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REFINEMENT OF ERDÖS-LAX INEQUALITY FOR POLAR DERIVATIVE OF A POLYNOMIAL

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ABSTRACT. In this paper, we present generalizations of Erdös-Lax type inequalities concerning the polar derivative of a polynomial over the disc of radius $k \geq 1$. Additionally, the results provide refinements of several known inequalities and lead to other interesting results as special cases.

Keywords. Coefficients, inequalities, polar derivative, zeros
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1. Introduction and preliminaries

Let $\mathcal{P}_n$ denote the set of all polynomials of degree at most n over the field $\mathbb{C}$ of complex numbers. The study of extremal properties for polynomials is a classical topic in analysis. One of the fundamental result in this regard is due to S. Bernstein [3] which states that if $P \in \mathcal{P}_n$ is of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The inequality (1.1) is sharp and equality holds if $P(z)$ is a multiple of z^n. For a subclass of polynomials in $\mathcal{P}_n$ which do not vanish in $|z| < 1$, inequality (1.1) can be sharpened further. In fact if $P \in \mathcal{P}_n$ is of degree n not vanishing in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

This sharp result was first conjectured by Erdös [5] in 1940, which was later verified by Lax [12] in 1944. It may be noted that the inequality (1.2) is also sharp with equality holding if $P(z) = a + bz^n$, where $|a| = |b| \neq 0$. In place of the open unit disc $|z| < 1$, R. P. Boas Jr. suggested to study the class of polynomials having no zeros in $|z| < k$, where $k > 0$ and obtain an inequality analogous to that of (1.2). This propound problem has been studied considerably by multifarious people, for perusal see Malik [13], Govil and Rahman [8], Govil, Qazi and Rahman [9] and others. In this direction Malik [13] in 1969 proved a partial extension of (1.2) for polynomial $P(z)$ not vanishing in $|z| < k$,

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where $k \geq 1$, by obtaining

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.3}$$

The result is best possible and equality holds for $P(z) = (z+k)^n$. Inequality (1.3) was further refined by Malik [10], who proved that for a polynomial $P(z)$ of degree n having no zeros in $|z| < k,\ k \geq 1$, we have the following inequality

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right\}. \tag{1.4}$$

Inequality (1.4) is best possible and equality holds for $P(z) = (z+k)^n$. Although inequality (1.4) is best possible, however if a polynomial $P(z)$ has even one zero on the boundary $|z| = k$, it does not provide any refinement of (1.3). In this essence, P. Kumar [11] obtained a refinement of (1.3) involving the extreme coefficients of $P(z)$ by proving that if $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, not vanishing in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \left\{ 1 - \frac{|a_0| - |a_n|k^n}{n(|a_0| + |a_n|k^n)} \right\} \max_{|z|=1} |P(z)|, \tag{1.5}$$

whenever $\max_{|z|=1} |P(z)| \geq 2|a_0|$ and (1.3) otherwise. Geometric function theory focuses on studing the extremal problems of functions of a complex variable and generalizing classical polynomial inequalities in various directions, for references (see [1, 3, 10, 11, 6]). One such generalization comprises moving from the domain of the ordinary derivative to the polar derivative of polynomials.

Definition 1.1. Let $P(z)$ be a polynomial of degree n and α be any real or complex number, then the polynomial $nP(z) + (\alpha - z)P'(z)$ is called the polar derivative of $P(z)$ with respect to the point α and is usually denoted by $D_\alpha P(z)$. It may be noted that $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z), \tag{1.6}$$

uniformly with respect to z for $|z| \leq R,\ R > 0$.

Aziz [1] was among the first to extend some of the above inequalities by replacing the classical derivative with the polar derivative of polynomials. In fact, he [1] extended (1.2) to the polar derivative of a polynomial and proved the following analogous result which states that if $P \in \mathcal{P}_n,\ P(z) \neq 0\ in\ |z| < 1$, then for every real or complex number α with $|\alpha| \geq 1$

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{2}(|\alpha| + 1) \max_{|z|=1} |P(z)|. \tag{1.7}$$

As a extension of (1.7), Aziz [1] was also able to prove that if $P(z)$ does not vanish in $|z| < k,\ k \geq 1$, then for every real or complex number α with $|\alpha| \geq 1$

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \left(\frac{|\alpha| + k}{1+k} \right) \max_{|z|=1} |P(z)|. \tag{1.8}$$

