

GENERALIZED TOPOLOGICAL D-ALGEBRA

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ABSTRACT. This paper introduces the concept of a generalized topological d-algebra, which extends the structure of a d-algebra by incorporating a generalized topology. We explore key properties such as closure operators, separation axioms, and continuity within this framework. Additionally, we define a generalized topology derived from a quasi-uniform structure based on the d^* -ideal of a d-algebra and establish some fundamental results related to it.

Keywords. Generalized topology, quasi-uniformity, d-algebra, topological d-algebra

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1. INTRODUCTION

One of the key ideas linking topology and algebra is the need for algebraic operations to be continuous in a topological sense. This continuity might be required in one variable, the other, or both, forming the foundation of what is known as topological algebra. The relationship between topology and algebra has been a major area of interest in modern mathematical research [1, 2, 3, 4].

Generalized topology (GT), introduced by Császár [5], extends classical topology by loosening the conditions on open sets. This flexible approach allows for a broader analysis of continuity, convergence, and separation properties, making it useful in fields such as multi-criteria decision-making, fuzzy logic, and algebraic structures [6, 11, 17, 16]. Meanwhile, d-algebras, first developed by Neggers and Kim [8], provide an algebraic framework for logical and computational models. So far, research on d-algebras has mainly focused on their algebraic properties and ideal structures [9]. Some studies have examined their topological aspects, such as topological d-algebras [12, 13] and uniform structures on d-algebras [14]. However, these approaches primarily rely on classical topology, which motivates us to generalize the concept of topological algebra by replacing classical topology with generalized topology.

In this paper, we introduce the concept of a GT-d-algebra, which integrates a generalized topology into a d-algebra. Our work presents several key results that build upon existing research. We examine closure operators in GT-d-algebras

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and establish the equivalence of separation axioms $(\mu-T_0, \mu-T_1, \mu-T_2)$. We also investigate μ -open filters and ideals, proving that left and right mappings are (μ, μ) -continuous. Additionally, we construct a μ -space (X, μ_I) using a quasi-uniformity based on the d^* -ideal of a d -algebra X and show that this space is μ -compact if and only if the quasi-uniformity is totally bounded. Lastly, we demonstrate that if the complement of a d^* -ideal is countable, then (X, μ_I) is μ -Lindelöf. These findings enhance our understanding of how topology and algebra interact, providing deeper theoretical insights and potential applications.

2. PRELIMINARIES

Let $X \neq \emptyset$, and let $\mathcal{P}(X)$ denote the power set of X . A subset $\mu \subseteq \mathcal{P}(X)$ is called a generalized topology on X [5] if it satisfies the following conditions:

- (1) $\emptyset \in \mu$,
- (2) The arbitrary union of elements of μ belongs to μ .

A pair (X, μ) is called a generalized topological space. In such a space, a subset $S \subseteq X$ is said to be μ -open if $S \in \mu$, and the complement of a μ -open set is called μ -closed. The μ -interior of S , denoted by $i_\mu(S)$, is the largest μ -open subset contained within S , while the μ -closure of S , denoted by $c_\mu(S)$, is the smallest μ -closed subset containing S . The union of all μ -open sets in X is denoted by M_μ .

Definition 2.1. [7] Let (V, μ) be a generalized topological space and $S \subseteq V$, then

- (1) $s \in i_\mu(S) \Leftrightarrow$ there exists a μ -open set P containing s such that $P \subseteq S$.
- (2) $s \in c_\mu(S) \Leftrightarrow$ for any μ -open set P containing s such that $P \cap S \neq \emptyset$.

Definition 2.2. [10] A generalized topological space (X, μ) is said to be:

- (1) $\mu-T_0$ if for each $s, t \in M_\mu$ and $s \neq t$, there exists an μ -open set containing one of them but not the other.
- (2) $\mu-T_1$ if for each $s, t \in M_\mu$ and $s \neq t$, there exist $S \in \mu_s$ and $T \in \mu_t$ such that $t \notin S$ and $s \notin T$.
- (3) $\mu-T_2$ if for each $s, t \in M_\mu$ and $s \neq t$, there exist $S \in \mu_s$ and $T \in \mu_t$ such that $S \cap T = \emptyset$.

