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ON HENSTOCK-KURZWEIL-STIELTJES-$\Diamond$-DOUBLE INTEGRALS OF INTERVAL-VALUED FUNCTIONS ON TIME SCALES

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ABSTRACT. In this paper, we define Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral for interval-valued functions on time scales. Some of the basic properties of this integral are proved.

1. INTRODUCTION AND PRELIMINARIES

The theory of integration cannot be left behind in view of its several contributions in the study of differential analysis, mechanics, physics, probability and statistics, and so on. In 1988, Stephen Hilger [5] in his Ph.D. thesis introduced the theory of time scales and its wide applications to so many branches of Mathematics. The theory has attracted many researchers and mathematicians because of its unification of both continuous and discrete systems with the minds of extending them to more general classes of dynamical systems.

The concept of Henstock integral for real valued functions was first defined by Henstock in 1963, and it is seen to be easier and simpler than the Lebesgue, Denjoy, Wiener and Perron integrals. In the last decades or more, there have been tremendous development in the study of Henstock integral. Many authors have investigated Henstock integrals dealing with certain valued functions on time scales. For further study of such valued-functions integrals and other important properties (see [4]). In 2006, Peterson and Thompson introduced Henstock delta integral on time scales and presented some of its basic properties [9]. Henstock-Kurzweil integrals on time scales was studied by Thompson [10]. The theory and application of delta and nabla derivatives and that of integration on time scales is relatively new, therefore receiving less attention from the researchers (see [1, 2, 7, 8, 9, 10]). Nevertheless, Thompson [10] studied a version of Henstock-Kurzweil integrals on time scales using covering arguments. He showed that the integral is expressible in some situations, as an ordinary integral in the Newton, the Lebsgue, and Henstock-Kurzweil senses.

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The theory of interval analysis has a long history which can be traced back to the celebrated book of Moore [6]. One of the initial applications of interval analysis was to compute the error bounds of the numerical solution of a finite state of machine and for providing reliable computations [6]. Other interesting applications include; in automatic error analysis, computer graphics, rental network optimization and many others. For more detailed results and applications of interval valued functions, we refer the reader to see ([3, 11, 12]).

In 2000, Wu and Gong [11] introduced the concept of the Henstock integrals for interval-valued single functions and fuzzy-number-valued functions and gave some of its simple properties. Yoon [12] also presented some basic properties of interval-valued Henstock-Stieltjes integral on time scales. Presently, Hamid and Elmuiz [3] introduced the notion of Henstock-Stieltjes integrals of interval-valued functions and fuzzy-number-valued functions and discuss some of their properties. Nonetheless, to our best knowledge, not much have been done for interval-valued functions on double time scales variable calculus for Henstock-Kurzweil-Stieltjes-$\Diamond$-integral.

In this paper, with the hope to extend recent results given in [11], we introduce the interval Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral on time scales. We show the interval Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral as generalization of Henstock-Kurzweil-Stieltjes integral. Also, some basic properties for interval Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral are given.

2. Main results

Firstly, we discuss the following new concepts in $\mathbb{T}_1 \times \mathbb{T}_2$. Let $a, b \in \mathbb{T}_1, c, d \in \mathbb{T}_2$, where $a < d, c < d$, and $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} = \{(t, s) : t \in [a, b), s \in [c, d), t \in \mathbb{T}_1, s \in \mathbb{T}_2\}$. Let $g_1, g_2 : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be two non-decreasing functions on $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$, respectively. Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be bounded on $\mathcal{R}$. Let P_1 and P_2 be two partitions of $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_n\} \subset [c, d]_{\mathbb{T}_2}$. Let $\{\xi_1, \xi_2, ..., \xi_n\}$ denote an arbitrary selection of points from $[a, b]_{\mathbb{T}_1}$ with $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}_1}, i = 1, 2, ..., n$. Similarly, let $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ denote an arbitrary selection of points from $[c, d]_{\mathbb{T}_2}$ with $\zeta_j \in [s_{j-1}, s_j)_{\mathbb{T}_2}, j = 1, 2, ..., k$.

