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SIMULTANEOUS APPROXIMATION OF NEW BASKAKOV OPERATORS

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Abstract. This paper provides direct results for a new Baskakov operator in simultaneous approximation. Also, we found a recurrence relation to calculate the moments for this operator. We discuss a uniform convergence, an asymptotic formula of the Voronovskaja type, and an error estimate about the modulus of continuity for this operator. Lastly, we provide some graphical examples of the convergence of the derivatives of the new Baskakov operator using the Maple algorithm.

Keywords. Baskakov operator, Simultaneous approximation, Modulus of continuity, Voronovskaja formula
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1. Introduction

In 1957 [1], Baskakov invented a sequence of positive linear operators known as Baskakov operators, which are based on the interval $[0,\infty)$, as following

$$B_n(f;x)=\sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right), \quad x\in[0,\infty), q_{n,k}(x)=\binom{n+k-1}{k}x^k(1+x)^{n+k}. \tag{1.1}$$

Numerous other investigators have conducted research in this area and have determined various approximation characteristics of numerous operators, we list a few of them as [2, 3, 5]. By using a new sequence of integral type based on two parameters, Mohammed and Hassan [8] established simultaneous approximation, thereby improving the approximation order. Asimultaneous approximation of a class of modified operators of the Baskakov type, which is a generalization of the conventional Baskakov sequence, was defined and explored by Mohammad and others [7] based on a parameter $s>\frac{-1}{2}$. The Baskakov Jain type operators containing two parameters, α and τ, were introduced by Kajla and Mohiuddine [6] in 2022. For $y>0$ and $f\in C_y(I)=\{f\in C(I):|f(x)|\le M(1+x^y),\forall x\in I\}$,

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where $I = [0, \infty)$ and M is positive constant dependent on f. Jabbar and Hassan [3] proposed a new modification of the Baskakov operators as follows :

$$B_n^1(f;x) = \sum_{k=0}^{\infty} q_{n,k}^1(x) f\left(\frac{k}{n}\right), \tag{1.2}$$

where $q_{n,k}^1(x) = \alpha(x,n)q_{n+1,k}(x) + \beta(x,n)q_{n+1,k-1}(x)$ and $\alpha(x,n) = a(n) + b(n)x$, $\beta(x,n) = a(n) - b(n) - b(n)x$. For $a(n) = b(n) = 1$ obviously (1.1) reduces to (1.2). In order to study the uniformly convergence, we consider that sequences $a(n)$ and $b(n)$ Verify the condition

$$2a(n) - b(n) = 1. \tag{1.3}$$

We will consider to cases for the sequences $a(n)$ and $b(n)$.

Case 1: Let $a(n) - b(n) \geq 0$ and $a(n) \geq 0$, using (1.3), we get $0 \leq a(n) \leq 1$ and $-1 \leq b(n) \leq 1$. In this case, the operator (1.2) is positive .

Case 2: Let $a(n) - b(n) < 0$ or $a(n) < 0$. If $a(n) - b(n) < 0$, we get $a(n) > 1$ and if $a(n) < 0$, we obtain $a(n) - b(n) > 1$. In this case, the operator (1.2) is not positive.

The work aims to examine a few direct results, such as error estimates in simultaneous .

2. Main Results

Lemma 2.1. *For the weighted function $q_{n,k}^1(x)$, we have*

$$x(1+x)(q_{n,k}^1)'(x) = (k-nx)q_{n,k}^1(x) - (a(n)-b(n))(xq_{n+1,k}(x) + (1+x)q_{n+1,k-1}(x))$$

Proof. Since $q_{n,k}^1(x) = \alpha(x,n)q_{n+1,k}(x) + \beta(x,n)q_{n+1,k-1}(x)$.

Hence,

$$\begin{aligned}(q_{n,k}^1)'(x) &= (a(n) + b(n)x)q_{n+1,k}'(x) + b(n)q_{n+1,k}(x) \\ &\quad + (a(n) - b(n) - b(n)x)q_{n+1,k-1}'(x) - b(n)q_{n+1,k-1}(x)\end{aligned} \tag{2.1}$$

Since, $x(1+x)q_{n,k}'(x) = (k - nx)q_{n,k}(x)$, thus

$$\begin{aligned}x(1+x)(q_{n,k}^1)'(x) &= (a(n) + b(n)x)(k - nx - x)q_{n+1,k}(x) + x(1+x)b(n)q_{n+1,k}(x) \\ &\quad + (a(n) - b(n) - b(n)x)(k - nx - x - 1)q_{n+1,k-1}(x) \\ &\quad - x(1+x)b(n)q_{n+1,k-1}(x)\end{aligned}$$

Therefore,

$$\begin{aligned}x(1+x)(q_{n,k}^1)'(x) &= (k - nx)q_{n,k}^1(x) - (xa(n) + x^2b(n) - xb(n) - x^2b(n))q_{n+1,k}(x) \\ &\quad - (xa(n) - xb(n) - x^2b(n) + a(n) - b(n) - b(n)x + xb(n) \\ &\quad + x^2b(n))q_{n+1,k-1}(x).\end{aligned}$$

This completes the proof. □

Theorem 2.2. *The m-th order moment for the operators (1.2) for $m \in \mathbb{N}^0$ is defined as*

$$T_{n,m}(x) = B_n^1((t-x)^m; x) = \sum_{k=0}^{\infty} q_{n,k}^1(x)\left(\frac{k}{n} - x\right)^m. \tag{2.2}$$