Definition 2.3. [8] A d -algebra is a non-empty set X that includes a constant 0 and has a binary operation denoted by $\star$, satisfying the following properties for all $s, t \in X$:

- (1) $s \star s = 0$,
- (2) $0 \star s = 0$,
- (3) If $s \star t = 0$ and $t \star s = 0$, then $s = t$.

Definition 2.4. A subset F of a d -algebra X is referred to as a *filter* on X if it fulfills the following criteria:

- (1) $0 \in F$.
- (2) For all $s, t \in X$, if $0 \neq s \in F$ and $s \star t \in F$, then $t \in F$.

Definition 2.5. [8] A subset I of a d-algebra X is termed an BCK-ideal of X if it satisfies these conditions:

- (1) $0 \in I$.
- (2) For all $s, t \in X$, if $0 \neq t \in I$ and $s \star t \in I$, then $s \in I$.

3. GT-D-ALGEBRA

Definition 3.1. A d-algebra $(X, \star, 0)$ equipped with a GT μ is called a generalized topological d-algebra (or GT-d-algebra) if for every μ -open set $G \in \mu_{s \star t}$, there exist $S \in \mu_s$ and $T \in \mu_t$, such that $S \star T \subseteq G$.

Theorem 3.2. Let $(X, \star, 0, \mu)$ be a GT-d-algebra. For any subset $S, T \subseteq X$, the following statements are true:

- (a) $c_\mu(S) \star c_\mu(T) \subseteq c_\mu(S \star T)$.
- (b) If $c_\mu(S) \star c_\mu(T)$ is closed, then $c_\mu(S) \star c_\mu(T) = c_\mu(S \star T)$.

Proof. (a) Let $z \in c_\mu(S) \star c_\mu(T)$ and $U \in \mu_z$. So there exists $z = s \star t$, where $s \in c_\mu(S)$ and $t \in c_\mu(T)$. Since X is GT-d-algebra and $s \star t \in U$, there exist $P \in \mu_s$ and $Q \in \mu_t$ such that $P \star Q \subseteq U$. Now $s \in c_\mu(S)$ and $t \in c_\mu(T)$ implies that $S \cap P \neq \emptyset$ and $T \cap Q \neq \emptyset$. Suppose that $s_1 \in S \cap P$ and $t_1 \in T \cap Q$, so $s_1 \star t_1 \in S \star T$ and $s_1 \star t_1 \in P \star Q \subseteq U$. Therefore $(S \star T) \cap U \neq \emptyset$. Hence $z \in c_\mu(S \star T)$.

(b) Let $c_\mu(S) \star c_\mu(T)$ be a μ -closed set and suppose $y \in c_\mu(S \star T)$. If possible let $y \notin c_\mu(S) \star c_\mu(T)$, then $y \in X - (c_\mu(S) \star c_\mu(T))$, which is an μ -open set. Therefore, $S \star T \subseteq c_\mu(S) \star c_\mu(T)$ implies

$$(S \star T) \cap \{X - (c_\mu(S) \star c_\mu(T))\} = \emptyset$$

which leads to a contradiction to the assumption $y \notin c_\mu(S \star T)$. Hence $c_\mu(S) \star c_\mu(T) = c_\mu(S \star T)$. $\square$

The following example illustrates that the equality in Theorem 3.2(a) does not always hold.

Example 3.3. Consider a d-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$\star$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

Now, define a generalized topology μ on X as:

$$\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, 0\}, \{0, b\}, \{0, a, b\}\}.$$

Then X is an GT-d-algebra. If $S = \{a, b\}$, $T = \{c\}$, then $c_\mu(S) = X$ and $c_\mu(T) = \{c\}$, $c_\mu(S) \star c_\mu(T) = \{0\}$ which is not μ -closed. Also $c_\mu(S \star T) = c_\mu(\{0\}) = \{0, c\}$. Thus $c_\mu(S) \star c_\mu(T) \neq c_\mu(S \star T)$, this confirms that the equality in Theorem 3.2(a) does not always hold.