Definition 2.1. A pair $\delta = (\delta_L, \delta_R)$ of real-valued functions defined on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ is said to be a Δ-gauge for $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ if

$$\begin{cases} \delta_L(t, s) > 0 \quad \text{on} \quad (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \\ \delta_R(t, s) > 0 \text{ on } \ [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \\ \delta_L(a) \geq 0, \ \delta_R(b) \geq 0 \ \delta_L(c) \geq 0 \ \delta_R(d) \geq 0 \\ \delta_R(t, s) \geq \sigma(t, s) - (t, s) \ \text{ for all } t \in [a, b)_{\mathbb{T}_1}, s \in [c, d)_{\mathbb{T}_2}. \end{cases}$$

A pair $\gamma = (\gamma_L, \gamma_R)$ of real-valued functions defined on $[a, b]_{\mathbb{T}_1} \times [c, d]_{\mathbb{T}_2}$ is said to be a $\bigtriangledown$-gauge for $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ if

$$\begin{cases} \gamma_L(t,s) > 0 \text{ on } (a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2} \\ \gamma_R(t,s) > 0 \text{ on } [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \\ \gamma_L(a) \geq 0,\ \gamma_R(b) \geq 0,\ \gamma_L(c) \geq 0,\ \gamma_R(d) \geq 0 \\ \gamma_R(t,s) \geq (t,s) - \rho(t,s) \text{ for all } t \in (a,b]_{\mathbb{T}_1}, s \in (c,d]_{\mathbb{T}_2}. \end{cases}$$

Given a Δ-gauge δ and a $\bigtriangledown$-gauge γ, the partitions $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2},\ j = 1, 2, ..., k$, is said to be:

- δ-fine if $g(\xi_i, \zeta_j) - \delta_L \leq g(t_{i-1}, s_{j-1}) < g(t_i, s_j) \leq g((\xi_i, \zeta_j) + \delta_R(\xi_i, \zeta_j))$ and
- γ-fine if $g(\xi_i, \zeta_j) - \gamma_L \leq g(t_{i-1}, s_{j-1}) < g(t_i, s_j) \leq g((\xi_i, \zeta_j) + \gamma_R(\xi_i, \zeta_j))$.

Now, we present newly the following definitions.

Definition 2.2. Let $\mathbb{T}_1, \mathbb{T}_2$ be two given time scales and let $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x,y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$ which is a complete metric space with the metric (distance) D, define $D(A,B) = D((a,c),(b,d)) = ((b-a)^2 + (d-c)^2)$ as the distance between A and B. Then

$$\begin{aligned} D(A,B) &= \max\left(\left|A^- - B^-\right|, \left|A^+ - B^+\right|\right) \\ &= \max\left(\left|(a^-, c^-) - (b^-, d^-)\right|, \left|(a^+, c^+) - (b^+, d^+)\right|\right) \\ &= \max((b^- - a^-)^2 + (d^- - c^-)^2)^{\frac{1}{2}}, ((b^+ - a^+)^2 + (d^+ - c^+)^2)^{\frac{1}{2}}. \end{aligned}$$

Throughout this work, we denote $I_{\mathbb{R}}$ by the set of interval-valued functions on real line.

Definition 2.3. Let $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ be an interval-valued function on $\mathcal{R}$ and let g be a non-decreasing function defined on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ with partitions $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, 2, ..., n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$ for $j = 1, 2, ..., k$. Then

$$S(P_1, P_2, F, g_1, g_2) = \sum_{i=1}^{n} \sum_{j=1}^{k} F(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))$$

is defined as Interval Henstock-Kurweil-Stieltjes-$\Diamond$-double sum of F with respect to functions g_1 and g_2.

Let $P = P_1 \times P_2$ and $\Diamond g_{1_i} \Diamond g_{2_j} = (g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))$, then the Henstock-Kurweil-Stieltjes-$\Diamond$-double sum of F with respect to functions g_1 and g_2 is denoted by $S(P, F, g)$ is written as

$$S(P, F, g_1, g_2) = \sum_{i=1}^{n} \sum_{j=1}^{k} F(\xi_i, \zeta_j) \Diamond g_{1_i} \Diamond g_{2_j},\ (i = 1, ..., n;\ j = 1, ..., k).$$

Definition 2.4. Let $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ be an interval-valued function on $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. We say that F is Interval Henstock-Kurzweil-Stieltjes-$\Diamond$-integrable with respect to non-decreasing functions $g_1 : [a,b)_{\mathbb{T}_1} \longrightarrow \mathbb{R}$ and $g_2 : [c,d)_{\mathbb{T}_2} \longrightarrow \mathbb{R}$ if there exists an interval $I_0 \in I_{\mathbb{R}}$ such that for every $\varepsilon > 0$, there are $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that

$$D\left(S(P,F,g), I_0\right) < \varepsilon$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, ..., n$ is a δ_1-fine (or γ_1) and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2},\ j = 1, 2, ..., k$ is a δ_2-fine (or γ_2) are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively.