Then, $T_{n,0}(x) = 2a(n) - b(n) = 1$, $T_{n,1}(x) = \frac{(1+2x)(a(n)-b(n))}{n}$ and there holds the recurrence relation

$$nT_{n,m+1}(x) = x(1+x)(T'_{n,m}(x)+mT_{n,m-1}(x))+(a(n)-b(n))(x\lambda_{n,m}(x)+(1+x)\mu_{n,m}(x)). \tag{2.3}$$

Where,

$$\lambda_{n,m}(x) = \sum_{k=0}^{\infty} q_{n+1,k}(x)\left(\frac{k}{n} - x\right)^m \quad \text{and} \quad \mu_{n,m}(x) = \sum_{k=0}^{\infty} q_{n+1,k-1}(x)\left(\frac{k}{n} - x\right)^m.$$

Consequently,

(i) *$T_{n,m}(x)$ is a polynomial in x of degree m.*

(ii) *For each $x \in [0, \infty)$, $T_{n,m}(x) = O\left(n^{-\lfloor\frac{m+1}{2}\rfloor}\right)$ where $[\delta]$ denotes the integer part of δ.*

Proof. It is easy to show that $T_{n,0}(x) = 2a(n)-b(n) = 1$ and $T_{n,1}(x) = \frac{(1+2x)(a(n)-b(n))}{n}$.

We next demonstrate (2.3).

For $x = 0$, it clearly holds. For $x \in (0, \infty)$, we have

$$T'_{n,m}(x) = \sum_{k=0}^{\infty}(q_{n,k}^1)'(x)\left(\frac{k}{n} - x\right)^m - m\sum_{k=0}^{\infty} q_{n,k}^1(x)\left(\frac{k}{n} - x\right)^{m-1} \tag{2.4}$$

Multiply both sides of the equation above by $x(1+x)$ and using Lemma 2.1, we obtain

$$x(1+x)(T'_{n,m}(x) + mT_{n,m-1}(x)) = n\sum_{k=0}^{\infty} q_{n,k}^1(x)\left(\frac{k}{n} - x\right)^{m+1}$$
$$-(a(n)-b(n))\left(x\sum_{k=0}^{\infty} q_{n+1,k}(x)\left(\frac{k}{n} - x\right)^m + (1+x)\sum_{k=0}^{\infty} q_{n+1,k-1}(x)\left(\frac{k}{n} - x\right)^m\right)$$

$$nT_{n,m+1}(x) = x(1+x)\left(T'_{n,m}(x) + mT_{n,m-1}(x)\right)+(a(n)-b(n))\left(x\lambda_{n,m}(x) + (1+x)\mu_{n,m}(x)\right)$$

Hence, the proof (2.3) is Complete.

It is evident from the values of $T_{n,0}(x)$ and $T_{n,1}(x)$ that the outcomes (i) and (ii) hold. For $m = 0$ and $m = 1$. The induction on m and (2.3) make it simple to show outcome (i). If result (ii) for m is accurate, then by (2.3), we have

$$nT_{n,m+1}(x) = O\left(n^{-\lfloor\frac{m+1}{2}\rfloor}\right) + O\left(n^{-\lfloor\frac{m}{2}\rfloor}\right) + O\left(n^{-\lfloor\frac{m+1}{2}\rfloor}\right) + O\left(n^{-\lfloor\frac{m+1}{2}\rfloor}\right).$$

$$= \begin{cases} O\left(n^{-\frac{m-1}{2}}\right), & \text{if } m \text{ is odd} \\ O\left(n^{-\frac{m}{2}}\right), & \text{if } m \text{ is even} \end{cases}$$

Hence,

$$T_{n,m+1}(x) = \begin{cases} O\left(n^{-\frac{m+1}{2}}\right), & \text{if } m \text{ is odd} \\ O\left(n^{-\frac{m+2}{2}}\right), & \text{if } m \text{ is even} \end{cases}$$

Thus, for every $x \in [0, \infty)$, $T_{n,m+1}(x) = O\left(n^{-\lfloor \frac{m+2}{2} \rfloor}\right)$.

So, the result (ii) is true for $m + 1$. Therefore, the result (ii) is proved by mathematical induction for all m. □

Lemma 2.3. *If* $\phi_{n,m}(x) = \sum_{k=0}^{\infty} k^m q_{n,k}^1(x)$, *we have*

$$\phi_{n,m+1}(x) = x(1+x)\phi'_{n,m}(x) + nx\phi_{n,m}(x) + (a(n)-b(n))(xF_{n,m}(x) + (1+x)H_{n,m}(x)) \tag{2.5}$$

where $F_{n,m}(x) = \sum_{k=0}^{\infty} k^m q_{n+1,k}(x)$, $H_{n,m}(x) = \sum_{k=0}^{\infty} k^m q_{n+1,k-1}(x)$.

Proof. By using the Lemma 2.2, we obtain

$$\phi'_{n,m}(x) = \sum_{k=0}^{\infty} k^m (q_{n,k}^1)'(x),$$

$$x(1+x)\phi'_{n,m}(x) = \sum_{k=0}^{\infty} k^m (k - nx) q_{n,k}^1(x) - (a(n) - b(n))k^m (xq_{n+1,k}(x) + (1+x)q_{n+1,k-1}(x)).$$

Therefore,

$$x(1+x)\phi'_{n,m}(x) = \phi_{n,m+1}(x) - nx\phi_{n,m}(x) - (a(n)-b(n))(xF_{n,m}(x) + (1+x)H_{n,m}(x))$$

This, completes the result. □

Lemma 2.4. *For the Linear positive operator* $B_n^1(f; x)$, *we have*

$$\begin{aligned} B_n^1(f;x) &= \left(\frac{a(n)(n+m-1)!}{n^m(n-1)!} + (1 + \frac{m(m+3)}{2n})(a(n) - b(n)) \right) x^m \\ &+ \left(\frac{a(n)(n+m-2)}{n^m(n-1)!} + (\frac{m(m+1)}{2n})(a(n) - b(n)) \right) x^{m-1} + O(n^{-2}). \end{aligned} \tag{2.6}$$

Proof. If $m = 1$, then $\phi_{n,1}(x) = n(2a(n) - b(n)) + (1 + 2x)(a(n) - b(n))$.