Theorem 3.4. *Let $(X, \star, 0, \mu)$ be a GT-d-algebra, if $\{0\}$ is μ -open, then $\{z\}$ is μ -open for all $z \in M_\mu$.*

Proof. Let $\{0\}$ is μ -open subset of X and $z \in M_\mu$. Since X is GT-d-algebra and $z \star z = 0$, so there exists $S \in \mu_z$ such that $S \star S \subseteq \{0\}$. Let $s \in S$ and $s \neq z$, then $s \star s = z \star s = 0$. Which is a contradiction hence, we conclude that $S = \{z\}$. $\square$

Theorem 3.5. *In a GT-d-algebra $(X, \star, 0, \mu)$, if $\{0\}$ is μ -closed, then it is μ - T_2 .*

Proof. Suppose $\{0\}$ is a μ -closed subset of X . Let s and t be any two distinct points in X . Then either $t \star s \neq 0$ or $s \star t \neq 0$. Without loss of generality, assume that $s \star t \neq 0$. By assumption, $X - \{0\}$ is a μ -open set that contains $s \star t$. Since X is GT-d-algebra, so there exists $S \in \mu_s$ and $T \in \mu_t$ such that $S \star T \subseteq X - \{0\}$, which implies that $S \cap T = \emptyset$. Hence, we conclude that $(X, \star, 0, \mu)$ is μ - T_2 . $\square$

Theorem 3.6. *If a GT-d-algebra $(X, \star, 0, \mu)$ is μ - T_0 , then it is μ - T_1 .*

Proof. Let $s, t \in X$ and $s \neq t$. Then either $s \star t \neq 0$ or $t \star s \neq 0$. Suppose that $s \star t \neq 0$. Since $(X, \star, 0, \mu)$ is a μ - T_0 space, there exists $W \in \mu$ containing one of $t \star s$ and 0 , but not the other.

Case 1: Let $0 \notin W$ and $x \star y \in W$. Since $(X, \star, 0, \mu)$ is a GT-d-algebra, there exists $S \in \mu_s$ and $T \in \mu_t$ such that $S \star T \subseteq W$.

Sub Case 1: If $S \cap T = \emptyset$, then X is μ - T_2 that is X is μ - T_1 .

Sub Case 2: If $S \cap T \neq \emptyset$, then $r \in S \cap T$, which implies that $0 = r \star r \in S \star T \subseteq W$, which is a contradiction.

Case 2: Let $0 \in W$ and $s \star t \notin W$, then we have $s \star s = 0 \in W$, so there exists $S \in \mu_s$ such that $S \star S \subseteq W$. Again, $t \star t = 0 \in W$, so there exists an $T \in \mu_t$ and such that $T \star T \subseteq W$. Therefore, S contains s but not t and the set T contains t but not s . Hence, $(X, \star, 0, \mu)$ is a μ - T_1 space. $\square$

Theorem 3.7. *If a GT-d-algebra $(X, \star, 0, \mu)$ is μ - T_1 , then it is μ - T_2 .*

Proof. Let $s, t \in X$ and $s \neq t$. Then either $s \star t \neq 0$ or $t \star s \neq 0$. Assume that $s \star t \neq 0$ without loss of generality. Since $(X, \star, 0, \mu)$ is a μ - T_1 space, there exists an μ -open set G containing $s \star t$ and $0 \notin G$. Since $(X, \star, 0, \mu)$ is a GT-d-algebra, there exists $S \in \mu_s$ and $T \in \mu_t$ such that $S \star T \subseteq G$. If $S \cap T \neq \emptyset$, then $x \in S \cap T$ implies that $0 = x \star x \in S \star T \subseteq G$. This contradict our hypothesis $0 \notin G$. Hence $S \cap T = \emptyset$. Therefore, $(X, \star, 0, \mu)$ is a μ - T_2 space. $\square$

Corollary 3.8. *If a GT-d-algebra $(X, \star, 0, \mu)$ then the following are equivalent*

- (a) $(X, \star, 0, \mu)$ is μ - T_0 .
- (b) $(X, \star, 0, \mu)$ is μ - T_1 .
- (c) $(X, \star, 0, \mu)$ is μ - T_2 .