We say that I_0 is the interval Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral of F with respect to g_1 and g_2 defined on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$, and write

$$\int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s) = I_0.$$

Now we give our main results.

Theorem 2.5. *If $F : (a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2} \to I_{\mathbb{R}}$ is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to two increasing functions g_1, g_2 on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$, then the Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral of F is unique.*

Proof. Suppose that I_1 and I_2 are both Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrals of F on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. With the assumption that I_1 and I_2 are not unique, then F is said to be Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$ if it satisfies the following point wise integrability criterion: for every $\varepsilon > 0$ there are $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) defined on $(a,b]_{\mathbb{T}_1}$ and $(c,d]_{\mathbb{T}_2}$ respectively, such that for every $\varepsilon > 0$, there are $\Diamond$-gauges δ_1^1 and δ_2^1 (or γ_1^1 and γ_2^1) for $[a,b)_{\mathbb{T}_1}$ and δ_1^2 and δ_2^2 (or γ_1^2 and γ_2^2) for $[c,d)_{\mathbb{T}_2}$ such that

$$D\left(S(P^1,F,g), I_1\right) < \frac{\varepsilon}{2} \text{ and } D\left(S(P^2,F,g), I_2\right) < \frac{\varepsilon}{2} \text{ for all pairs } P^1 = P_1^1 \times P_2^1$$
$$\text{and } P^2 = P_1^2 \times P_2^2 \text{ of } \delta_1\text{-fine (or } \gamma_1)$$

and for every $\varepsilon > 0$ and $i \in \{1,2\}$, there are $\Diamond$-gauges δ_1^i and δ_2^i (or γ_1^i and γ_2^i) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that

$$D(S(P^i,F,g), I_i) < \frac{\varepsilon}{2}$$

provided that $P^i = P_1^i \times P_2^i$ is a pair of δ_1^i-fine (or γ_1^i) and δ_2^i-fine (or γ_2^i) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively.
Let $\delta_1 = \min\{\delta_1^1, \delta_1^2\}$ i.e. $(\delta_1)_L = \min\{(\delta_1^1)_L, (\delta_1^2)_L\}$ and $(\delta_1)_R = \min\{(\delta_1^1)_R, (\delta_1^2)_R\}$ and
$\delta_2 = \min\{\delta_2^1, \delta_2^2\}$ i.e. $(\delta_2)_L = \min\{(\delta_2^1)_L, (\delta_2^2)_L\}$ and $(\delta_2)_R = \min\{(\delta_2^1)_R, (\delta_2^2)_R\}$, δ_1 and δ_2 are $\Diamond$-gauges for $(a,b]_{\mathbb{T}_1}$ and $(c,d]_{\mathbb{T}_2}$ respectively, and given a pair $P = P_1 \times P_2$ of δ_1-fine and δ_2-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$, P_1 is a δ_1^1-fine

and δ_1^2-fine partition of $(a,b]_{\mathbb{T}_1}$, P_2 is a δ_2^1-fine and δ_2^2-fine partition of $[c,d)_{\mathbb{T}_2}$, hence

$$\begin{aligned} D(I_1, I_2) &\leq D((I_1, S(P,F,g) + S(P,F,g), I_2)) \\ &\leq D(S(P,F,g), I_1) + D(S(P,F,g), I_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

since for all $\varepsilon > 0$, there are $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2), then it follows that $I_1 = I_2$.
Hence, the Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral of F on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ is unique. $\square$

Remark 2.6. (i). For $a, b \in \mathbb{T}_1$ with $a = b$ or $c, d \in \mathbb{T}_2$ with $c = d$ then

$$\int_a^b \int_c^d F(t,s) \Diamond g_1(t) \Diamond g_2(s) = 0.$$

(ii). For $a, b \in \mathbb{T}_1$ with $a \leq b$ and $c, d \in \mathbb{T}_2$ with $c \leq d$. It is easy to see that if $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ and $g_1 = g_2 = g$, f is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable if and only if f is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable in $\mathbb{R}$ in the classical sense, and the

$$\int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s) = \int\int_{\mathcal{R}} F(t,s) dg(t) dg(s),$$

where the integral on the right hand side is the ordinary Henstock-Kurzweil-Stieltjes double integral.

