Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 5 (2022) 1-11

GROUP EMBEDDING OF GENERALIZED BOL LOOPS

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ABSTRACT. In this paper, special embedding of generalized Bol loops in groups is obtained. It is found that if the generalized Bol loop is specially embeddable in a group, each loop isotopic to it is also specially embeddable in the group. A generalized Bol loop obtained from a special subset of a group is specially embeddable in the homomorphic image of the group. The universal embedding of generalized Bol loops into free groups on n generators is obtained.

1. Introduction and preliminaries

In 1937 Gerrit Bol [6] introduced Bol loops in the context of web geometry wherein the loops that satisfy Bol and Moufang identities are shown to be related. Robinson [23] in his Ph.D. thesis investigated the algebraic properties of loops that satisfy; Bol, Moufang and Bruck identities. The isotopic and holomorphic theory of Bol loops were also developed and other notions on Bol loops among which is the special embedding of Bol loops in groups are also investigated. The theory of Bol loops is believed to have evolved through the thesis. Some later results on Bol loops can be found in [1, 3], [6, 7, 8, 9], [25, 26].

In the 1980s, many researchers among whom are Burn [10, 11, 12], Solarin and Sharma [29, 30, 31] became preoccupied by the construction of finite Bol loops and in the current era, such construction has caught the attention of Chein and Goodaire [13, 14, 15], Foguel at. al. [22], Kinyon and Phillips [17, 18]. One of the most important results in the theory of Bol loops is the solution of the open problem on the existence of a simple Bol loop which was finally laid to rest by Nagy [19, 20, 21].

In 1978, the algebraic properties of half-Bol loops was introduced and studied by Sharma and Sabinin [27, 28]. Also Ajimal (see [2, 4]) introduced and studied the algebraic properties of a dual notion of half-Bol loop called generalized Bol loops. Since the introduction of generalised Bol loops, several studies have been undertaken to reveal the structure and properties of generalized Bol loops. Some interesting results on generalized Bol loops can be found in [2, 3, 4, 5].

Date: Received: Dec 10, 2021; Accepted: Jan 8, 2022.

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2010 *Mathematics Subject Classification.* Primary 20N02; Secondary 20N05.

Key words and phrases. Generalized Bol loops, special embedding, universal embedding, homomorphic image.

In the present study, we extend the notion of special embedding of Bol loops in groups [24] to generalized Bol loops. Thus, through this study, we are able to demonstrates how generalized Bol loops can be realized by introducing appropriate binary operation on a selected subset (special subset) of a group. It must however be noted that there is nothing unique about the group in which the loop is embedded. This realization is referred to as special embedding of generalized Bol loops in groups. The embedding is then examined relative to isotopes, subloops and homomorphic images. The study is concluded by looking at universal embedding of generalized Bol loops into free groups on n generators.

This section provides necessary background and some useful results needed throughout the study.

In what follows, all mappings are assumed to be single valued. If f is a mapping of set G into itself or some other sets and $x \in G$, then xf denotes the unique image of x under f.

Definition 1.1. A mapping f of a set G is a permutation of G if and only if f is one-to-one mapping of G onto G. If $(G, \cdot)$ is a groupoid and $x \in G$, then mappings R_x and L_x are defined by $yR_x = yx$ and $yL_x = xy$ for all $y \in G$. The groupoid $(G, \cdot)$ is a quasigroup if and only if $(G, \cdot)$ is closed and for each $x \in G$, R_x and L_x are permutation of G.

Definition 1.2. A loop $(Q, .)$ satisfying the identical relation

$$(xy.z)y = x(yz.y) \tag{1.1}$$

$\forall\ x, y, z \in Q$ is called a right Bol loop while a loop $(Q, .)$ satisfying the identical relation

$$y(z.yx) = (y.zy)x \tag{1.2}$$

$\forall\ x, y, z \in Q$ is called a left Bol loop.

Loops satisfying (1.1) and (1.2) are named after Gerit Bol [6], who showed that the identities (1.1) and (1.2) corresponds to certain configurations in 3-webs. Bol also showed that there is a duality between the identities (1.1) and (1.2).

Definition 1.3. A loop is a generalized Bol loop if it satisfies the relation

$$(xy \cdot z)y^\alpha = x(yz \cdot y^\alpha)$$

where y^α is the image of y under some mapping α of the loop onto itself.

The dual of the relation in Defintion 1.3 is given by

$$(y^\alpha \cdot zy)x = y^\alpha(z \cdot yx)$$

and it is called half-Bol loops.

Definition 1.4. Let $(L, \circ)$ and $(G, \cdot)$ be two distinct loops and let U, V, W be a bijection $(U, V, W) : (L, \circ) \longrightarrow (G, \cdot)$. For all $x, y \in G$, (U, V, W) is called a loop isotopism if and only if $(x \circ y)W = xV \cdot yU$. Then, $(L, \circ)$ and $(G, \cdot)$ are called loop isotope.

Definition 1.5. A group is an ordered pair of a set G and one binary operation on that set G such that

(1) the operation is associative;
(2) there is an identity element;
(3) (in the multiplicative notation) every element x of G has an inverse (there is an element y of G such that $xy = yx = 1$).