Theorem 3.9. *Let $(X, \star, 0, \mu)$ be a GT-d-algebra and F be a filter on X . If $0 \in i_\mu(F)$, then F is μ -open.*

Proof. Suppose $0 \in i_\mu(F)$. By the definition of μ -interior, there exists a set $G \in \mu_0$ such that $0 \in G \subseteq F$. Let $z \in F$. Since $z \star z = 0 \in G$ and X is a GT-d-algebra, it follows that there exists a set $P \in \mu_z$ such that $P \star P \subseteq G$.

Now, consider any $x \in P$. Since $P \star P \subseteq G$ and $G \subseteq F$, we have

$$z \star x \in P \star P \subseteq G \subseteq F.$$

Given that F is a filter, this implies that $x \in F$. Consequently, $P \subseteq F$. This demonstrates that z is an μ -interior point of F . Hence, F is μ -open. $\square$

Theorem 3.10. *Let $(X, \star, 0, \mu)$ be a GT-d-algebra and J be a BCK-ideal on X . If $0 \in i_\mu(J)$, then J is μ -open.*

Proof. Let $(X, \star, 0, \mu)$ be a GT-d-algebra. Suppose $0 \in i_\mu(J)$. By the definition of μ -interior, there exists a set $G \in \mu_0$ such that $0 \in G \subseteq J$. Let $z \in J$. Since X is a GT-d-algebra and $z \star z = 0 \in G$, there exists a set $P \in \mu_z$ such that $P \star P \subseteq G$. Now, for any $x \in P$, we have

$$x \star z \in P \star P \subseteq G \subseteq J.$$

Since J is an BCK-ideal, it follows that $x \in J$. Therefore, $P \subseteq J$. This shows that z is an μ -interior point of J . Hence, J is μ -open. $\square$

Definition 3.11. A mapping $f : (X, \mu) \rightarrow (Z, \kappa)$ is said to be (μ, κ) -continuous if for each $x \in X$ and $T \in \mu_{f(x)}$, there exists a $S \in \mu_x$ such that $f(S) \subseteq T$.

Definition 3.12. [13] Let $(X, \star, 0, \mu)$ be a d-algebra, and let $s \in X$. The *left map* $\mathcal{L}_s : X \rightarrow X$ is defined by $\mathcal{L}_s(x) = s \star x$ for all $x \in X$, and the *right map* $\mathcal{R}_s : X \rightarrow X$ is given by $\mathcal{R}_s(x) = x \star s$ for all $x \in X$.

Theorem 3.13. *Let $(X, \star, 0, \mu)$ be a GT-d-algebra, then every*

- (1) $\mathcal{L}_s : (X, \star, 0, \mu) \rightarrow (X, \star, 0, \mu)$ is (μ, μ) -continuous for all $s \in X$.
- (2) $\mathcal{R}_s : (X, \star, 0, \mu) \rightarrow (X, \star, 0, \mu)$ is (μ, μ) -continuous for all $s \in X$.

Proof. (1) Let $s \in X$, define a left mapping $\mathcal{L}_s : (X, \star, 0, \mu) \rightarrow (X, \star, 0, \mu)$ by $\mathcal{L}_s(z) = s \star z$, for all $z \in X$. Let P be a μ -open set containing $s \star z$. Since X is an GT-d-algebra, then there exist $S \in \mu_s$ and $T \in \mu_t$ such that $S \star T \subseteq P$. Therefore

$$\mathcal{L}_s(T) = s \star T \subseteq S \star T \subseteq P$$

This shows that the left mapping $\mathcal{L}_s$ is (μ, μ) -continuous.