(iii). If $F(t,s) = F^-(t,s) = F^+(t,s)$ for all $t, s \in (a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$. It is clear that Definition 2.4 implies the real-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$.

Proposition 2.7. *Assume $a, b \in \mathbb{T}_1$ with $a \leq b$ and $c, d \in \mathbb{T}_2$ with $c \leq d$. Every constant function $F(t,s) = A \in I_{\mathbb{R}}$ for $(t,s) \in [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable over $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ and*

$$\int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s) = A[g_1(b) - g_1(a)][g_2(d) - g_2(c)].$$

Proof. Let $a, b \in \mathbb{T}_1$ with $a < b$ and $c, d \in \mathbb{T}_2$ with $c < d$. Consider a partitions $P_1 = \{([t_{i-1}, t_i], \xi_i) : 1 \leq i \leq n\}$ of $(a,b]_{\mathbb{T}_1}$ and $P_2 = \{([s_{j-1}, s_j], \zeta_j) : 1 \leq j \leq k\}$

of $(c,d]_{\mathbb{T}_2}$. Then

$$\begin{aligned} S(P,F,g) &= \sum_{i=1}^{n}\sum_{j=1}^{k} A\Diamond g_{1_i}\Diamond g_{2_j} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k} A(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})) \\ &= A\sum_{i=1}^{n}\sum_{j=1}^{k}(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})) \\ &= A[g_1(b)-g_1(a)][g_2(d)-g_2(c)]. \end{aligned}$$

Hence F is $\Diamond$-double integrable and result of proposition 2.7 holds. $\square$

Theorem 2.8. *An interval-valued function $F:(a,b]_{\mathbb{T}_1}\times(c,d]_{\mathbb{T}_2}\to I_{\mathbb{R}}$ is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to g_1,g_2 on $(a,b]_{\mathbb{T}_1}\times(c,d]_{\mathbb{T}_2}$ if and only if F^- and F^+ are interval Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $(a,b]_{\mathbb{T}_1}\times(c,d]_{\mathbb{T}_2}$ such that*

$$\begin{aligned} &\int_a^b\int_c^d F(t,s)\Diamond g_1(t)\Diamond g_2(s) \\ = &\left[\int_a^b\int_c^d F^-(t,s)\Diamond g_1(t)\Diamond g_2(s), \int_a^b\int_c^d F^+(t,s)\Diamond g_1(t)\Diamond g_2(s)\right], \end{aligned}$$

where $F(t,s)=[F^-(t,s),F^+(t,s)]$.

Proof. Let $I_0=[I_0^-,I_0^+]$ be an interval with property that for each $\varepsilon>0$ there exists a gauge $\delta(t,s)$ with respect to two increasing functions $g_1:[a,b)_{\mathbb{T}_1}\longrightarrow\mathbb{R}$ and $g_2:[c,d)_{\mathbb{T}_2}\longrightarrow\mathbb{R}$ such that $P_1=\{t_0,t_1,...,t_n\}\subset[a,b]_{\mathbb{T}_1}$ with tag points $\xi_i\in[t_{i-1},t_i]_{\mathbb{T}_1}$ for $i=1,...,n$ and $P_2=\{s_0,s_1,...,s_k\}\subset[c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j\in[s_{j-1},s_j]_{\mathbb{T}_2}$, $j=1,2,...,k$ are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$. Then

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k}F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),I_0\right)<\varepsilon.$$

Let $\varepsilon>0$, there are $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $D(S(P,F,g),I_0)<\varepsilon$ provided that $P=P_1\times P_2$ is such that P_1 is a δ_1-fine (or γ_1) partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a δ_2-fine (or γ_2) partition of $[c,d)_{\mathbb{T}_2}$.