Definition 1.6. A subgroup of a group G (written multiplicatively) is a subset H of G such that

(1) $1 \in H$;
(2) $x \in H$ implies $x^{-1} \in H$;
(3) $x, y \in H$ implies $xy \in H$

Definition 1.7. A subgroup H of G is normal if $ghg^{-1} \in H$ for each $g \in G$ and $h \in H$. Every subgroup of an abelian group is normal. If f is a homomorphism from G to H, then the kernel of f, $ker f = \{x \in G : (x)f = 1\} = f^{-1}(1)$ is easily seen to be a normal subgroup of G.

Definition 1.8. A homomorphism of a groupoid A into a groupoid B (written multiplicatively) is a mapping ϕ of A into B such that $(xy)\phi = (x)\phi(y)\phi$ for all $x, y \in A$.

In [16], it was proved that every free groups of a finite rank can be obtained as a multiplication group of a loop. This fact stated in the results below becomes our motivating factor to consider universal embedding of generalized Bol loops into free groups.

Theorem 1.9 ([16], Theorem 1)**.** *For every free group F of a finite rank $n \geq 1$ there exists a commutative loop Q with multiplication group isomorphic to F.*

Corollary 1.10 ([16], Corollary 2)**.** *For every free group F there exists a loop Q with multiplication group isomorphic to F.*

2. Special embedding of generalized Bol loops

Theorem 2.1. *Let G be a multiplicative group with 1 as the identity element of G. Let H be a non empty subset of G such that $\alpha : H \longrightarrow H$ preserves inverse. If*

i. *H generates G*
ii. *for each $y^\alpha \in H$ there exists an automorphism θ_{y^α} of G such that $((y^\alpha)^{-1}z)\theta_{y^\alpha}^{-1} \in H$ whenever $z \in H$ (in particular $1 \in H$)*
iii. *$\theta_1 = I$*

We define

$$x \circ y = y \cdot x\theta_y \; for \; all \; x, y \in \; H. \tag{2.1}$$

Under the above assumptions, $(H, \circ)$ is a generalized Bol loop if and only if

iv. $x \circ y \in H$ whenever $x, y \in H$
v. $\theta_{(y \circ z) \circ y^\alpha} = \theta_y \theta_z \theta_{y^\alpha}$ $y, z \in H$.

Proof. From (2.1), it is evident that

$$[(x \circ y) \circ z] \circ y^\alpha = x \circ [(y \circ z) \circ y^\alpha] \text{ for all } x, y, z \in H, \tag{2.2}$$

if and only if $x\theta_y\theta_z\theta y^\alpha = x\theta_{(y \circ z) \circ y^\alpha}$ for all $x, y, z \in H$.

Since H generate G and each $\theta_u, u \in H$, is an automorphism of G, it follows that

$$x\theta_y\theta_z\theta y^\alpha = x\theta_{(y\circ z)\circ y^\alpha} \text{ holds for all } x, y, z \in H$$

if and only if

$$\theta_y\theta_z\theta y^\alpha = \theta_{(y\circ z)\circ y^\alpha} \text{ for all } x, y, z \in H.$$

is satisfied. Conversely, assume that i., ii., iii., iv. and v. hold. By ii., $1 \in H$. For all $x, y \in H$ $x \circ 1 = 1 \cdot x\theta_1 = x$ and $1 \circ y = y \cdot 1\theta_y = y$. so 1 is also identity element for $(H, \circ)$. For $y, z \in H$ $((y^\alpha)^{-1}z)\theta^{-1}_{y^\alpha} \in H$ and $((y^\alpha)^{-1}z)\theta^{-1}_{y^\alpha} \circ y^\alpha = y^\alpha \cdot ((y^\alpha)^{-1}z)\theta^{-1}_{y^\alpha}\theta_{y^\alpha} = z$.

Moreover, for $x, y, z \in H$, $x \circ y^\alpha = z$ implies that

$$y^\alpha \cdot x\theta_{y^\alpha} = z \text{ and } x = ((y^\alpha)^{-1}z)\theta^{-1}_{y^\alpha}.$$

Thus, for $y, z \in H$, there exists a unique $x \in H$ such that $x \circ y^\alpha = z$. Hence, $(H, \circ)$ satisfies the right cancellation law. For $x \in H$, let x' be that unique element in H such that $x' \circ x = 1$. For $x, y, z \in H$, assume that $x \circ y = x \circ z$. Then

$$x' \circ [(x \circ y) \circ x^\alpha] = x' \circ [(x \circ z) \circ x^\alpha]$$

By (2.2)

$$[(x' \circ x) \circ y] \circ x^\alpha = [(x' \circ x) \circ z] \circ x^\alpha$$

So, $y \circ x^\alpha = z \circ x^\alpha$ and by right cancellation law $y = z$. Thus, we have been able to show that $(H, \circ)$ satisfy the left cancellation law.