(2) Let $s \in X$, define a right mapping $\mathcal{R}_s : (X, \star, 0, \mu) \rightarrow (X, \star, 0, \mu)$ by $\mathcal{R}_s(z) = z \star s$ for all $z \in X$. Let Q be a μ -open set containing $z \star s$. Since X is an GT-d-algebra, then there exist $T \in \mu_z$ and $S \in \mu_s$ such that $T \star S \subseteq Q$. Therefore

$$\mathcal{R}_s(T) = T \star s \subseteq T \star S \subseteq Q$$

This shows that the right mapping $\mathcal{R}_s$ is (μ, μ) -continuous. $\square$

4. QUASI-UNIFORMITY IN D-ALGEBRAS

In this section, we first consider a quasi-uniformity based on the d^* -ideal of a d-algebra. Next, we construct an induced generalized topology via quasi-uniformity and study its basic properties.

Definition 4.1. [9] Let $(X, \star, 0)$ be a d-algebra. A nonempty subset $\mathcal{I}$ of X is called a d^* -ideal of X if it meets the following conditions:

- (D₁) If $r \star s \in \mathcal{I}$ and $s \in \mathcal{I}$, then $r \in \mathcal{I}$,
- (D₂) If $r \in \mathcal{I}$ and $s \in X$, then $r \star s \in \mathcal{I}$, i.e., $\mathcal{I} \star X \subseteq \mathcal{I}$,
- (D₃) If $r \star s \in \mathcal{I}$ and $s \star t \in \mathcal{I}$, then $r \star t \in \mathcal{I}$,
- (D₄) If $r \star s \in \mathcal{I}$ and $s \star r \in \mathcal{I}$, then $(r \star t) \star (s \star t) \in \mathcal{I}$ and $(t \star r) \star (t \star s) \in \mathcal{I}$ for all $r, s, t \in X$.

Theorem 4.2. [9] Let $\mathcal{I}$ be a d^* -ideal of a d-algebra $(X, \star, 0)$, then the relation $\sim$ defined by

$$r \sim s \iff r \star s \in \mathcal{I} \text{ and } s \star r \in \mathcal{I}$$

is a congruence relation on X .

Let D be a nonempty set and let S, T be any subsets of $D \times D$. Define:

$$\Delta = \{(s, s) \in D \times D \mid s \in D\},$$

$$S^{-1} = \{(s, t) \in D \times D \mid (t, s) \in S\},$$

$$S \circ T = \{(s, t) \in D \times D \mid \text{for some } z \in D, (z, t) \in S \text{ and } (s, z) \in T\}.$$

Definition 4.3. [15] A nonempty collection $\mathcal{Q}$ of subsets of $D \times D$ is said to be a quasi-uniformity on D if it satisfies the following conditions:

- (Q₁) $\Delta \subseteq S$ for any $S \in \mathcal{Q}$.
- (Q₂) If $S \in \mathcal{Q}$, then there exists a $T \in \mathcal{Q}$ such that $T \circ T \subseteq S$.
- (Q₃) If $S \in \mathcal{Q}$ and $S \subseteq T$, then $T \in \mathcal{Q}$.

The pair $(D, \mathcal{Q})$ is called a quasi-uniform structure.

Let $(X, \star, 0)$ be a d-algebra and $\mathcal{I}$ be a d^* -ideal of X . Then define a set $S_{\mathcal{I}}$ with the help of a relation $\sim$ as

$$S_{\mathcal{I}} = \{(s, t) \in X \times X \mid s \sim t\}.$$

Theorem 4.4. Let $\mathcal{Q}_{\mathcal{I}} = \{S \mid S_{\mathcal{I}} \subseteq S\}$, then $(X, \mathcal{Q}_{\mathcal{I}})$ is a quasi-uniform structure.