It is easy to notice that

$$S(P,F,g)=[S(P,F^-,g),S(P,F^+,g)]$$

hence

$$D(S(P,F,g),I_0)<\varepsilon\Leftrightarrow\max\{|S(P,F^-,g)-I_0^-|,|S(P,F^+,g)-I_0^+|\}<\varepsilon$$

which implies

$$\begin{cases}|S(P,F^-,g)-I_0^-|<\varepsilon\\ |S(P,F^+,g)-I_0^+|<\varepsilon.\end{cases}$$

Therefore by remark 2.6.(iii), we say that F^-, F^+ is interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ and

$$I_0^- = \int\int_{\mathcal{R}} F^-(t,s)\Diamond g_1(t)\Diamond g_2(s)$$
$$I_0^+ = \int\int_{\mathcal{R}} F^+(t,s)\Diamond g_1(t)\Diamond g_2(s).$$

Conversely, let F^-, F^+ be Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2} \to \mathbb{R}$, then there exists $J_1, J_2 \in \mathbb{R}$ with the property that given $\varepsilon > 0$, there are gauges δ_1 and δ_2 for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{k} F^-(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})) - J_1\right| < \varepsilon,$$
$$\left|\sum_{i=1}^{n}\sum_{j=1}^{k} F^+(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})) - J_2\right| < \varepsilon,$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, ..., n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, ..., k$ are δ_1-fine (or γ_1) and δ_2-fine (or γ_2) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively. If $F^-(\xi_i,\zeta_j) \le F^+(\xi_i,\zeta_j)$, then $J_1 \le J_2$. We define $\delta(t,s) = \min\{\delta_1(t,s), \delta_2(t,s)\}$ and $I_0 = [J_1, J_2]$ to have

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})), I_0\right) < \varepsilon,$$

where $I_0 = [J_1, J_2]$.
Hence $F : (a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2} \to I_{\mathbb{R}}$ is interval Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to g_1, g_2 on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$. $\square$

Theorem 2.9. *(Bolzano Cauchy Criterion) Let $F : (a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2} \to I_{\mathbb{R}}$ be an interval-valued function over a rectangle $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$ and consider two non-decreasing functions $g_1 : [a,b)_{\mathbb{T}_1} \longrightarrow \mathbb{R}$ and $g_2 : [c,d)_{\mathbb{T}_2} \longrightarrow \mathbb{R}$. Then, F is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$ with respect to g_1 and g_2 if and only if for each $\varepsilon > 0$ there exists $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively, such that $D(S(P^1,F,g), S(P^2,F,g), F, g)) < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of δ_1 (or γ_1)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and δ_2 (or γ_2)-fine partitions of $[c,d)_{\mathbb{T}_2}$.*

Proof. Suppose F is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$ with respect to g_1 and g_2 , and let

$$I_0 = \int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s).$$

Let $\varepsilon > 0$. There are $\Diamond$-gauges δ_1 and δ_2 for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $D(S(P,F,g), I_0) < \dfrac{\varepsilon}{2}$ provided that $P = P_1 \times P_2$ where P_1 is a δ_1 (or γ_1) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d)_{\mathbb{T}_2}$. Therefore, if $P = P_1 \times P_2$ and $P = P_1^{'} \times P_2^{'}$ are pairs of δ_1 (or γ_1) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d)_{\mathbb{T}_2}$, then

$$\begin{aligned} D(S(P,F,g),S(P^1,F,g)) &\leq D(S(P,F,g),I_0) + D(I_0,S(P^1,F,g)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Conversely, suppose that for all $\varepsilon > 0$ there are $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $D(S(P^1,F,g),S(P^2,F,g)) < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of δ_1 (or γ_1)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and δ_2 (or γ_2)-fine partitions of $[c,d)_{\mathbb{T}_2}$.

Let $n \in \mathbb{N}$. Taking $\varepsilon = \dfrac{1}{n}$, there are $\Diamond$-gauges $\delta_{1,n}$ and $\delta_{2,n}$ (or $\gamma_{1,n}$ and $\gamma_{2,n}$) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $D(S(P^{'},F,g),S(P^2,F,g)) < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or $\gamma_{2,n}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$.