Inequality (1.8) is the best possible and equality holds for $P(z) = (z+k)^n$ with real $\alpha \geq 1$, and $k \geq 1$. By introducing the minimum value of $|P(z)|$ on the boundary of open disc $|z| < k$, Dewan et al. ([4], cor 1) minimized the bound on the right side of inequality (1.8) and proved that if $P(z)$ is polynomial of degree n having no zeros in $|z| < k, \ k \geq 1$, then for every real or complex number α with $|\alpha| \geq 1$, then

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{1+k} \left\{ (|\alpha| + k) \max_{|z|=1} |P(z)| - (|\alpha| - 1) \min_{|z|=k} |P(z)| \right\}. \tag{1.9}$$

Though inequality (1.9) sharpens inequality (1.8), it has the disadvantage that if there is even one zero on $|z| = k$, then $\min_{|z|=k} |P(z)| = 0$ and so the inequality (1.9) fails to give any progress over (1.8). On behalf of preceding points a query accrues: is there any approach to achieve the information on the moduli of zeros in order to refine inequality (1.8) for the class of polynomials satisfying the hypothesis of Erdös-Lax inequality? Could we secure the bound in case the polynomial has at least one zero outside the closed disc $|z| \leq k, \ k \geq 1$. In this paper we attempt to present the answer in affirmative to the above query.

2. Main results

First, we sharpen the bound given in (1.8) by involving the extreme coefficients of $P(z)$. In fact, we prove the following result.

Theorem 2.1. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, not vanishing in $|z| < k, \ k \geq 1$, then for any complex number α with $|\alpha| \geq 1$*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{1+k} \left\{ (|\alpha| + k) - (|\alpha| - 1) \left(\frac{|a_0| - |a_n| k^n}{n(|a_0| + |a_n| k^n)} \right) \right\} \max_{|z|=1} |P(z)|, \tag{2.1}$$

whenever $\max_{|z|=1} |P(z)| \geq 2|a_0|$ and (1.8) otherwise. The result is best possible and equality holds for the polynomial $z^n + cz^{n-1} + z + c$ with $k = 1$ and $c \geq 1$.

Remark 2.2. If we divide inequality (2.1) by $|\alpha|$ and then let $|\alpha| \to \infty$, it reduces to inequality (1.5).

Remark 2.3. Since $P(z)$ has all zeros in $|z| \geq k, \ k \geq 1$, therefore $|a_0| \geq |a_n| k^n$, which gives for $|\alpha| \geq 1$

$$\frac{(|\alpha| - 1)}{1+k} \left(\frac{|a_0| - |a_n| k^n}{|a_0| + |a_n| k^n} \right) \geq 0. \tag{2.2}$$

By this observation we see that (2.1) is a refinement of inequality (1.8)

Remark 2.4. It can be inferred that inequality (2.1) provides the refinement of (1.8) even if only one zero of $P(z)$ is away from the boundary $|z| = k$. Further it can be easily observed that $\frac{x - k^n}{x + k^n}$ is a non-decreasing function of x, $x \in \mathbb{R}$. This shows that as the zeros of $P(z)$ go farther and farther from the disc $|z| < k$, the value of the term $\frac{|a_0| - |a_n| k^n}{n(|a_0| + |a_n| k^n)}$ increasing significantly and thus the bound given in (2.1) would be much proximate to the value of $\max_{|z|=1} |D_\alpha P(z)|$ than the one given in (1.9).

Example 2.5. Consider a polynomial $P(z) = 2(z+\frac{3}{2})(z+3)(z+4) = 2z^3+17z^2+45z+36$. Clearly all zeros of $P(z)$ lie outside $|z| < 1.4$ and $\max_{|z|=1}|P(z)| = 100$. Taking $\alpha = 3.5$ and $k = 1.4$, so that $P(z) \neq 0$ for $|z| < k$, we get from inequality (1.8) that

$$\max_{|z|=1}|D_\alpha P(z)| \leq 612.500,$$

while Theorem 2.1 yields

$$\max_{|z|=1}|D_\alpha P(z)| \leq 535.8915.$$

This shows that Theorem 2.1 provides a sharper bound.

Next, we provide the improvement of inequality (2.1) by introducing the minimum of $|P(z)|$ on $|z| = k$.