Now, if $x, z \in H$, then $x' \circ (z \circ x^\alpha) \in H$, by iv., so there is a unique $y \in H$ so that

$$y \circ x^\alpha = x' \circ (z \circ x^\alpha)$$

Then

$$x' \circ [(x \circ y) \circ x^\alpha] = x' \circ (z \circ x^\alpha).$$

and by (2.2),

$$x' \circ [(x \circ y) \circ x^\alpha] = x' \circ (z \circ x^\alpha)$$

By cancellation laws $x \circ y = z$. So for $x, z \in H$ there is unique $y \in H$ such that $x \circ y = z$.

Consequently, $(H, \circ)$ is a generalized Bol loop. □

Definition 2.2. A generalized Bol loop $(K, \cdot)$ is specially embeddable in a group G if and only if there exists a special subset H of G so that $(K, \cdot)$ and $(H, \circ)$ are isomorphic.

Lemma 2.3. *Let $(K, \cdot)$ be a generalized Bol loop and let G_λ be the group generated by all L_x for all $x \in K$. If $G_{\lambda,\rho}$ is the group generated by all L_x, R_x for all $x \in K$, then G_λ is a normal subgroup of $G_{\lambda,\rho}$.*

Proof. Since $(K, \cdot)$ is a generalized Bol loop,

$$R_{y^\alpha}L_xR_{(y^\alpha)^{-1}} = L_{(y)^{-1}}L_{xy}$$

for all $x, y \in K$. That is $R_{y^\alpha}L_xR_{(y^\alpha)^{-1}} \in G_\lambda$ for all $x, y \in K$.

Replacing y by y^{-1}, we have

$$R_{(y^\alpha)^{-1}}L_xR_{y^\alpha} = L_{(y^{-1})^{-1}}L_{(xy^{-1})} \in G_\lambda \ \forall\ x, y \in K \text{ since } R_{(y^{-1})} = R_{(y)^{-1}}$$

Taking inverses,

$$R_{y^\alpha}L_{(x)^{-1}}R_{(y^\alpha)^{-1}} \in G_\lambda$$

and

$$R_{(y^\alpha)^{-1}}L_xR_{y^\alpha} \in G_\lambda \ \forall\ x, y \in K$$

Clearly,

$$L_{(y^\alpha)^{-1}}L_x^{\pm 1}L_{y^\alpha} \in G_\lambda$$

and

$$L_{y^\alpha}L_x^{\pm 1}L_{(y^\alpha)^{-1}} \in G_\lambda$$

for all $x, y \in K$. Hence G_λ is normal in $G_{\lambda,\rho}$. □

Theorem 2.4. *If $(K, \cdot)$ is a generalized Bol loop then $(K, \cdot)$ is specially embeddable in a group.*

Proof. Define G_λ and $G_{\lambda,\rho}$ as in the preceding lemma. Let $H = \{L_x | x \in K\}$ and let $G = G_\lambda$. For each $L_x \in H$, define $\bar{\theta}_{L_x}$ by

$$Y\bar{\theta}_{L_x} = R_xYR_{(x)^{-1}}$$

for all $Y \in G_{\lambda,\rho}$. Each $\bar{\theta}_{L_x}$ is an inner automorphism of $G_{\lambda,\rho}$.
For each $L_x \in H$, let θ_{L_x} be the restriction of $\bar{\theta}_{L_x}$ to G. Clearly H generates G and in view of (2.3), each θ_{L_x} is an automorphism of G. Since $(K, \cdot)$ is a generalized Bol loop

$$L_{xy} = L_yR_{y^\alpha}L_xR_{(y^\alpha)^{-1}} \text{ for all } x, y \in K. \tag{2.3}$$

So,

$$[L_{(y)^{-1}}L_z]\theta_{L_y}^{-1} = R_{(y^\alpha)^{-1}}L_{(y)^{-1}}L_zR_{y^\alpha}$$

from (2.3) with $x = y^{-1}$ we obtain,

$$L_{(y)^{-1}} = R_{y^\alpha}L_{(y^{-1})}R_{(y^\alpha)^{-1}} \text{ for all } y \in K.$$

Hence,

$$\begin{aligned}[L_{(y)^{-1}}L_z]\theta_{L_y}^{-1} &= L_{(y)^{-1}}R_{(y^\alpha)^{-1}}L_zR_{y^\alpha} \\ &= L_{(zy^{-1})} \in H\end{aligned}$$

Also, $Y\theta_{L_1} = Y$ for all $Y \in G$. Define ϕ by $\phi(x) = L_x$ for all $x \in K$. Then ϕ is a one-to-one mapping of K onto H. For $x, y \in K$,

$$\begin{aligned}\phi(x) \circ \phi(y) &= L_x \circ L_y \\ &= L_y \cdot L_x\theta_{L_y} \\ &= L_yR_{y^\alpha}L_xR_{(y^\alpha)^{-1}} \\ &= L_{xy} = \phi(xy)\end{aligned}$$

So, ϕ is an isomorphism of $(K, \cdot)$ onto $(H, \circ)$. Thus $(H, \circ)$ is a generalized Bol loop. □

2.1. **On isotope, subloop and homomorphic image of the special embedding.** In this subsection we treated the special embedding relative to isotope, subloop and homomorphic image of the generalized Bol loops.