Proof. Since the relation $\sim$ is reflexive, $\Delta \in S_{\mathcal{I}}$. Therefore, $\mathcal{Q}_{\mathcal{I}}$ satisfies condition Q₁. Let $S \in \mathcal{Q}_{\mathcal{I}}$. Since the relation $\sim$ is transitive, $S \circ S \subseteq S$. This shows that $\mathcal{Q}_{\mathcal{I}}$ also satisfies condition Q₂. Condition Q₃ is trivially satisfied. Hence, $(X, \mathcal{Q}_{\mathcal{I}})$ is a quasi-uniform structure. $\square$

Next, we define a set $S_{\mathcal{I}}[z]$ using the above quasi-uniform structure $(X, \mathcal{Q}_{\mathcal{I}})$ as

$$S_{\mathcal{I}}[z] := \{y \in X \mid (z, y) \in S_{\mathcal{I}}\}.$$

Theorem 4.5. Let $\mathcal{I}$ be a d^* -ideal of a d-algebra $(X, \star, 0)$. Then:

- (1) If $\mathcal{I} = \{0\}$, then $S_{\mathcal{I}}[z] = \{z\}$

- (2) $S_{\mathcal{I}}[z] = \mathcal{I}$ for all $z \in \mathcal{I}$.
 (3) $S_{\mathcal{I}}[z] \cap \mathcal{I} = \emptyset$ for all $z \in X - \mathcal{I}$.

Proof. (1)

$$\begin{aligned} S_{\mathcal{I}}[z] &= \{s \mid (z, s) \in S_{\mathcal{I}}\} \\ &= \{s \mid s \star z \in \mathcal{I} \text{ and } z \star s \in \mathcal{I}\} \\ &= \{s \mid s \star z = 0 \text{ and } z \star s = 0\} \\ &= \{z\} \end{aligned}$$

(2) Let $z \in \mathcal{I}$ and $s \in S_{\mathcal{I}}[z]$. Then $(z, s) \in S_{\mathcal{I}}$, implying $z \star s \in \mathcal{I}$ and $s \star z \in \mathcal{I}$. Therefore, $s \in \mathcal{I}$ by (D_1) . Thus, $S_{\mathcal{I}}[z] \subseteq \mathcal{I}$. For reverse inequality let $t \in \mathcal{I}$, also $0, z \in \mathcal{I}$. Then using D_2 and D_3 , we have $t \star z \in \mathcal{I}$, $z \star t \in \mathcal{I}$ i.e. $(z, t) \in S_{\mathcal{I}}$. Thus $t \in S_{\mathcal{I}}[z]$. Hence $S_{\mathcal{I}}[z] \supseteq \mathcal{I}$.

(3) Let $z \in X - \mathcal{I}$ and suppose $S_{\mathcal{I}}[z] \cap \mathcal{I} \neq \emptyset$. Then there exists $s \in S_{\mathcal{I}}[z]$ and $s \in \mathcal{I}$ such that $(z, s) \in S_{\mathcal{I}}$. This implies $z \star s \in \mathcal{I}$ and $s \star z \in \mathcal{I}$. Since X is a d^* -ideal, by (D_1) , $s \in \mathcal{I}$, which is a contradiction. Therefore, $S_{\mathcal{I}}[z] \cap \mathcal{I} = \emptyset$. $\square$

Corollary 4.6. Let $\mathcal{I}$ be a d^* -ideal of a d -algebra $(X, \star, 0)$. Then $S_{\mathcal{I}}[0] = \mathcal{I}$

Theorem 4.7. Let $\mathcal{I}$ be a d^* -ideal of a d -algebra $(X, \star, 0)$. Then:

- (1) $\mathcal{I} = \bigcup_{z \in \mathcal{I}} S_{\mathcal{I}}[z]$.
 (2) $X - \mathcal{I} = \bigcup_{z \in X - \mathcal{I}} S_{\mathcal{I}}[z]$.

Proof. (1) Let $z \in \mathcal{I}$. Since $z \in S_{\mathcal{I}}[z]$ for all $z \in \mathcal{I}$, we have $\mathcal{I} \subseteq S_{\mathcal{I}}[z]$. Thus, $\mathcal{I} \subseteq \bigcup_{z \in \mathcal{I}} S_{\mathcal{I}}[z]$. Let $z \in \mathcal{I}$ and $s \in S_{\mathcal{I}}[z]$, then $z \star s \in \mathcal{I}$ and $s \star z \in \mathcal{I}$. Since $\mathcal{I}$ is a d^* -ideal, $s \in \mathcal{I}$. Therefore, $\mathcal{I} \supseteq S_{\mathcal{I}}[z]$. Thus, $\mathcal{I} \supseteq \bigcup_{z \in \mathcal{I}} S_{\mathcal{I}}[z]$. Hence, we conclude that $\mathcal{I} = \bigcup_{z \in \mathcal{I}} S_{\mathcal{I}}[z]$.