Theorem 2.6. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n not vanishing in $|z| < k$, $k \geq 1$, then for any complex number β with $|\beta| \leq 1$ and $|\alpha| \geq 1$, we have*

$$\begin{aligned}&\max_{|z|=1}\left|D_\alpha\left(P(z)+\frac{\beta m z^n}{k^n}\right)\right| \\ &\leq \frac{n}{1+k}\left[(|\alpha|+k)-(|\alpha|-1)\left(\frac{|a_0|-|a_nk^n+\beta m|}{n(|a_0|+|a_nk^n+\beta m|)}\right)\right]\max_{|z|=1}\left|P(z)+\frac{\beta m z^n}{k^n}\right|\end{aligned} \tag{2.3}$$

whenever $\max_{|z|=1}|P(z)| \geq 2|a_0|$ *and* $m = \min_{|z|=k}|P(z)|$*. The result is best possible and the extremal polynomial is* $z^n + cz^{n-1} + z + c$*, where* $k = 1$ *and* $c \geq 1$*.*

Remark 2.7. For $\beta = 0$, (2.3) reduces to (2.1).

Dividing both sides (2.3) by $|\alpha|$ and then letting $|\alpha| \to \infty$, we get the following result for classical derivative.

Corollary 2.8. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n not vanishing in $|z| < k$, $k \geq 1$*

$$\max_{|z|=1}\left|P'(z)+\frac{\beta m n z^{n-1}}{k^n}\right| \leq \frac{n}{1+k}\left\{1-\frac{|a_0|-|a_nk^n+\beta m|}{n(|a_0|+|a_nk^n+\beta m|)}\right\}\max_{|z|=1}\left|P(z)+\frac{\beta m z^n}{k^n}\right|. \tag{2.4}$$

Remark 2.9. For $\beta = 0$, (2.4) reduces to (1.5).

Theorem 2.10. *Let $P(z) = z^s\sum_{j=0}^{n-s} c_j z^j$ has all its zeros in $|z| \geq k$, $k \geq 1$, except for s-fold zeros at origin, then for every complex α with $|\alpha| \geq 1$, we have*

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \leq \frac{1}{1+k}\Bigg[&sk(|\alpha|-1)+n(|\alpha|+k)-(n-s)(|\alpha|-1)\times \\ &\left(\frac{|c_0|-|c_{n-s}|k^{n-s}}{(n-s)(|c_0|+|c_{n-s}|k^{n-s})}\right)\Bigg]\max_{|z|=1}|P(z)|\end{aligned} \tag{2.5}$$

whenever $\max_{|z|=1}|P(z)| \geq 2|c_0|$.

Remark 2.11. For $s = 0$, Theorem 2.10 reduces to Theorem 2.1.

Dividing both sides to inequality (2.5) by $|\alpha|$ and then letting $|\alpha| \to \infty$, we get the following result.

Corollary 2.12. *Let $P(z) = z^s \sum_{j=0}^{n-s} c_j z^j$ has all its zeros in $|z| \geq k,\ k \geq 1$, except for s-fold zeros at origin, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{1}{1+k}\left[sk + n + (n-s)\left(\frac{|c_0| - |c_{n-s}|k^{n-s}}{(n-s)(|c_0| + |c_{n-s}|k^{n-s})}\right)\right]. \tag{2.6}$$

Theorem 2.13. *Let $P(z) = z^s \sum_{j=0}^{n-s} c_j z^j$ has all its zeros in $|z| \geq k$, where $k \geq 1$, except for s-fold zeros at origin, then for every real or complex β with $|\beta| \leq 1$ and $|\alpha| \geq 1$, we have*

$$\max_{|z|=1}\left|D_\alpha\left(\frac{P(z)}{z^s} + \frac{\beta m z^{n-s}}{k^{n-s}}\right)\right| \leq$$
$$\frac{n-s}{1+k}\left[(|\alpha|+k) - (|\alpha|-1)\left(\frac{|a_0| - |a_{n-s}k^{n-s} + \beta m|}{(n-s)(|a_0| + |a_{n-s}k^{n-s} + \beta m|)}\right)\right]\max_{|z|=1}\left|P(z) + \frac{\beta m z^n}{k^{n-s}}\right|. \tag{2.7}$$

whenever $\max_{|z|=1}|P(z)| \geq 2|a_0|$.

Remark 2.14. For $s = 0$, Theorem 2.13 reduces to Theorem 2.6.

Dividing both sides to inequality (2.7) by $|\alpha|$ and then letting $|\alpha| \to \infty$, we get the following result for classical derivative.