$\phi_{n,1}(x) = (na(n) + n(a(n) - b(n)) + 2(a(n) - b(n)))x + (a(n) - b(n))$

If $m = 2$, then

$$\begin{aligned} \phi_{n,2}(x) &= n^2x^2(2a(n) - b(n)) + n(6x^2 + 4x)a(n) - n(5x^2 + 3x)b(n) \\ &+ (1 + 4x + 4x^2)(a(n) - b(n)). \\ &= (n(n+1)a(n) + (n^2 + 5n + 4)(a(n) - b(n)))x^2 \\ &+ (na(n) + (3n + 4)(a(n) - b(n)))x + (a(n) - b(n)) \end{aligned}$$

If $m = 3$, then

$$\begin{aligned}\phi_{n,3}(x) &= n^3x^3(2a(n) - b(n)) + n^2(12x^3 + 9x^2)a(n) - n^2(9x^3 + 6x^2)b(n)\\ &+ n(22x^3 + 27x^2 + 8x)a(n)\\ &- n(20x^3 + 24x^2 + 7x)b(n) + (1 + 8x + 18x^2 + 12x^3)(a(n) - b(n))\\ &= (n(n+1)(n+2)a(n) + (n^3 + 9n^2 + 20n + 12)(a(n) - b(n)))x^3\\ &+ (3n(n+1)a(n) + (6n^2 + 24n + 18)(a(n) - b(n)))x^2\\ &+ (na(n) + (7n+8)(a(n) - b(n)))x + (a(n) - b(n))\end{aligned}$$

In general,

$$\begin{aligned}\phi_{n,m}(x) = &\left(n(n+1)...(n+m-1)a(n) + \left(n^m + \frac{m(m+3)}{2n} + ...\right)(a(n) - b(n))\right)x^m\\ &+ \left(\frac{m(m-1)}{2}a(n)(n(n-1)...(n+m-2)) + \left(\frac{m(m+1)}{2} + ...\right)(a(n) - b(n))\right)x^{m-1}\\ &+ ... + C_{m-1}x + (a(n) - b(n)).\end{aligned}$$

Hence, the proof is Complete. □

Lemma 2.5. *There exist polynomials $\theta_{i,j,r}(x)$ such that*

$$\begin{aligned}\frac{d^r}{dx^r}q^1_{n,k}(x) = &\sum_{\substack{2i+j\le r\\ i,j\ge 0}} \theta_{i,j,r}(x)n^i\left[(x+x^2)^{-r}(k-nx)^j q^1_{n,k}(x)\right.\\ &\left.+(b(n)-a(n))(k-nx)^{j-1}((1+x)^{-r}q_{n+1,k}(x) + x^{-r}q_{n+1,k-1}(x))\right]\end{aligned} \tag{2.7}$$

Proof. Assuming that the outcome holds true for r, use the induction hypothesis and

$$x(1+x)(q^1_{n,k})'(x) = (k-nx)q^1_{n,k}(x) + (b(n)-a(n))(xq_{n+1,k}(x) + (1+x)q_{n+1,k-1}(x))$$

To prove that this result is true for $r+1$.

$$\frac{d^{r+1}}{dx^{r+1}}q^1_{n,k}(x) = \frac{d}{dx}\left(\frac{d^r}{dx^r}q^1_{n,k}(x)\right)$$

$$\begin{aligned}&= \sum_{\substack{2i+j\le r\\ i,j\ge 0}} \theta'_{i,j,r}(x)(x+x^2)^{-r}n^i(k-nx)^j q^1_{n,k}(x) - \sum_{\substack{2i+j\le r\\ i,j\ge 0}} r(1+2x)\theta_{i,j,r}(x)(x+x^2)^{-r-1}n^i(k-nx)^j q^1_{n,k}(x)\\ &- \sum_{\substack{2i+j\le r\\ i,j\ge 0}} j\theta_{i,j,r}(x)(x+x^2)^{-r}n^{i+1}(k-nx)^{j-1}q^1_{n,k}(x) + \sum_{\substack{2i+j\le r\\ i,j\ge 0}} \theta_{i,j,r}(x)(x+x^2)^{-r}n^i(k-nx)^{j+1}q^1_{n,k}(x)\\ &+ (b(n)-a(n))\sum_{\substack{2i+j\le r\\ i,j\ge 0}} \theta_{i,j,r}(x)n^i(k-nx)^j(x^{-r}(1+x)^{-r-1}q_{n+1,k}(x) + x^{-r-1}(1+x)^{-r}q_{n+1,k-1}(x))\\ &+ (b(n)-a(n))\sum_{\substack{2i+j\le r\\ i,j\ge 0}} \theta'_{i,j,r}(x)n^i(k-nx)^{j-1}((1+x)^{-r}q_{n+1,k}(x) + x^{-r}q_{n+1,k-1}(x))\end{aligned}$$