By replacing $\delta_{i,n}$ by $\min\{\delta_{i,1},\delta_{i,2},...,\delta_{i,n}\}$ with $i \in \{1,2\}$, we may assume that $\delta_{i,n+1} \leq \delta_{i,n}$. Thus, for all $j > n$ $\delta_{i,j} \leq \delta_{i,n}$ so any pair $P^n = P_1^n \times P_2^n$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or $\gamma_{2,n}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$ is also a pair of $\delta_{1,j}$ (or $\gamma_{1,j}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,j}$ (or $\gamma_{2,j}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$, hence

$$D(S(P^n,F,g),S(P^j,F,g)) < \frac{1}{j}.$$

This shows that $\{S(P^n,F,g)\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Let I_0 be the limit of $\{S(P^n,F,g)\}_{n\in\mathbb{N}}$. For all $\varepsilon > 0$, choosing $N > \dfrac{2}{\varepsilon}$, for $\Diamond$-gauges $\delta_{1,N}$ and $\delta_{2,N}$ (or $\gamma_{1,N}$ and $\gamma_{2,N}$) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively,

$$\begin{aligned} D(S(P,F,g),I_0) &\leq D(S(P,F,g),S(P^N,F,g)) + D(S(P^N,F,g),I_0) \\ &< \frac{1}{N} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

for pair $P = P_1 \times P_2$ such that P_1 is a $\delta_{1,N}$ (or $\gamma_{1,N}$) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a $\delta_{2,N}$ (or $\gamma_{2,N}$) fine partition of $[c,d)_{\mathbb{T}_2}$. □

We present the following algebraic properties of the interval Henstock-Kurzweil-Stieltjes-$\Diamond$-Double integral on Time Scales.

Theorem 2.10. *Let $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ be an interval-valued function and let g_1 and g_2 be non-decreasing functions respectively on $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$. If F is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to g_1 and g_2*

on $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. *Then,*

i. $\displaystyle\int\int_{\mathcal{R}} \lambda \Diamond g_1(t) \Diamond g_2(s) = \lambda(g_1(b) - g_1(a))(g_2(d) - g_2(c)),$ λ *is a constant;*

ii. $\displaystyle\int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s) = 0$ *when* g_1 *or* g_2 *are constants;*

iii. $\displaystyle\int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s) = f(a,c)(g_1^{\tau_1}(a) - g_1(a))(g_2^{\tau_2}(c) - g_2(c))$
with $b = \tau_1(a)$ *and* $\tau_2(c)$*;*

iv. $\displaystyle\int\int_{\mathcal{R}} \lambda F(t,s)\mu[\Diamond g_1(t) \Diamond g_2(s)] = \int\int_{\mathcal{R}} \lambda\mu F(t,s) \Diamond g_1(t) \Diamond g_2(s);$ λ *and* μ *are constants.*

The proof of Theorem 2.10 is straightforward by using our definition 2.4.

Theorem 2.11. *Let* F, G *be interval-valued Henstock-Kurzweil-Stieltjes-*$\Diamond$*-double integrable with respect to increasing functions* g_1 *and* g_2 *on* $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ *and* $\lambda, \mu \in \mathbb{R}$. *Then*

i. $\lambda F + \mu G$ *is also interval-valued Henstock-Kurzweil-Stieltjes-*$\Diamond$*-double integrable with respect to* g_1 *and* g_2 *on* $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ *and*

$$\begin{aligned} & \int\int_{\mathcal{R}} (\lambda F(t,s) + \mu G(t,s)) \Diamond g_1(t) \Diamond g_2(s) \\ = \ & \lambda \int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s) + \mu \int\int_{\mathcal{R}} G(t,s) \Diamond g_1(t) \Diamond g_2(s). \end{aligned}$$

ii. *Let* $F(t,s) = G(t,s)$ *almost everywhere on* $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. *Then*

$$\int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s) = \int\int_{\mathcal{R}} G(t,s) \Diamond g_1(t) \Diamond g_2(s).$$

Proof. (i). If F, G be interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to g_1 and g_2 on $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$, then F^-, F^+, G^-, G^+ be interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to g_1 and g_2 on $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ by Theorem 2.8. Hence

$\lambda F^- + \mu G^-, \lambda F^- + \mu G^+, \lambda F^+ + \mu G^-, \lambda F^+ + \mu G^+$ be interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to increasing functions g_1 and g_2 on $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$.