Corollary 2.15. *Let $P(z) = z^s \sum_{j=0}^{n-s} c_j z^j$ has all its zeros in $|z| \geq k$, where $k \geq 1$, except for s-fold zeros at origin, then*

$$\max_{|z|=1}\left|zP'(z) - sP(z) + \frac{\beta m z^n}{k^{n-s}}(n-s)\right| \leq$$
$$\frac{n-s}{1+k}\left\{1 - \frac{|a_0| - |a_n k^{n-s} + \beta m|}{(n-s)(|a_0| + |a_n k^{n-s} + \beta m|)}\right\}\max_{|z|=1}\left|P(z) + \frac{\beta m z^n}{k^{n-s}}\right|. \tag{2.8}$$

Remark 2.16. For s=0, (2.8) reduces to (2.4).

3. Lemmas

For the proofs of our results, we require the following lemma due to Govil and Rahman [8].

Lemma 3.1. *If $P(z)$ is a polynomial of degree n, then*

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1}|P(z)| \qquad \text{for } |z| = 1,$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

4. Proofs of Main Results

Proof of Theorem 2.1. Let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then for $|z| = 1$ we have

$$|Q'(z)| = |nP(z) - zP'(z)|. \tag{4.1}$$

Now for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, we have for $|z| = 1$

$$\begin{aligned}
|D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\
&\leq |nP(z) - zP'(z)| + |\alpha||P'(z)| \\
&= |Q'(z)| + |\alpha||P'(z)| \qquad \text{using (4.1)} \\
&\leq n \max_{|z|=1} |P(z)| - |P'(z)| + |\alpha||P'(z)| \\
&\leq n \max_{|z|=1} |P(z)| + (|\alpha| - 1) \max_{|z|=1} |P'(z)|.
\end{aligned}$$

Since $P(z) \neq 0$ for $|z| < k,\ k \geq 1$, therefore from inequality (1.5), it follows that

$$|D_\alpha P(z)| \leq n \max_{|z|=1} |P(z)| + \frac{n(|\alpha| - 1)}{1+k} \left\{ 1 - \frac{|a_0| - |a_n|k^n}{n(|a_0| + |a_n|k^n)} \right\} \max_{|z|=1} |P(z)|.$$

Or equivalently

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{1+k} \left\{ (|\alpha| + k) - (|\alpha| - 1) \left(\frac{|a_0| - |a_n|k^n}{n(|a_0| + |a_n|k^n)} \right) \right\} \max_{|z|=1} |P(z)|.$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.6. Since $P(z) \neq 0$ for $|z| < k$, therefore by the application of minimum modulus principle it follows that

$$m = \min_{|z|=k} |P(z)| \leq |P(z)|, \qquad \text{for } |z| \leq k. \tag{4.2}$$

We define $H(z) = P(z) + \frac{\beta m}{k^n} z^n$, then $H(z)$ does not vanish in $|z| < k$. For if there exists a point z_0 in $|z| < k$ such that $H(z_0) = 0$, then

$$|P(z_0)| = \frac{|\beta| m}{k^n} |z_0|^n < m,$$

which is contradicts (4.2). Hence $H(z) \neq 0$ for $|z| < k,\ k \geq 1$. Therefore applying Theorem 2.1 to the polynomial $H(z)$, we get for $|\alpha| \geq k,\ k \geq 1$ and $|\beta| \leq 1$

$$\max_{|z|=1} |D_\alpha H(z)| \leq \frac{n}{1+k} \left\{ (|\alpha| + k) - (|\alpha| - 1) \left(\frac{|a_0| - |a_n k^n + \beta m|}{n(|a_0| + |a_n k^n + \beta m|)} \right) \right\} \max_{|z|=1} |H(z)|,$$

that is

$$\begin{aligned}
&\max_{|z|=1} \left| D_\alpha \left(P(z) + \frac{\beta m}{k^n} z^n \right) \right| \\
&\leq \frac{n}{1+k} \left\{ (|\alpha| + k) - (|\alpha - 1) \left(\frac{|a_0| - |a_n k^n + \beta m|}{n(|a_0| + |a_n k^n + \beta m|)} \right) \right\} \max_{|z|=1} \left| P(z) + \frac{\beta m}{k^n} z^n \right|.
\end{aligned}$$

This completes the proof of Theorem 2.6. □

Proof of Theorem 2.10. Let $P(z) = z^s G(z)$, where $G(z) = \sum_{j=0}^{n-s} c_j z^j$ has all its zeros in $|z| \geq k,\ k \geq 1$. Now for $|\alpha| \geq 1$, we have

$$\begin{aligned}
D_\alpha P(z) &= nP(z) + (\alpha - z)P'(z) \\
&= nz^s G(z) + (\alpha - z) \left\{ s z^{s-1} G(z) + z^s G'(z) \right\} \\
&= (n - s) z^s G(z) + (\alpha - z) z^s G'(z) + s\alpha z^{s-1} G(z),
\end{aligned}$$

which gives for $|z|=1$

$$\begin{aligned}|D_\alpha P(z)| &\leq |D_\alpha G(z)| + s|\alpha||P(z)|\\ &\leq s|\alpha|\max_{|z|=1}|P(z)| + \max_{|z|=1}|D_\alpha G(z)|.\end{aligned}$$