Theorem 2.5. *If the generalized Bol loop $(K,\cdot)$ is specially embeddable in a group G, then each loop isotopic to $(K,\cdot)$ is also specially embeddable in G.*

Proof. Let H be that special subset of G such that $(H,\circ)$ and $(K,\cdot)$ are isomorphic. Let α be a self map on H such that $f^\alpha \in H$ and let $L = H\theta_{f^\alpha}$. For each $x \in L$, define $\bar{\theta}_x$ by

$$\bar{\theta}_x = \theta_{f^\alpha}^{-1}\theta_{x\theta_{f^\alpha}^{-1}\circ f^\alpha}$$

Since for each $x \in L$, $x\theta_{f^\alpha}^{-1} \circ f^\alpha \in H$, $\bar{\theta}_x$ is an automorphism of G. In a similar manner to the definition of " $\circ$ " in H define " $*$ " in L by

$$x * y = y \cdot x\bar{\theta}_y$$

for all $x, y \in L$. Now let $(H,\oplus)$ be that principal isotope of $(H,\circ)$ defined by

$$x \oplus y = xR_\circ(f^\alpha) \circ yL_\circ(f^\alpha)^{-1} \text{ for all } x, y \in H.$$

We shall prove that θ_{f^α} is an isomorphism of $(H,\oplus)$ onto $(L,*)$. Let $f^{\alpha(-1)}$ be the inverse of f^α in the generalized Bol loop $(H,\circ)$. Since 1 is the identity for $(H,\circ)$ as well as for the group G.

$$1 = f^\alpha \cdot f^{\alpha(-1)}\theta_{f^\alpha}$$

So,

$$f^{\alpha(-1)} = (f^\alpha)^{-1}\theta_{f^\alpha}^{-1}.$$

Also by (v),

$$\theta_{(f\circ f^{(-1)})\circ f^\alpha} = \theta_f\theta_{f^{(-1)}}\theta_{f^\alpha}.$$

So, $\theta_{f^{(-1)}} = \theta_f^{-1}$. Thus, we have

$$f^{\alpha(-1)} = f^{-1}\theta_f^{-1},\ \theta_{f^{(-1)}} = \theta_f^{-1}$$

and using the fact that $L_{(y)^{-1}} = R_{(y^\alpha)}L_{(y^{-1})}R_{(y^\alpha)^{-1}}$, we have

$$(L_\circ)_{(f)^{-1}} = (R_\circ)_{(f^\alpha)}(L_\circ)_{(f^{(-1)})}(R_\circ)_{(f^\alpha)^{(-1)}}$$

Now for $x, y \in H$,

$$\begin{aligned}
(x \oplus y)\theta_f &= [x(R_\circ)_{(f)} \circ y(L_\circ)_{(f)^{-1}}]\theta_f \\
&= [(x \circ f) \circ y(R_\circ)_{(f^\alpha)}(L_\circ)_{(f^{(-1)})}(R_\circ)_{(f^\alpha)^{(-1)}}]\theta_f \\
&= [(x \circ f) \circ \{[f^{(-1)} \circ (y \circ f^\alpha)] \circ (f^\alpha)^{(-1)}\}]\theta_f \\
&= [\{[(x \circ f) \circ f^{(-1)}] \circ (y \circ f^\alpha)\} \circ (f^\alpha)^{(-1)}]\theta_f \\
&= \{[x \circ (y \circ f^\alpha)] \circ (f^\alpha)^{(-1)}\}\theta_f \\
&= \{f^{\alpha(-1)} \cdot (f^\alpha \cdot y\theta_{f^\alpha} \cdot x\theta_{y \circ f^\alpha})\theta_{(f^\alpha)^{(-1)}}\}\theta_f \\
&= \{((f^\alpha)^{-1}\theta_{f^\alpha}^{-1})(f^\alpha \cdot y\theta_{f^\alpha} \cdot x\theta_{y \circ f^\alpha})\theta_{(f^\alpha)^{(-1)}}\}\theta_f \\
&= (f^\alpha)^{-1} \cdot f^\alpha \cdot y\theta_{f^\alpha} \cdot x\theta_{y \circ f^\alpha} \\
&= y\theta_{f^\alpha} \cdot x\theta_{y \circ f^\alpha}
\end{aligned}$$

But also, for $x, y \in H$,

$$x\theta_{f^\alpha} * y\theta_{f^\alpha} = y\theta_{f^\alpha} \cdot (x\theta_{f^\alpha})\theta_y\theta_{f^\alpha} = y\theta_{f^\alpha} \cdot x\theta_{f^\alpha}\theta_{f^\alpha}^{-1}\theta_{y \circ f^\alpha} = y\theta_{f^\alpha} \cdot x\theta_{y \circ f^\alpha}$$

So,

$$(x \oplus y)\theta_{f^\alpha} = x\theta_{f^\alpha} * y\theta_{f^\alpha} \text{ for all x, y} \in H.$$

Since $(L, *)$ is isomorphic to the isotope $(H, \oplus)$ of $(H, \circ)$, it follows that $(L, *)$ is also a generalized Bol loop and since L generates G, L is a special subset of G. □

Theorem 2.6. *If a generalized Bol loop $(K, \cdot)$ is specially embeddable in a group G, then each subloop of $(K, \cdot)$ is specially embeddable in a subgroup of G*