Case 1. If $\lambda = \mu = 0$. It is trivial.

Case 2. If $\lambda > 0$ and $\mu > 0$, then

$$\begin{aligned}
& \int\int_{\mathcal{R}} (\lambda F(t,s) + \mu G(t,s))^{-} \Diamond g_1(t) \Diamond g_2(s) \\
= & \int\int_{\mathcal{R}} \left(\lambda F^{-}(t,s) + \mu G^{-}(t,s)\right) \Diamond g_1(t) \Diamond g_2(s) \\
= & \lambda \int\int_{\mathcal{R}} \left(F^{-}(t,s)\right) \Diamond g_1(t) \Diamond g_2(s) + \mu \int\int_{\mathcal{R}} \left(G^{-}(t,s)\right) \Diamond g_1(t) \Diamond g_2(s) \\
= & \lambda \left(\int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s)\right)^{-} + \mu \left(\int\int_{\mathcal{R}} G(t,s) \Diamond g_1(t) \Diamond g_2(s)\right)^{-} \\
= & \left(\lambda \int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s) + \mu \int\int_{\mathcal{R}} G(t,s) \Diamond g_1(t) \Diamond g_2(s)\right)^{-}.
\end{aligned}$$

Case 3. If $\lambda < 0$ and $\mu < 0$, then

$$\begin{aligned}
& \int\int_{\mathcal{R}} (\lambda F(t,s) + \mu G(t,s))^{-} \Diamond g_1(t) \Diamond g_2(s) \\
= & \int\int_{\mathcal{R}} \left(\lambda F^{+}(t,s) + \mu G^{+}(t,s)\right) \Diamond g_1(t) \Diamond g_2(s) \\
= & \lambda \int\int_{\mathcal{R}} \left(F^{+}(t,s)\right) \Diamond g_1(t) \Diamond g_2(s) + \mu \int\int_{\mathcal{R}} \left(G^{+}(t,s)\right) \Diamond g_1(t) \Diamond g_2(s) \\
= & \lambda \left(\int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s)\right)^{+} + \mu \left(\int\int_{\mathcal{R}} G(t,s) \Diamond g_1(t) \Diamond g_2(s)\right)^{+} \\
= & \left(\lambda \int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s) + \mu \int\int_{\mathcal{R}} G(t,s) \Diamond g_1(t) \Diamond g_2(s)\right)^{-}.
\end{aligned}$$

Case 4. If $\lambda > 0$ and $\mu < 0$, (or $\lambda < 0$, and $\mu > 0$), then

$$\begin{aligned}
& \int\int_{\mathcal{R}} (\lambda F(t,s) + \mu G(t,s))^{-} \Diamond g_1(t) \Diamond g_2(s) \\
= & \int\int_{\mathcal{R}} \left(\lambda F^{-}(t,s) + \mu G^{+}(t,s)\right) \Diamond g_1(t) \Diamond g_2(s) \\
= & \lambda \int\int_{\mathcal{R}} \left(F^{-}(t,s)\right) \Diamond g_1(t) \Diamond g_2(s) + \mu \int\int_{\mathcal{R}} \left(G^{+}(t,s)\right) \Diamond g_1(t) \Diamond g_2(s) \\
= & \lambda \left(\int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s)\right)^{-} + \mu \left(\int\int_{\mathcal{R}} G(t,s) \Diamond g_1(t) \Diamond g_2(s)\right)^{+} \\
= & \lambda \left(\int\int_{\mathcal{R}} F(t,s) \Diamond g_1(t) \Diamond g_2(s) + \mu \int\int_{\mathcal{R}} G(t,s) \Diamond g_1(t) \Diamond g_2(s)\right)^{-}.
\end{aligned}$$

Similarly, considering the four cases above, we have

$$\begin{aligned}&\int\int_{\mathcal{R}}(\lambda F(t,s)+\mu G(t,s))^{+}\Diamond g_1(t)\Diamond g_2(s)\\=&\left(\lambda\int\int_{\mathcal{R}}F(t,s)\Diamond g_1(t)\Diamond g_2(s)+\mu\int\int_{\mathcal{R}}G(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)^{+}.\end{aligned}$$