(2) Let $z \in X - \mathcal{I}$. Since $z \in S_{\mathcal{I}}[z]$ for all $z \in X - \mathcal{I}$, we have $X - \mathcal{I} \subseteq S_{\mathcal{I}}[z]$. Thus, $X - \mathcal{I} \subseteq \bigcup_{z \in X - \mathcal{I}} S_{\mathcal{I}}[z]$. By the theorem, $S_{\mathcal{I}}[z] \subseteq X - \mathcal{I}$ for all $z \in X - \mathcal{I}$. Therefore, $\bigcup_{z \in X - \mathcal{I}} S_{\mathcal{I}}[z] \subseteq X - \mathcal{I}$. Hence, $X - \mathcal{I} = \bigcup_{z \in X - \mathcal{I}} S_{\mathcal{I}}[z]$. $\square$

4.1. Induced $\mu_{\mathcal{I}}$ Generalized Topology.

Theorem 4.8. Let $(X, \star, 0)$ be a d -algebra and $\mathcal{I}$ is a d^* ideal of X , then

$$\mu_{\mathcal{I}} = \{ T \subseteq X \mid \forall t \in T, \exists S_{\mathcal{I}} \in \mathcal{Q}_{\mathcal{I}}, S_{\mathcal{I}}[t] \subseteq T \}$$

is a generalized topology on X .

Proof. The proof is obvious from the definition. $\square$

Example 4.9. Consider a d -algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$\star$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Since $\mathcal{I} = \{0, a\}$ is a d^* -ideal, we define the relation

$$r \sim s \iff r \star s \in \mathcal{I} \text{ and } s \star r \in \mathcal{I},$$

which implies that

$$(s, t) \in S_{\mathcal{I}}.$$

Consequently, we obtain

$$S_{\mathcal{I}} = \Delta \cup \{(a, 0), (0, a), (b, c), (c, b)\}.$$

Now, computing the equivalence classes:

$$S_{\mathcal{I}}[0] = \{0, a\}, \quad S_{\mathcal{I}}[a] = \{0, a\}, \quad S_{\mathcal{I}}[b] = \{b, c\}, \quad S_{\mathcal{I}}[c] = \{b, c\}.$$

Therefore, the induced generalized topology on X is given by

$$\mu_{\mathcal{I}} = \{\emptyset, X, \{0, a\}, \{b, c\}\}.$$

Theorem 4.10. *Any d^* -ideal of a d -algebra $(X, \star, 0)$ is a μ -clopen subset of $(X, \mu_{\mathcal{I}})$.*

Proof. Let $\mathcal{I}$ be a d^* -ideal in a GT- d -algebra X . By definition, for any $z \in \mathcal{I}$, the set $S_{\mathcal{I}}[z]$ is μ -open in $(X, \mu_{\mathcal{I}})$. By Theorem 4.7, it follows that $\mathcal{I}$ is μ -open in $(X, \mu_{\mathcal{I}})$. Furthermore, by Theorem 4.7, $\mathcal{I}$ is also μ -closed in $(X, \mu_{\mathcal{I}})$. Thus, $\mathcal{I}$ is μ -clopen in $(X, \mu_{\mathcal{I}})$. $\square$

Definition 4.11. A quasi-uniform space $(X, \mathcal{Q})$ is called totally bounded if for each $S \in \mathcal{Q}$, there exists a finite set $\{z_1, z_2, \dots, z_n\} \subseteq X$ such that $X = \bigcup_{i=1}^n S[z_i]$.