Proof. Let H be that special subset of G such that $(H, \circ)$ and $(K, \cdot)$ are isomorphic. Let $\bar{H}$ be any subloop of $(H, \circ)$ and let $\bar{G}$ be the subgroup of G generated by $\bar{H}$. For each $x^\alpha \in H$ and consequently, for each $x^\alpha \in \bar{H}$, there corresponds an automorphism θ_{x^α} of G so that $(i), (ii), (iii), (iv)$ & (v) are satisfied. For $y, z \in \bar{H}$, $((y^\alpha)^{-1}z)\theta_{y^\alpha}^{-1} \circ y^\alpha = z$ and thus $((y^\alpha)^{-1}z)\theta_{y^\alpha}^{-1} \in \bar{H}$. Hence, to prove that $\bar{H}$ is a special subset of $\bar{G}$ it suffices to show that $\bar{G}\theta_{x^\alpha} = \bar{H}$ for each $x^\alpha \in \bar{H}$.
For $x, y \in \bar{H}$, let $\alpha : \bar{H} \longrightarrow \bar{H}$, we have $y\theta_{x^\alpha} = (x^\alpha)^{-1}(y \circ x^\alpha)$. Since, $\bar{H}$ is a subloop of $(H, \circ)$, $y \circ x^\alpha \in H$. So, $y\theta_{x^\alpha} \in \bar{G}$ for all $x, y \in H$. Also, for $x, y \in \bar{H}$, $y^{-1}\theta_{x^\alpha} = (y\theta_{x^\alpha})^{-1} \in \bar{G}$. Now let $z \in \bar{G}$. The $z = y_1^{\epsilon_1}, y_2^{\epsilon_2}, \cdots, y_n^{\epsilon_n}$ where $y_i \in \bar{H}$ and $\epsilon_i = 1$ or -1. Let $x^\alpha \in \bar{H}$, then

$$z\theta_{x^\alpha} = y_1^{\epsilon_1}\theta_{x^\alpha}, y_2^{\epsilon_2}\theta_{x^\alpha}, \cdots, y_n^{\epsilon_n}\theta_{x^\alpha}$$

But, each $y_i^{\epsilon_i}\theta_{x^\alpha} \in \bar{G}$. So, $\bar{G}\theta_{x^\alpha} \subseteq \bar{G}$ for all $x^\alpha \in \bar{H}$. For $x^\alpha \in \bar{H}$, let $x^{\alpha(-1)}$ be the inverse of x^α in the loop $(\bar{H}, \circ)$.
Recall that $\theta_{x^{\alpha(-1)}} = \theta_{x^\alpha}^{-1}$. So, $\bar{G}\theta_{x^{\alpha(-1)}} \subseteq \bar{G}$ or $\bar{G} \subseteq \bar{G}\theta_{x^\alpha}$. Consequently, $\bar{G}\theta_{x^\alpha} = \bar{G}$ for all $x^\alpha \in \bar{H}$ and the proof is complete. □

Theorem 2.7. *Let H be a special subset of a group $(G, \cdot)$. If θ is a homomorphism of the group $(G, \cdot)$ onto the group $G\theta$ with kernel K such that*

(1) $x^\alpha = y^\alpha$ *whenever* $x^\alpha, y^\alpha \in H$ *and* $x^\alpha\beta = y^\alpha\theta$

(2) $k\theta_{y^\alpha} \subseteq K$ *for all* $y^\alpha \in H$,

then the generalized Bol loop $(H, \circ)$ *is specially embeddable in* $G\theta$

Proof. Define a one-to-one map, θ, of H onto $H\theta$ to be restriction of θ to H. Also, define $\bar{\theta}_{y^\alpha\theta}$ for $y^\alpha \in H$ by

$$(x^\alpha\theta)\bar{\theta}_{y^\alpha\theta} = (x^\alpha\theta_{y^\alpha})\theta \text{ for all } x^\alpha \in G.$$

Before we proceed, we shall establish that each $\bar{\theta}_{y^\alpha\theta}$, $y^\alpha \in H$ is a well defined map of $G\theta$ into $G\theta$. Now suppose that $x^\alpha\theta = z^\alpha\theta$ for $x, z \in G$. Then set $z^\alpha = x^\alpha \cdot k$ for some $k \in K$ and

$$\begin{aligned}(z^\alpha\theta)\bar{\theta}_{y^\alpha\theta} &= (z^\alpha\theta_{y^\alpha})\theta = ((x^\alpha \cdot k)\theta_{y^\alpha})\theta = (x^\alpha\theta_{y^\alpha} \cdot k\theta_{y^\alpha})\theta \\ &= (x^\alpha\theta_{y^\alpha})\theta \cdot (k\theta_{y^\alpha})\theta \\ &= (x^\alpha\theta_{y^\alpha})\theta \text{ (Since } k\theta_{y^\alpha} \in K \text{ by 2 of Theorem 2.7)} \\ &= (x^\alpha\theta)\bar{\theta}_{y^\alpha\theta}\end{aligned}$$