Hence by Theorem 2.8, if $\lambda F+\mu G$ is interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to g_1 and g_2 on $\mathcal{R}=[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ then

$$\begin{aligned}&\int\int_{\mathcal{R}}(\lambda F(t,s)+\mu G(t,s))\Diamond g_1(t)\Diamond g_2(s)\\=&\ \lambda\int\int_{\mathcal{R}}F(t,s)\Diamond g_1(t)\Diamond g_2(s)+\mu\int\int_{\mathcal{R}}G(t,s)\Diamond g_1(t)\Diamond g_2(s).\end{aligned}$$

□

The proof of (ii) is just a straightforward application of (i) of the Theorem 2.11 since $F(t,s)=G(t,s)$.

Theorem 2.12. *Let $F:[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}\to I_{\mathbb{R}}$ be an interval-valued function and let g_1,g_2 be an increasing function on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$. If $F(t,s)\le G(t,s)$ nearly everywhere on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ and F,G are Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$, then*

$$\int\int_{\mathcal{R}}F(t,s)\Diamond g_1(t)\Diamond g_2(s)\le\int\int_{\mathcal{R}}G(t,s)\Diamond g_1(t)\Diamond g_2(s).$$

The proof of Theorem 2.12 follows easily by applying Theorem 2.11.

Theorem 2.13. *Let $F,G:[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}\to I_{\mathbb{R}}$ be interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable functions $(IVHS)$ and $D(F,G)$ is (LS) Lebesgue-Stieltjes-$\Diamond$-double integrable on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$. Then*

$$\begin{aligned}&D\left((IVHS)\int\int_{\mathcal{R}}F(t,s)\Diamond g_1(t)\Diamond g_2(s),\ (IVHS)\int\int_{\mathcal{R}}G(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)\\\le&\ (LS)\int\int_{\mathcal{R}}D(F(t,s),\ G(t,s))\Diamond g_1(t)\Diamond g_2(s),\end{aligned}$$

where the integrals on the left-hand side is the interval Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral, and that on the right-hand side is the Lebesgue-Stieltjes integral.

Proof. Let F^-,F^+,G^-,G^+ be interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$.

Denote $(IVHS)\int\int_{\mathcal{R}}F(t,s)\Diamond_1 g_1(t)\Diamond_2 g_2(s)$ by interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral and $LS\int\int_{\mathcal{R}}F(t,s)\Diamond g_1(t)\Diamond g_2(s)$ by Lebesgue-Stieltjes-$\Diamond$-double integral.

Then

$D\left((IVHS)\int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s),\ (IVHS)\int\int_{\mathcal{R}} G(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)$

$= \max\left(|\left((IVHS)\int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)^{-} - \left((IVHS)\int\int_{\mathcal{R}} G(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)^{-}|,\right.$

$\left.|\left((IVHS)\int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)^{+} - \left((IVHS)\int\int_{\mathcal{R}} G(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)^{+}|\right.$

$$\begin{aligned}
&= \max(|(IVHS)\int\int_{\mathcal{R}}(F^-(t,s)-G^-(t,s))\Diamond g_1(t)\Diamond g_2(s)|,\\
&\quad |(IVHS)\int\int_{\mathcal{R}}(F^+(t,s)-G^+(t,s))\Diamond g_1(t)\Diamond g_2(s)|)\\
&\le \max((LS)\int\int_{\mathcal{R}}|(F^-(t,s)-G^-(t,s))|\Diamond g_1(t)\Diamond g_2(s),\\
&\quad (LS)\int\int_{\mathcal{R}}|(F^+(t,s)-G^+(t,s))|\Diamond g_1(t)\Diamond g_2(s))\\
&\le ((LS)\int\int_{\mathcal{R}}\max(|(F^-(t,s)-G^-(t,s))|,|(F^+(t,s)-G^+(t,s))|\Diamond g_1(t)\Diamond g_2(s)\\
&= (LS)\int\int_{\mathcal{R}} D(F(t,s),G(t,s))\Diamond g_1(t)\Diamond g_2(s).
\end{aligned}$$

We see that $D\left(IVHS\int\int_{\mathcal{R}} F,\ \ IVHS\int\int_{\mathcal{R}} G\right) \le LS\int\int_{\mathcal{R}} D(F,G)$, as desired. $\square$

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