$$
\begin{aligned}
&-(b(n)-a(n))\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(j-1)\theta_{i,j,r}(x)n^{i+1}(k-nx)^{j-2}((1+x)^{-r}q_{n+1,k}(x)+x^{-r}q_{n+1,k-1}(x))\\
&+(b(n)-a(n))\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\theta_{i,j,r}(x)n^{i}(k-nx)^{j-1}(-r(1+x)^{-r-1}q_{n+1,k}(x)-rx^{-r-1}q_{n+1,k-1}(x))\\
&-(b(n)-a(n))\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\theta_{i,j,r}(x)n^{i}(k-nx)^{j}((1+x)^{-r-1}q_{n+1,k}(x)+x^{-r-1}q_{n+1,k-1}(x))
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\theta'_{i,j,r}(x)(x+x^2)^{-r}n^{i}(k-nx)^{j}q^{1}_{n,k}(x)\\
&-\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}r(1+2x)\theta_{i,j,r}(x)(x+x^2)^{-(r+1)}n^{i}(k-nx)^{j}q^{1}_{n,k}(x)\\
&-\sum_{\substack{2(i-1)+(1+j)\leq r\\ i,j\geq 0}}(1+j)\theta_{i-1,j+1,r}(x)(x+x^2)^{-r}n^{i}(k-nx)^{j}q^{1}_{n,k}(x)\\
&+\sum_{\substack{2i+(j-1)\leq r\\ i,j\geq 0}}\theta_{i,j-1,r}(x)(x+x^2)^{-(r+1)}n^{i}(k-nx)^{j}q^{1}_{n,k}(x)\\
&+(b(n)-a(n))\sum_{\substack{2i+(j-1)\leq r\\ i,j\geq 0}}\theta_{i.j-1}(x)n^{i}(k-nx)^{j-1}(x^{-r}(1+x)^{-r-1}q_{n+1,k}(x)\\
&+x^{-r-1}(1+x)^{-r}q_{n+1,k-1}(x))\\
&+(b(n)-a(n))\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\theta'_{i,j,r}(x)n^{i}(k-nx)^{j-1}((1+x)^{-r}q_{n+1,k}(x)+x^{-r}q_{n+1,k-1}(x))\\
&-(b(n)-a(n))\sum_{\substack{2(i-1)+(j+1)\leq r\\ i,j\geq 0}}(j)\theta_{i-1,j+1,r}(x)n^{i}(k-nx)^{j-1}((1+x)^{-r}q_{n+1,k}(x)+x^{-r}q_{n+1,k-1}(x))\\
&-(b(n)-a(n))\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}r\theta_{i,j,r}(x)n^{i}(k-nx)^{j-1}((1+x)^{-r-1}q_{n+1,k}(x)+x^{-r-1}q_{n+1,k-1}(x))\\
&-(b(n)-a(n))\sum_{\substack{2i+(j-1)\leq r\\ i,j\geq 0}}\theta_{i,j-1}(x)n^{i}(k-nx)^{j-1}((1+x)^{-r-1}q_{n+1,k}(x)+x^{-r-1}q_{n+1,k-1}(x))
\end{aligned}
$$

The last expression has the required from

$$
\begin{aligned}
\frac{d^{r+1}}{dx^{r+1}}q^{1}_{n,k}(x)&=\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\theta_{i,j,r+1}(x)n^{i}\left[(x+x^2)^{-r-1}(k-nx)^{j}q^{1}_{n,k}(x)\right.\\
&\left.+(b(n)-a(n))(k-nx)^{j-1}((1+x)^{-r-1}q_{n+1,k}(x)+x^{-r-1}q_{n+1,k}(x))\right]
\end{aligned}
$$

Where

$$\begin{aligned}\theta_{i,j,r+1}(x) &= \theta'_{i,j,r}(x)\left(x+x^2\right) - r(1+2x)\theta_{i,j,r}(x) - (1+j)\left(x+x^2\right)\theta_{i-1,j+1,r}(x)\\ &\quad + \theta_{i,j-1,r}(x) + \theta_{i,j-1,r}(x)\left(x^{-r}+(1+x)^{-r}\right) + \theta'_{i,j,r}(x)(1+2x)\\ &\quad - (j)\theta_{i-1,j+1,r}(x)(1+2x) - r\theta_{i,j,r} - \theta_{i,j-1,r}\end{aligned}$$

and $2i+j \le r+1, i,j \ge 0$.

Hence, the result is true for $r+1$. For $r=0$, the result is trivial. Therefore, by induction the result holds for all r. □

3. Simultaneous Approximation

The derivatives of the operators (2.1) are first demonstrated in this section to be approximation processes for corresponding order derivative of the function,, i.e., we prove that

$$(B_n^1)^{(r)}(f;x) \to f^{(r)}(x), \quad \text{as } n\to\infty,\ r=1,2,3,...$$

Theorem 3.1. *If $f \in C_y(I)$ and $f^{(r)}$ located at the point $x \in (0,\infty)$, thus*

$$\lim_{n\to\infty}(B_n^1)^{(r)}(f;x) = f^{(r)}(x) \tag{3.1}$$

Additionally, if $f^{(r)}$ is continuous and located on $(a+\eta, b+\eta) \subset (0,\infty)$, $\eta>0$, then (3.1) holds uniformly in $x\in[a,b]$.

Proof. By the Taylor's expansion, we obtain

$$f(t) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$.