Hence, $\bar{\theta}_{y^\alpha\theta}$ is well-defined for each $y^\alpha \in H$.
Now, let $x^\alpha, z^\alpha \in G$ and $y^\alpha \in H$,

$$\begin{aligned}(x^\alpha\theta \cdot z^\alpha\theta)\bar{\theta}_{y^\alpha\theta} &= ((x^\alpha \cdot z^\alpha)\theta)\bar{\theta}_{y^\alpha\theta} = ((x^\alpha \cdot z^\alpha)\theta_{y^\alpha})\theta \\ &= (x^\alpha\theta_{y^\alpha} \cdot z^\alpha\theta_{y^\alpha})\theta = (x^\alpha\theta_{y^\alpha})\theta \cdot (z^\alpha\theta_{y^\alpha})\theta \\ &= (x^\alpha\theta)\bar{\theta}_{y^\alpha\theta} \cdot (z^\alpha\theta)\bar{\theta}_{y^\alpha\theta}.\end{aligned}$$

Thus, for each $y^\alpha \in H$, it follows that $\bar{\theta}_{y^\alpha\theta}$ is a self homomorphism of the group $G\theta$.
Also for $x^\alpha \in G$ and $y^\alpha \in H$ observe that

$$((x^\alpha\theta_{(y^\alpha)^{-1}})\theta)\bar{\theta}_{y^\alpha\theta} = (x^\alpha\theta_{(y^\alpha)^{-1}}\theta_{y^\alpha})\theta = x^\alpha\theta.$$

Thus, $\bar{\theta}_{y^\alpha\theta}$ maps $G\theta$ onto $G\theta$. For $x^\alpha, y^\alpha \in H$ define $x^\alpha\theta \;\bar{\circ}\; y^\alpha\theta = y^\alpha\theta \cdot (x^\alpha\theta\bar{\theta}_{y^\alpha\theta})$ and observe that

$$\begin{aligned}(x^\alpha \circ y^\alpha) &= (y^\alpha \cdot x^\alpha\theta_{y^\alpha})\theta = y^\alpha\theta \cdot (x^\alpha\theta_{y^\alpha})\theta \\ &= y^\alpha\theta \cdot (x^\alpha\theta)\bar{\theta}_{y^\alpha\theta} = x^\alpha\theta \;\bar{\circ}\; y^\alpha\theta\end{aligned}$$

Thus $(H, \circ)$ and $(H\theta, \bar{\circ})$ are isomorphic generalized Bol loops. Since $H\beta$ generates $G\theta$ implies that $H\theta$ is a special subset of $G\theta$, $(H, \circ)$ is specially embeddable in $G\theta$. □

2.2. **Universal embedding of generalized Bol loops into free groups.** Motivated by Corollary 2 of [16] stated as Corollary 2.1 above, we are poised to investigate the universal embedding of generalized Bol loops into free groups.

Theorem 2.8. *A generalized Bol loops of order* $n+1$ *is specially embeddable into free group on* n *generators.*

Proof. Let $(K, \cdot)$ be a generalized Bol loop of order $n+1$. Let $K = \{a_0, a_1, \cdots a_n\}$ with a_0 denoting the identity element of $(K, \cdot)$. Now let G be the free group on n free generators $x_1, x_2, \cdots, x_n$ and let x_0 be the identity element of G. Now let

$H = \{x_0, x_1, \cdots, x_n\}$, clearly H generates G. Define a binary operation "$\circ$" on H by

$$x_i \circ x_j^\alpha = x_k \text{ if and only if } a_i \circ a_j^\alpha = a_k$$

for all $i, j, k = 0, 1, \cdots, n$. Then $(H, \circ)$ is a generalized Bol loop isomorphic to $(K, \cdot)$. For each fixed integer j, $0 \leq j \leq n$, let us define a mapping π_j by

$$x_i \pi_j = x_i \circ x_j^\alpha$$

for $i = 0, 1, \cdots, n$. Since $(H, \circ)$ is a loop, π_j is a permutation of H.
We shall show that for each $x_j^\alpha \in H$, there exists an automorphism $\theta_{x_j^\alpha}$ of G such that

$$x_i \theta_{x_j^\alpha} = (x_j^\alpha)^{-1}(x_i \pi_j) \text{ for all } x_i \in H \tag{2.4}$$

Then we shall have $x_i \circ x_j^\alpha = x_j^\alpha \cdot x_i \theta_{x_j^\alpha}$ for all $x_i, x_j^\alpha \in H$ and it follows that H is a special subset of G. Let $\theta_{x_0} = I$ and note that (2.4) with $j = 0$ is satisfied for all $x_i \in H$.
Now for $p, q = 1, 2, \cdots, n$, $p \neq q$, there exists automorphisms P_{pq}, V_p, W_{pq} of G such that

$$\begin{aligned}
P_{pq} &: x_p \longrightarrow x_q, x_q \longrightarrow x_p, x_k \longrightarrow x_k \text{ for } k \neq p, q \\
V_p &: x_p \longrightarrow x_p^{-1}, x_k \longrightarrow x_k \text{ for } k \neq p \\
W_{pq} &: x_q \longrightarrow x_p x_q, x_k \longrightarrow x_k \text{ for } k \neq q
\end{aligned}$$

