_{j_1}^\alpha, \dots, x_{j_{m-1}}^\alpha\}$ , there exists an automorphism of  $G$  of type  $P_{pq}$  such that  $x_j^\alpha \theta_j = x_{j_1}^\alpha, x_{j_1}^\alpha \theta_j = x_{j_2}^\alpha, \dots, x_{j_{m-3}}^\alpha \theta_j = x_{j_{m-2}}^\alpha, x_{j_{m-2}}^\alpha \theta_j = x_j^\alpha$  and  $y^\alpha \theta_j = y^\alpha \pi_j$  for all  $y^\alpha \in H - \{x_j^\alpha, x_{j_1}^\alpha, \dots, x_{j_{m-1}}^\alpha\}$ . Then  $\theta_j$  followed by  $n-1$  applications of automorphisms of type  $W_q (q \neq j)$  and application of  $V_j$  yields automorphism  $\theta_{x_j^\alpha}$  satisfying (??).  $\square$

Before we proceed, we provide a partial converse of Theorem (??) in the lemma stated below.

**Lemma 2.9.** *Let  $H$  be a special subset of a group  $G$  and let  $\theta$  be a homomorphism with kernel  $K$  of  $G$  onto the group  $G\theta$ . If  $H\theta$  is a special subset of  $G\theta$  and if the restriction of  $\theta$  to  $H$  is a homomorphism of the generalized Bol loop  $(H, \circ)$  onto the generalized Bol loop  $(H\theta, \circ)$ , then*

$$k\theta_{y^\alpha} \subseteq K \text{ for all } y^\alpha \in H.$$

*Proof.* for all  $x, y \in H$  and  $\alpha : H \longrightarrow H$ , we have

$$y^\alpha \theta \cdot x\theta_{y^\alpha} \theta = (y^\alpha \cdot x\theta_{y^\alpha}) \theta = (x \circ y^\alpha) \theta = x\theta \circ y^\alpha \theta = y^\alpha \theta \cdot x\theta\theta_{y^\alpha} \theta.$$

Consequently,  $x\theta_{y^\alpha} \theta = x\theta\theta_{y^\alpha} \theta$  for all  $x, y \in H$  and  $\alpha : H \longrightarrow H$ . Since  $H$  generates  $G$ ,

$$z\theta_{y^\alpha} \theta = z\theta\theta_{y^\alpha} \theta \text{ for all } z \in G \text{ and all } y^\alpha \in H. \quad (2.5)$$

From (??) with  $z \in K$ , we have

$$z\theta_{y^\alpha} \theta = x\theta\theta_{y^\alpha} \theta = 1\theta_{y^\alpha} \theta = 1 \text{ for all } y^\alpha \in H.$$

That is  $k\theta_{y^\alpha} \subseteq K$  for all  $y^\alpha \in H$ .  $\square$

We are now in a position to investigate universal embedding of generalized Bol loops into free groups. Before then, we define the notion of universal embedding.

**Definition 2.10.** A generalized Bol loop  $(K, \cdot)$  is universally embeddable in a group  $G$  if and only if the following two conditions are satisfied

- (1)  $(K, \cdot)$  is specially emdeddable in  $G$
- (2) If  $(K, \cdot)$  is specially embeddable in a group  $G'$ , then there is a homomorphism  $\theta$  of  $G$  onto  $G'$  such that 1. and 2. of Theorem (??) hold.

**Theorem 2.11.** *A generalized Bol loop of order  $n + 1$  is universally embeddable in the free group on  $n$  generators.*

*Proof.* Let  $(K, \cdot)$  be a generalized Bol loop of order  $n + 1$ , from Theorem (??), we see that condition (1.) of Definition (??) is satisfied.

Assume that  $(K, \cdot)$  is specially embeddable in a group  $G'$ , then there is a special subset  $H'$  of  $G'$  such that  $(H', \circ)$  and  $(K, \cdot)$  are isomorphic.

Let  $H' = \{a_0, a_1, a_2, \dots, a_n\}$  with  $a_0$  denoting the identity element of  $(H', \circ)$ . Now let  $G$  be free group on  $n$  free generators  $x_1, x_2, \dots, x_n$ . Let  $H = \{x_0, x_1, \dots, x_n\}$  where  $x_0$  is the identity element of  $G$ . Introduce the binary operation “ $\circ$ ” on  $H$  by

$$x_1 \circ x_j^\alpha = x_k \text{ if and only if } a_1 \circ a_j^\alpha = a_k \text{ for } i, j, k = 0, 1, \dots, n.$$

Then by the proof of Theorem (??),  $H$  is a special subset of  $G$ . Since  $G'$  is generated by  $a_1, a_2, \dots, a_n$ , there is a homomorphism  $\theta$  of  $G$  onto  $G'$  so that  $x_i\theta = a_i$  for  $i = 0, 1, \dots, n$ . Furthermore, the restriction of  $\theta$  to  $H$  is an isomorphism of  $(H, \circ)$  onto  $(H', \circ)$ . Thus by Lemma (??), condition 2. of Theorem (??) holds. Condition 1. of Theorem (??) is obvious.  $\square$

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