The fore

$$\begin{aligned}(B_n^1)^{(r)}(f;x) &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(B_n^1)^{(r)}((t-x)^i;x) + (B_n^1)^{(r)}(\varepsilon(t,x)(t-x)^r;x)\\ &= I_1 + I_2.\end{aligned}$$

Now, by applying Lemma 2.4, obtain

$$I_1 = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}(B_n^1)^{(r)}(t^j;x).$$

Consequently,

$$\begin{aligned}I_1 &= \frac{f^{(r)}(x)}{r!}(B_n^1)^{(r)}(t^r;x),\\ &= \frac{f^{(r)}(x)}{r!}\left[\frac{a(n)(n+r-1)!}{n^r(n-1)!} + \left(1+\frac{r(r+3)}{2n}\right)(a(n)-b(n))\right]r!\\ &= f^{(r)}(x)\left[2a(n)-b(n)+\frac{r(r+3)}{2n}(a(n)-b(n))\right] \to f^{(r)}(x), \text{ as } n\to\infty.\end{aligned}$$

Next, making use of Lemma 2.5, we have

$$
\begin{aligned}
I_2 &= \sum_{k=0}^{\infty} \left(q_{n,k}^1\right)^{(r)}(x)\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^r \\
&= \sum_{\substack{2i+j\le r\\ i,j\ge 0}} \theta_{i,j,r}(x)n^i \sum_{k=0}^{\infty}\Big[\left(x+x^2\right)^{-r}(k-nx)^j q_{n,k}^1(x) \\
&+(b(n)-a(n))(k-nx)^{j-1}\left((1+x)^{-r}q_{n+1,k}(x)+x^{-r}q_{n+1,k}(x)\right)\Big]\varepsilon\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^r
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|I_2| &\le \sum_{\substack{2i+j\le r\\ i,j\ge 0}} \frac{|\theta_{i,j,r}(x)|\, n^{i+j}}{(x+x^2)^r}\sum_{k=0}^{\infty} q_{n,k}^1(x)\left|\frac{k}{n}-x\right|^{j+r}\left|\varepsilon\left(\frac{k}{n},x\right)\right| \\
&+ \sum_{\substack{2i+j\le r\\ i,j\ge 0}} \frac{|\theta_{i,j,r}(x)|\, n^{i+j-1}}{(x+x^2)^r}\sum_{k=0}^{\infty}\left(x^r q_{n+1,k}(x)+(1+x)^r q_{n+1,k-1}(x)\right)\left|\frac{k}{n}-x\right|^{j+r-1}\left|\varepsilon\left(\frac{k}{n},x\right)\right| \\
&= I_3+I_4.
\end{aligned}
$$

Since $\varepsilon\left(\frac{k}{n},x\right)\to 0$ as $\frac{k}{n}\to x$. There is a $\delta>0$ such that $\left|\varepsilon\left(\frac{k}{n},x\right)\right|<\varepsilon$ for each given $\varepsilon>0$,
whenever $0<\left|\frac{k}{n}-x\right|<\delta$. For $\left|\frac{k}{n}-x\right|\ge\delta$, there is a constant $C>0$ that makes it possible for $\left|\left(\frac{k}{n}-x\right)^r\varepsilon\left(\frac{k}{n},x\right)\right|\le C\left|\frac{k}{n}-x\right|^{\beta}, \beta\in Z$. Thus,

$$
\begin{aligned}
I_3 &\le M\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+j}\left(\sum_{\left|\frac{k}{n}-x\right|<\delta}\varepsilon q_{n,k}^1(x)\left|\frac{k}{n}-x\right|^{j+r}+\sum_{\left|\frac{k}{n}-x\right|\ge\delta} C q_{n,k}^1(x)\left|\frac{k}{n}-x\right|^{j+\beta}\right) \\
&= I_5+I_6.
\end{aligned}
$$

Let $\sup_{\substack{2i+j\le r\\ i,j\ge 0}}\frac{|\theta_{i,j,r}(x)|}{(x+x^2)^r}=M$. By utilizing Schwarz inequality , we obtain

$$
\begin{aligned}
I_5 &\le \varepsilon M(x)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+j}\left(\sum_{k=0}^{\infty} q_{n,k}^1(x)\left(\frac{k}{n}-x\right)^{2j}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty} q_{n,k}^1(x)\left(\frac{k}{n}-x\right)^{2r}\right)^{\frac{1}{2}} \\
&\le \varepsilon M(x)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+j} O\left(n^{\frac{-j}{2}}\right) O\left(n^{\frac{-r}{2}}\right) \\
&= \varepsilon M(x)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i} O\left(n^{\frac{j}{2}}\right) O\left(n^{\frac{-r}{2}}\right)=\varepsilon O(1).
\end{aligned}
$$

Next, again using Schwarz inequality, we get

$$
\begin{aligned}
I_6 &\leq C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+j} \sum_{\left|\frac{k}{n}-x\right|\geq\delta} C q^1_{n,k}(x) \left|\frac{k}{n}-x\right|^{j+\beta} \\
&\leq C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+j} \left(\sum_{k=0}^{\infty} q^1_{n,k}(x)\left(\frac{k}{n}-x\right)^{2j}\right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} q^1_{n,k}(x)\left(\frac{k}{n}-x\right)^{2\beta}\right)^{\frac{1}{2}} \\
&\leq C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+j} O\left(n^{\frac{-j}{2}}\right) O\left(n^{\frac{-\beta}{2}}\right) \\
&= C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i} O\left(n^{\frac{j}{2}}\right) O\left(n^{\frac{-\beta}{2}}\right) = O\left(n^{\frac{r-\beta}{2}}\right) = o(1), \text{ for } \beta > r.
\end{aligned}
$$

Hence in view of the arbitrariness of $\varepsilon > 0$. It consequently follows that $I_3 = o(1)$. Also $I_4 \to 0$ as $n \to \infty$ and hence $I_2 = o(1)$. Adding together the I_1 and I_2 estimates, we obtain (3.1). □

We next establish an asymptotic formula of the voronvskaja type for operators $(B^1_n)^{(r)}(f;x)$.