For $x_j^\alpha \in H, j \neq 0$ of order m in $(H, \circ)$, let

$$x_{j_1}^\alpha = x_{x_j}^\alpha \pi_j,\ x_{j_2}^\alpha = x_{x_{j_1}}^\alpha \pi_j\ \cdots, x_{j_{m-1}}^\alpha = x_0.$$

Since π_j is a permutation on $H - \{x_j^\alpha, x_{j_1}^\alpha, \cdots, x_{j_{m-1}}^\alpha\}$, there exists an automorphism of G of type P_{pq} such that $x_j^\alpha \theta_j = x_{j_1}^\alpha$, $x_{j_1}^\alpha \theta_j = x_{j_2}^\alpha, \cdots, x_{j_{m-3}}^\alpha \theta_j = x_{j_{m-2}}^\alpha$, $x_{j_{m-2}}^\alpha \theta_j = x_j^\alpha$ and $y^\alpha \theta_j = y^\alpha \pi_j$ for all $y^\alpha \in H - \{x_j^\alpha, x_{j_1}^\alpha, \cdots, x_{j_{m-1}}^\alpha\}$. Then θ_j followed by $n-1$ applications of automorphisms of type $W_q (q \neq j)$ and application of V_j yields automorphism $\theta_{x_j^\alpha}$ satisfying (2.4). □

Before we proceed, we provide a partial converse of Theorem (2.7) in the lemma stated below.

Lemma 2.9. *Let H be a special subset of a group G and let θ be a homomorphism with kernel K of G onto the group $G\theta$. If $H\theta$ is a special subset of $G\theta$ and if the restriction of θ to H is a homomorphism of the generalized Bol loop $(H, \circ)$ onto the generalized Bol loop $(H\theta, \circ)$, then*

$$k\theta_{y^\alpha} \subseteq K \textit{ for all } y^\alpha \in H.$$

Proof. for all $x, y \in H$ and $\alpha : H \longrightarrow H$, we have

$$y^\alpha\theta \cdot x\theta_{y^\alpha}\theta = (y^\alpha \cdot x\theta_{y^\alpha})\theta = (x \circ y^\alpha)\theta = x\theta \circ y^\alpha\theta = y^\alpha\theta \cdot x\theta\theta_{y^\alpha\theta}.$$

Consequently, $x\theta_{y^\alpha}\theta = x\theta\theta_{y^\alpha\theta}$ for all $x, y \in H$ and $\alpha : H \longrightarrow H$. Since H generates G,

$$z\theta_{y^\alpha}\theta = z\theta\theta_{y^\alpha\theta} \text{ for all } z \in G \text{ and all } y^\alpha \in H. \tag{2.5}$$

From (2.5) with $z \in K$, we have

$$z\theta_{y^\alpha}\theta = x\theta\theta_{y^\alpha\theta} = 1\theta_{y^\alpha\theta} = 1 \text{ for all } y^\alpha \in H.$$

That is $k\theta_{y^\alpha} \subseteq K$ for all $y^\alpha \in H$. □

We are now in a position to investigate universal embedding of generalized Bol loops into free groups. Before then, we define the notion of universal embedding.

Definition 2.10. A generalized Bol loop $(K, \cdot)$ is universally embeddable in a group G if and only if the following two conditions are satisfied

(1) $(K, \cdot)$ is specially emdeddable in G
(2) If $(K, \cdot)$ is specially embeddable in a group G', then there is a homomorphism θ of G onto G' such that 1. and 2. of Theorem (2.7) hold.

Theorem 2.11. *A generalized Bol loop of order $n + 1$ is universally embeddable in the free group on n generators.*

Proof. Let $(K, \cdot)$ be a generalized Bol loop of order $n+1$, from Theorem (2.8),we see that condition (1.) of Definition (2.10) is satisfied.
Assume that $(K, \cdot)$ is specially embeddable in a group G', then there is a special subset H' of G' such that $(H', \circ)$ and $(K, \cdot)$ are isomorphic.
Let $H' = \{a_0, a_1, a_2, \cdots, a_n\}$ with a_0 denoting the identity element of $(H', \circ)$. Now let G be free group on n free generators $x_1, x_2, \cdots, x_n$. Let $H = \{x_0, x_1, \cdots, x_n\}$ where x_0 is the identity element of G. Introduce the binary operation "$\circ$" on H by

$$x_1 \circ x_j^\alpha = x_k \text{ if and only if } a_1 \circ a_j^\alpha = a_k \text{ for } i, j, k = 0, 1, \cdots, n.$$

Then by the proof of Theorem (2.8), H is a special subset of G. Since G' is generated by $a_1, a_2, \cdots, a_n$, there is a homomorphism θ of G onto G' so that $x_i\theta = a_i$ for $i = 0, 1, \cdots, n$. Furthermore, the restriction of θ to H is an isomorphism of $(H, \circ)$ onto $(H', \circ)$. Thus by Lemma (2.9), condition 2. of Theorem (2.7) holds. Condition 1. of Theorem (2.7) is obvious. □