Applying Theorem 2.1 to the $(n-s)$th degree polynomial $G(z)$, we obtain

$$\begin{aligned}|D_\alpha P(z)| &\leq s|\alpha|\max_{|z|=1}|P(z)| + \frac{n-s}{k}\Bigg[(|\alpha|+k)-(|\alpha|-1)\times\\ &\qquad \left\{\frac{|c_0|-|c_{n-s}|k^{n-s}}{(n-s)(|c_0|+|c_{n-s}|k^{n-s})}\right\}\Bigg]\max_{|z|=1}|G(z)|\\ &= s|\alpha|\max_{|z|=1}|P(z)| + \frac{n-s}{k}\Bigg[(|\alpha|+k)-(|\alpha|-1)\times\\ &\qquad \left\{\frac{|c_0|-|c_{n-s}|k^{n-s}}{(n-s)(|c_0|+|c_{n-s}|k^{n-s})}\right\}\Bigg]\max_{|z|=1}|z^sG(z)|.\end{aligned}$$

After solving, we get

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \leq \frac{1}{1+k}\Bigg[&sk(|\alpha|-1)+n(|\alpha|+k)-(n-s)(|\alpha|-1)\times\\ &\left(\frac{|c_0|-|c_{n-s}|k^{n-s}}{(n-s)(|c_0|+|c_{n-s}|k^{n-s})}\right)\Bigg]\max_{|z|=1}|P(z)|.\end{aligned}$$

which is the desired result. □

Proof of Theorem 2.13. We have $P(z)=z^sG(z)$ where $G(z)=\sum_{j=0}^{n-s}c_jz^j$ has all its zeros in $|z|\geq k,\ k\geq 1$. Applying Theorem 2.6 to the $(n-s)$th degree polynomial G(z), we obtain for $|\alpha|\geq 1$

$$\begin{aligned}&\max_{|z|=1}\left|D_\alpha\left(G(z)+\frac{\beta m z^{n-s}}{k^{n-s}}\right)\right|\\ &\leq \frac{n-s}{1+k}\left[(|\alpha|+k)-(|\alpha|-1)\left(\frac{|a_0|-|a_{n-s}k^{n-s}+\beta m|}{(n-s)(|a_0|+|a_{n-s}k^{n-s}+\beta m|)}\right)\right]\max_{|z|=1}\left|G(z)+\frac{\beta m z^{n-s}}{k^{n-s}}\right|\\ &= \frac{n-s}{1+k}\left[(|\alpha|+k)-(|\alpha|-1)\left(\frac{|a_0|-|a_{n-s}k^{n-s}+\beta m|}{(n-s)(|a_0|+|a_{n-s}k^{n-s}+\beta m|)}\right)\right]\max_{|z|=1}\left|z^sG(z)+\frac{\beta m z^{n}}{k^{n-s}}\right|.\end{aligned}$$

After solving, we get

$$\begin{aligned}\max_{|z|=1}\left|D_\alpha\left(\frac{P(z)}{z^s}+\frac{\beta m z^{n-s}}{k^{n-s}}\right)\right| \leq \frac{n-s}{1+k}\Bigg[&(|\alpha|+k)-(|\alpha|-1)\times\\ &\left(\frac{|a_0|-|a_{n-s}k^{n-s}+\beta m|}{(n-s)(|a_0|+|a_{n-s}k^{n-s}+\beta m|)}\right)\Bigg]\max_{|z|=1}\left|P(z)+\frac{\beta m z^{n}}{k^{n-s}}\right|\end{aligned} \tag{4.3}$$

which is the desired result. □

5. Conclusion

The results obtained in this paper present generalizations of Erdös-Lax type polynomial inequalities concerning the polar derivative of a polynomial over a disc of arbitrary radius greater than or equal to one. Besides this we have also establish a similar inequality for polynomials having a multiple zero at origin. These types of inequalities have significant applications in approximation theory, geometric function theory, and other areas of mathematical analysis.

Conflict of Interest: The authors declare that there are no conflict of interests regarding the publication of this paper.

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