Theorem 3.2. *If $f \in C_\gamma(I)$ and $f^{(r+2)}$ located at the point $x \in (0,\infty)$. Thus,*

$$
\begin{aligned}
&\lim_{n\to\infty} n\left(\left(B^{M,1}_n\right)^{(r)}(f;x) - f^{(r)}(x)\right) \\
&\qquad = \frac{r(r+3)}{2}(\mu-\theta) f^{(r)}(x) + \frac{(1+2x)(r+2(\mu-\theta))}{2} f^{(r+1)}(x) \\
&\qquad + \left(\frac{(r+2)(r+1)(\mu-\theta)}{4}x^2 + \frac{x(1+\mu x)}{2}\right) f^{(r+2)}(x) \qquad (3.2)
\end{aligned}
$$

Where $\lim_{n\to\infty} a(n) = \mu$ and $\lim_{n\to\infty} b(n) = \theta$. Additionally, if $f^{(r+2)}$ is continuous and located on $(a+\eta, b+\eta) \subset (0,\infty), \eta > 0$, then (3.2) holds uniformly in $x \in [a,b]$.

Proof. Through Taylor's expansion , we obtain

$$
\begin{aligned}
\left(B^1_n\right)^{(r)}(f;x) &= \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \left(B^1_n\right)^{(r)}\left((t-x)^i;x\right) + \left(B^1_n\right)^{(r)}\left(\varepsilon(t,x)(t-x)^r;x\right) \\
&= I_1 + I_2
\end{aligned}
$$

when $\varepsilon(t,x) \to 0$ as $t \to x$. Then

$$
\begin{aligned}
I_1 &= \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \left(B_n^1\right)^{(r)} (t^r; x) \\
&= \frac{f^{(r)}(x)}{r!} \left(B_n^1\right)^{(r)} (t^r; x) + \frac{f^{(r+1)}(x)}{(r+1)!} \left(\binom{r+1}{r} (-x) \left(B_n^1\right)^{(r)} (t^r; x) + \left(B_n^1\right)^{(r)} \left(t^{r+1}; x\right) \right) \\
&+ \frac{f^{(r+2)}(x)}{(r+2)!} \left(\binom{r+2}{r} x^2 \left(B_n^1\right)^{(r)} (t^r; x) + \binom{r+2}{r+1} (-x) \left(B_n^1\right)^{(r)} \left(t^{r+1}; x\right) \right. \\
&\left. + \left(B_n^1\right)^{(r)} \left(t^{r+2}; x\right) \right) \qquad (3.3)
\end{aligned}
$$

Lemma (2.4) enables us to have

$$
\left(B_n^1\right)^{(r)} (t^r; x) = \left((2a(n) - b(n)) + \frac{r(r+3)}{2n} (a(n) - b(n)) \right) r! \qquad (3.4)
$$

$$
\begin{aligned}
\left(B_n^1\right)^{(r)} \left(t^{r+1}; x\right) &= \left(\frac{a(n)(n+r)}{n} + \left(1 + \frac{(r+1)(r+4)}{2n}\right) (a(n) - b(n)) \right) x(r+1)! \\
&+ \left(\frac{a(n) r(r+1)}{2n} + \left(\frac{(r+1)(r+2)}{2n} \right) (a(n) - b(n)) \right) r! \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
\left(B_n^1\right)^{(r)} \left(t^{r+2}; x\right) &= \left(\frac{a(n)(n+r+1)(n+r)}{n^2} + \left(1 + \frac{(r+2)(r+5)}{2n}\right) (a(n) - b(n)) \right) x^2 \frac{(r+2)!}{2} \\
&+ \left(\frac{a(n)(r+2)(r+1)(n+r)}{2n^2} + \left(\frac{(r+3)(r+2)}{2n} \right) (a(n) - b(n)) \right) x(r+1)! \\
&\qquad (3.6)
\end{aligned}
$$

Equation (3.3) can be obtained by substituting equations (3.4), (3.5), and (3.6).

$$I_1 = f^{(r)}(x)\left(1+\frac{r(r+3)}{2n}(a(n)-b(n))\right)$$
$$+ f^{(r+1)}(x)\left((2a(n)-b(n))(-x)-\frac{r(r+3)}{2n}(a(n)-b(n))x\right)$$
$$+ f^{(r+1)}(x)\left(a(n)x+\frac{a(n)rx}{n}+(a(n)-b(n))x+\frac{(r+1)(r+4)}{2n}\right.$$
$$\left.(a(n)-b(n))x+\frac{a(n)r}{2n}+\frac{(r+2)}{2n}(a(n)-b(n))\right)$$
$$+ f^{(r+2)}(x)\left((2a(n)-b(n))\frac{x^2}{2}+\frac{r(r+3)}{4n}(a(n)-b(n))x^2\right)$$
$$+ f^{(r+2)}(x)\left(-a(n)x^2-\frac{a(n)rx^2}{n}-(a(n)-b(n))x^2-\frac{(r+1)(r+4)}{2n}(a(n)-b(n))x^2-\frac{a(n)rx}{2n}\right.$$
$$\left.-\frac{(r+2)}{2n}(a(n)-b(n))x\right)$$
$$+ f^{(r+2)}(x)\left(\frac{a(n)\left(n^2+2nr+r^2+n+r\right)}{2n^2}x^2+\left(1+\frac{(r+2)(r+5)}{2n}\right)\right.$$
$$\left.(a(n)-b(n))\frac{x^2}{2}+\frac{a(n)\left(r^2+nr+n+r\right)}{2n^2}x+\left(\frac{(r+3)}{2n}\right)(a(n)-b(n))x\right)$$

Hence.