References

1. J.O. Adeniran, A.R.T. Solarin, *A note on automorphic inverse property loops.* Zbornik Radfova, Coll. of Sci. Papers, **20** (1997), 47–52.
2. J.O. Adeniran, Solarin, A.R.T., *A note on generalised Bol Identity.* Scientific Annal of A.I.I Cuza Univ., Vol. **45** No. 1 (1999), 19–26.
3. J.O. Adeniran, *On holomorphic theory of a class of left Bol loops.* Scientiic Annal of A.I.I Cuza Univ., Vol. **51** No. 1 (2005), 23–28.
4. J. O. Adéníran, T. G. Jaíyéolá and K. A. Ìdòwú, *Holomorph of generalized Bol loops.* Novi Sad J. Math. **44**(1) (2014), 37–51.
5. J. O. Adeniran, S. A. Akinleye and T. Alakoya, *On the core and some isotopy characterizations of generalized Bol loops*, Transactions of the Nigerian Association of Mathematical Physics, Vol.**1**. (2015), 99-104
6. G. Bol, *Gewebe and Gruppen.* Math. Ann. **114** (1937), 414–431.
7. R.H. Bruck, *Contributions to the theory of loops.* Trans. Amer. Soc. **55** (1944), 245–354.
8. R.H. Bruck, *A survey of binary systems.* Berlin-Göttingen-Heidelberg: Springer-Verlag, 1966.
9. R.H. Bruck, L.J. Paige, *Loops whose inner mappings are automorphisms.* Annals of Math., Vol. **63** No. 2 (1956), 308–323.
10. R.P. Burn, *Finite Bol loops.* Math. Proc. Camb. Phil. Soc. **84** (1978), 377–385.
11. R.P. Burn, *Finite Bol loops II.* Math. Proc. Camb. Phil. Soc. **88** (1981), 445–455.
12. R.P. Burn, *Finite Bol loops III.* Math. Proc. Camb. Phil. Soc. **97** (1985), 219–223.

13. O. Chein, E.G. Goodaire, *A new construction of Bol loops: the "odd" case*. Quasigroups and Related Systems, Vol. **13** No. 1 (2005), 87–98.
14. Chein, O., Goodaire, E.G., Bol loops of nilpotence class two. Canad. J. Math., Vol. **59** No. 2 (2007), 296–310.
15. O. Chein, E.G. Goodaire, *Bol loops with a large left nucleus*, Comment. Math. Univ. Carolin. Vol. **49** No. 2(2008), 171–196.
16. A. Drapal, *Generating free groups by loop translation*, Europ. J. Combinatorics, **16** (1995), 561–565.
17. M.K. Kinyon, J.D. Phillips, *Commutants of Bol loops of odd order*. Proc. Amer. Math. Soc. **132** (2004), 617–619.
18. Kinyon, M.K., Phillips, J.D., Vojtěchovský, P., *When is the commutant of a Bol loop a subloop*?. Trans. Amer. Math. Soc., **360** (2008), 2393–2408.
19. G.P. Nagy, *A class of finite simple Bol loops of exponent 2*. Trans. Amer. Math. Soc. **361** (2009), 5331–5343.
20. G.P. Nagy, *A class of simple proper Bol loop*. Manuscripta Mathematica, Vol. **127** No. 1 (2008), 81–88.
21. G.P. Nagy, *Some remarks on simple Bol loops*. Comment. Math. Univ. Carolin., Vol. **49** No. 2 (2008), 259–270.
22. T. Foguel, M.K. Kinyon, J.D. Phillips, *On twisted subgroups and Bol loops of odd order*. Rocky Mountain J. Math. **36** (2006), 183–212.
23. D.A. Robinson, *Bol loops*. Ph.D Thesis, University of Wisconsin, Madison, 1964.
24. D.A. Robinson, *A special embedding of Bol loops in groups*. Acta Mathematica Academiae Scientiarum Hungaricae Tomus **30** (1–2) (1977), 95–103.
25. B.L. Sharma, *Left loops which satisfy the left Bol identity*. Proc. Amer. Math. Soc., **61** (1976), 189–195.
26. B.L. Sharma, *Left loops which satisfy the left Bol identity (II)*. Ann. Soc. Sci. Bruxelles, Sr. I **91** (1977), 69–78.
27. B.L. Sharma, L.V. Sabinin, *On the existence of Half Bol loops*. Scientific Annal of A.I.I Cuza Univ., Vol. **22** No. 2 (1976), 147–148.
28. B.L. Sharma, L.V. Sabinin, *On the Algebraic properties of half Bol Loops*. Ann. Soc. Sci. Bruxelles Sr. I, Vol. **93** No. 4 (1979), 227–240.
29. A.R.T. Solarin, B.L. Sharma, *On the Construction of Bol loops*. Scientific Annal of A.I.I Cuza Univ., Vol. **27** No. 1 (1981), 13–17.
30. A.R.T. Solarin, B.L. Sharma, *Some examples of Bol loops*. Acta Carol, Math. and Phys., Vol. **25** No. 1 (1984), 59–68.
31. Solarin, A.R.T., Sharma, B.L., *On the Construction of Bol loops II*. Scientific Annal of A.I.I Cuza Univ., Vol. **30**, No. 2 (1984), 7–14.

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