Definition 4.12. A μ -space (X, μ) is said to be μ -compact if every cover of X by μ -open subsets has a finite subcover.

Theorem 4.13. *Let $\mathcal{I}$ be a d^* -ideal of a d -algebra $(X, \star, 0)$. Then the following statements are equivalent:*

- (1) *The space $(X, \mu_{\mathcal{I}})$ is μ -compact.*
- (2) *The space $(X, \mathcal{Q})$ is totally bounded.*
- (3) *There exists a finite set $R = \{z_1, z_2, \dots, z_n\} \subseteq X$ such that for every $s \in X$, there is some $z_i \in R$ with $s \star z_i \in \mathcal{I}$ and $z_i \star s \in \mathcal{I}$.*

Proof. **(1 $\Rightarrow$ 2):** This follows directly from the definition.

(2 $\Rightarrow$ 3): Since $(X, \mu_{\mathcal{I}})$ is totally bounded, there exists a finite set $\{z_1, z_2, \dots, z_n\} \subseteq X$ such that $X = \bigcup_{i=1}^n S_{\mathcal{I}}[z_i]$. For any $s \in \mathcal{I}$, there exists an index i such that

$s \in S_{\mathcal{I}}[z_i]$, implying $s \star z_i \in \mathcal{I}$ and $z_i \star s \in \mathcal{I}$.

(3 $\Rightarrow$ 1): Given a finite set R satisfying the condition, for any $s \in \mathcal{I}$, there exists $z_i \in R$ such that $s \star z_i \in \mathcal{I}$ and $z_i \star s \in \mathcal{I}$. Thus, $s \in S_{\mathcal{I}}[z_i]$, which implies

$X = \bigcup_{i=1}^n S_{\mathcal{I}}[z_i]$. Given a covering of X by μ -open sets, $X = \bigcup_{\xi \in \Lambda} \mathcal{H}_{\xi}$, for each $z_i \in X$, there exists $\xi_i \in \Lambda$ such that $z_i \in \mathcal{H}_{\xi_i}$. Since $\mathcal{H}_{\xi_i}$ is μ -open, $S_{\mathcal{I}}[z_i] \subseteq \mathcal{H}_{\xi_i}$, hence $X = \bigcup_{i=1}^n \mathcal{H}_{\xi_i}$. This proves that $(X, \mu_{\mathcal{I}})$ is μ -compact. $\square$

Theorem 4.14. *Let $\mathcal{I}$ be a d^* -ideal of a d -algebra $(X, \star, 0)$ such that $X - \mathcal{I}$ is countable. Then $(X, \mu_{\mathcal{I}})$ is μ -Lindelöf.*

Proof. Let $\{\mathcal{H}_{\xi}\}_{\xi \in \Lambda}$ be a μ -open cover of X . Since $X - \mathcal{I}$ is countable, let $X - \mathcal{I} = \{z_n\}_{n \in \mathbb{N}}$. For each $z_i \in X - \mathcal{I}$, there exists some $\xi_i \in \Lambda$ such that $z_i \in \mathcal{H}_{\xi_i}$. Moreover, for 0, there exists some $\xi \in \Lambda$ such that $0 \in \mathcal{H}_{\xi}$. Since $U_{\mathcal{I}}[0] = \mathcal{I}$, it follows that $\mathcal{I} \subseteq \mathcal{H}_{\xi}$. Combining these, we have:

$$X = \bigcup_{i \in \mathbb{N}} \mathcal{H}_{\xi_i} \cup \mathcal{H}_{\xi}.$$

This shows that X is covered by the countable collection $\{\mathcal{H}_{\xi_i} \mid i \in \mathbb{N}\} \cup \{\mathcal{H}_{\xi}\}$. Hence, $(X, \mu_{\mathcal{I}})$ is μ -Lindelöf. $\square$

Corollary 4.15. *Let $\mathcal{I}$ be a d^* -ideal of a d -algebra $(X, \star, 0)$ such that $X - \mathcal{I}$ is a finite set. Then $(X, \mu_{\mathcal{I}})$ is μ -compact.*

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