$$\left(B_n^1\right)^{(r)}(f;x)-f^{(r)}(x)=\frac{r(r+3)}{2n}(a(n)-b(n))f^{(r)}(x)$$
$$+ f^{(r+1)}(x)\left(\frac{a(n)r(1+2x)}{2n}+\frac{(r+2)(1+2x)}{2n}(a(n)-b(n))\right)$$
$$+ f^{(r+2)}(x)\left(\frac{a(n)\left(x^2+x\right)+(a(n)-b(n))x}{2n}+\frac{(r+2)(r+1)}{4n}x^2(a(n)-b(n))\right.$$
$$\left.+\frac{a(n)\left(r^2+r\right)\left(x^2+x\right)}{2n^2}\right)$$

Thus to prove (3.2), it is sufficient to show that for each $x \in (0,\infty), nI_2 \to 0$ as $n \to \infty$, which follows on proceeding along the lines of proof of $I_2 \to 0$ as $n \to \infty$ in theorem 3.1. This, completes the result

□

Theorem 3.3. *If $f \in C_\gamma(I)$ and $r \le q \le r+2$. If $f^{(q)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then for sufficiently large n.*

$$\left\|\left(B_n^1\right)^{(r)}(f(t);x)-f^{(r)}(x)\right\|_{C[a,b]} \le \frac{C_1}{n}\sum_{i=r}^{q}\left\|f^{(i)}\right\|_{C[a,b]}+\frac{C_2}{\sqrt{n}}\omega_{f^{(q)}}\left(\frac{1}{\sqrt{n}}\right)+O\left(n^{-2}\right)$$

where C_1 and C_2 are constants independent of f and n.

Proof. By our hypothesis,

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^{i}+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^{q}\chi(t)+h(x,t)(1-\chi(t))$$

where $\chi(t)$ is the characteristic function of $(a-\eta,b+\eta)$, ξ lies between t,x and

$$h(x,t)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^{i}$$

Nown

$$\begin{aligned}\left(B_n^1\right)^{(r)}(f;x)-f^{(r)}(x)&=\left(\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\left(B_n^1\right)^{(r)}\left((t-x)^{i};x\right)-f^{(r)}(x)\right)\\&+\left(B_n^1\right)^{(r)}\left(\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^{q}\chi(t);x\right)++\left(B_n^1\right)^{(r)}\left(h(x,t)(1-\chi(t));x\right)\\&=I_1+I_2+I_3\end{aligned}$$

Lemma 2.4 is used to obtain

$$I_1=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\left(B_n^1\right)^{(r)}\left(t^{j};x\right)-f^{(r)}(x)$$

$$=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^{r}}{dx^{r}}\left(\left(\frac{a(n)(n+j-1)!}{n^{j}(n-1)!}+\left(1+\frac{j(j+3)}{2n}\right)(a(n)-b(n))\right)x^{j}\right.$$

$$\left.+\left(\frac{a(n)(n+j-2)!}{n^{j}(n-1)!}+\left(\frac{mj(m+1)}{2n}\right)(a(n)-b(n))\right)x^{j-1}+O\left(n^{-2}\right)\right)-f^{(r)}(x)$$

Hence.

$$\left\|I_1\right\|_{C[a,b]}\leq\frac{C_1}{n}\sum_{i=r}^{q}\left\|f^{(i)}\right\|_{C[a,b]}+O\left(n^{-2}\right),\ \text{unifromly on } [a,b]$$

Next , we estimate I_2 as

$$|I_2| \leq \sum_{k=0}^{\infty} \left| \left(q_{n,k}^1\right)^{(r)}(x) \right| \frac{\left|f^{(q)}(\xi) - f^{(q)}(x)\right|}{q!} \left| \frac{k}{n} - x \right|^q \chi(t)$$
$$\leq \frac{\omega_f(\delta)}{q!} \sum_{k=0}^{\infty} \left| \left(q_{n,k}^1\right)^{(r)}(x) \right| \left(1 + \frac{\left|\frac{k}{n} - x\right|}{\delta}\right) \left| \frac{k}{n} - x \right|^q$$
$$= \frac{\omega_f(\delta)}{q!} \sum_{k=0}^{\infty} \left| \left(q_{n,k}^1\right)^{(r)}(x) \right| \left(\left| \frac{k}{n} - x \right|^q + \frac{\left|\frac{k}{n} - x\right|^{q+1}}{\delta} \right)$$

Now . for $s = 0, 1, 2, \ldots,$ we have

$$\sum_{k=0}^{\infty} \left(q_{n,k}^1\right)(x)|k - nx|^j \left| \frac{k}{n} - x \right|^s = \sum_{k=0}^{\infty} n^j \left(q_{n,k}^1\right)(x) \left| \frac{k}{n} - x \right|^j \left| \frac{k}{n} - x \right|^s$$
$$\leq n^j \left(\sum_{k=0}^{\infty} q_{n,k}^1(x) \left(\frac{k}{n} - x\right)^{2j} \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} q_{n,k}^1(x) \left(\frac{k}{n} - x\right)^{2s} \right)^{\frac{1}{2}}$$
$$\leq O\left(n^{\frac{j}{2}}\right) O\left(n^{\frac{-s}{2}}\right) = O\left(n^{\frac{j-s}{2}}\right), \text{ unifromly on } [a, b] \tag{3.7}$$

Therefore by Lemma (2.5) and (3.7), we get

$$\sum_{k=0}^{\infty} \left| \left(q_{n,k}^1\right)^{(r)}(x) \right| \left| \frac{k}{n} - x \right|^s \leq \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \frac{|\theta_{i,j,r}(x)|\, n^{i+j}}{(x + x^2)^r} \sum_{k=0}^{\infty} q_{n,k}^1(x) \left| \frac{k}{n} - x \right|^k \left| \frac{1}{n} - x \right|^s$$
$$+ \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \frac{|\theta_{i,j,r}(x)|\, n^{i+j-1}}{(x + x^2)^r} (a(n) - b(n)) \sum_{k=0}^{\infty} \left(x^r q_{n+1,k}(x) + (1 + x)^r q_{n+1,k-1}(x)\right) \left| \frac{k}{n} - x \right|^{j+s-1}$$
$$\leq M \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i O\left(n^{\frac{j-s}{2}}\right) = O\left(n^{\frac{r-s}{2}}\right) \tag{3.8}$$

uniformly on [a, b] , where $M = \sup_{\substack{i+j \leq r \\ i,j \geq 0}}^{\sup} {}_{x \in [a,b]} \frac{|\theta_{i,jr}(x)|}{(x+x^2)^r}$. Choosing $\delta = \frac{1}{\sqrt{n}}$ and making use of (3.8), we get

$$|I_2| \leq \frac{\omega_{f^{(q)}}\left(\frac{1}{\sqrt{n}}\right)}{q!} \sum_{k=0}^{\infty} \left(O\left(n^{\frac{r-s}{2}}\right) + \sqrt{n} O\left(n^{\frac{r-s-1}{2}}\right) \right) \leq C_2 \omega_{f^{(q)}} \left(\frac{1}{\sqrt{n}}\right) n^{\frac{r-s}{2}}$$

Since $t \in [0, \infty)/(a - \eta, b + \eta)$, we can choose $\delta > 0$ in such a way that $\left|\frac{k}{n} - x\right| \geq \delta$ for all $x \in [a, b]$. Thus, by Lemma 2.5, we have

$$|I_3| \leq \sum_{k=0}^{\infty} \left| \left(q_{n,k}^1\right)^{(r)}(x) \right| \left| h\left(\frac{k}{n}, x\right) \right|$$

We can identify a constant $C > 0$ such that $\left|h\left(\frac{k}{n}, x\right)\right| \leq C_2 \left|\frac{k}{n} - x\right|^{\beta}$, for $\left|\frac{k}{n} - x\right| \geq \delta$. Now applying Schwarz inequality , it is east to see that $I_3 = O\left(n^{-s}\right)$, for any $s > 0$, uniformly on [a, b]. When the estimations of I_1, I_2 and I_3 are combined, the necessary outcome is obtained. $\square$

4. Numerical results:

The theoretical result that was stated in the earlier sections will be examined using numerical examples in this section . The convergence of the derivatives of the operator $B_n^1(f;x)$ to the derivatives of the function $f(x)$.

Example 4.1. Let $f(x) = \sin(2x) \qquad x \in [0, 4]$.

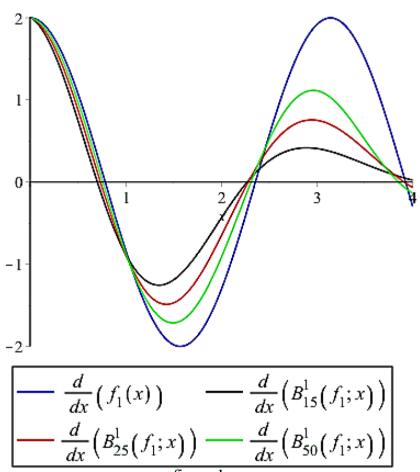

figure 1

Example 4.2. Let $f_2(x) = \cos(2x) \qquad x \in [0, 3]$.

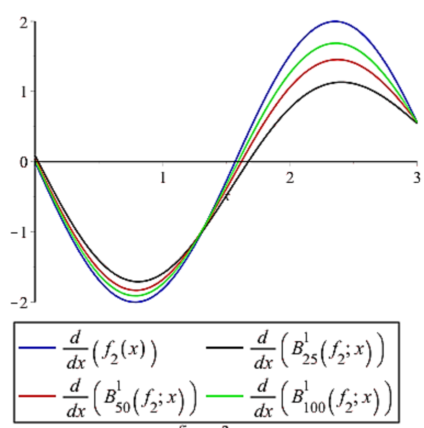

figure 2

Example 4.3. Let $f_3(x) = (1 - x^2)\sin(2\pi x) \qquad xin[0, 1]$.

Example 4.4. Let $f_4(x) = (1 - x)^2 \cos(2\pi x) \qquad xin[0, 1]$.

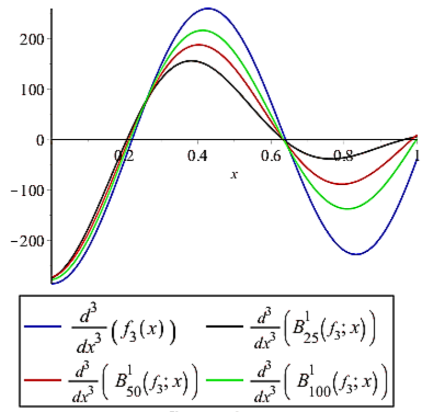

figure 3

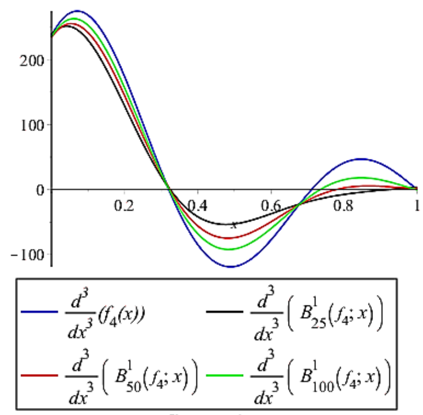

figure 4

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