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ADVANCED DEPENDENCE METRICS AND TAIL COEFFICIENTS IN A BOUNDED COPULA CONSTRUCTION

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Abstract. In this work, we study some properties of the family of copulas introduced previously by Bagré et al. We discuss from dependence metrics such as Spearman's rho, the medial correlation coefficient and distances separating this family of copulas from the Fréchet bounds (lower and upper bounds) and the product copula, respectively. The tail cœfficients are also described. We show that the ranges of the Spearman rho and the medial correlation of this class are respectively -0.4784 to 0.4784 and -0.3333 to 0.3333. Furthermore, we prove that this class of copulas is bounded between Gumbel's bivariate logistic copula and modified Ali-Mikhail-Haq copula. In particular, we study the properties of two examples of this new family. In addition to theoretical properties, we present graphical illustrations such as level curves and scatter plots of observations from each of the examples.

Keywords. Copula, Copula with horizontal or vertical section of a homographic function, Dependence metrics, Tail cœfficients
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1. Introduction and preliminaries

Copulas have gained in importance owing to their ability to capture complex dependence features between random variables. However, both the construction of a new copula and the study of these dependence properties are not easy tasks. Studying the dependence properties of a newly constructed copula is essential to highlight its flexibility in real-life applications. Indeed, the study provides a comprehensive view of the range of dependencies that the copula can capture, as well as types of dependency such as positive quadrant dependency, negative quadrant dependency, upper or lower tail dependencies. Knowledge of all these properties is useful when choosing the copula for empirical data. To this end, in this article we present dependence properties such as Spearman's measure of association (Spearman rho), the medial correlation coefficient or Blomgqvist's beta, as well as the upper and lower tail dependence coefficients of the new class of copula presented in [3] called copula with horizontal or vertical section of a

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homographic function. We also present the distances of this new copula class from the Fréchet lower bound (C^-), the independent copula (P) and the Fréchet upper bound (C^+), respectively. For a given copula, its distances from the above copulas (C^-, P and C^+) tell us the sign and degree of dependence this copula can capture. We recall that the diameter of the space of two-dimensional copulas is 1 (i.e. $d(C^-, C^+) = 1$) and that $d(C^-, P) = d(P, C^+) = \dfrac{\sqrt{3}}{3} \approx 0.57$ (for more details see[12]). When $d(C, C^-) < 0.57$ then C captures negative dependencies. On the other hand, if $d(C, C^+) < 0.57$ then C captures positive dependencies. When $d(C, P) = 0$, the copula C is the independent copula. We show that the ranges of Spearman's rho and Blomqvist's Beta for the class of copulas with horizontal or vertical section of a homographic function are $[-0.4784; 0.4784]$ and $[-0.3333; 0.3333]$ respectively. Thus, this new copula class captures more dependence compared to some copulas in the literature such as the Ali-Mikhail-Haq copula whose ranges of Spearman's rho and Blomqvist's Beta are respectively $[33 - 48\ln 2; 4\pi^2 - 39] \approx [-0.2711; 0.4784]$ (see [8]) and $\left[-\dfrac{1}{5}; \dfrac{1}{3}\right] \approx [-0.2000; 0.3333]$, and the modified Ali-Mikhail-Haq copula (see [6]) whose Spearman rho and Blomqvist Beta ranges are $[-0.4784; 0]$ and $[-0.3333; 0]$ respectively. Moreover, the family of copulas with horizontal or vertical section of a homographic function captures asymmetric dependences. For further details, we have studied properties of two examples, the first of which captures the full range of dependencies that a copula from the class of copulas with horizontal or vertical section of a homographic function can capture. Graphical illustrations such as level curves and scatter plots for each of the examples were presented.

2. Materials and methods

In this section, we recall some notions and existing results that we will use in this work. We also briefly review the class of copulas with horizontal or vertical section of a homographic function, the study of whose properties is the subject of this article. In the following, let $I = [0; 1]$ and $I^2 = I \times I$.

2.1. Definitions and Properties.

Definition 2.1. [2, 5, 10] A bivariate copula is a function $C(u, v), (u, v) \in I^2$ satisfying the following properties :

(1) For every $u, v \in I$,

$$C(0, v) = C(u, 0) = 0, C(1, v) = v \text{ and } C(u, 1) = u, \tag{2.1}$$

(2) For every $u_1, u_2, v_1, v_2 \in I$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$, we have

$$C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0 \tag{2.2}$$

The conditions (2.1) are the boundary conditions and the (2.2) referred to the 2-increasing property.

In the setting of the absolutely continuous case, the property (2.2) is equivalent

to :
For every $(u,v) \in I^2$, we have

$$\frac{\partial^2 C(u,v)}{\partial v \partial u} \geq 0 \tag{2.3}$$

Definition 2.2. [10, 7, 9] Let X and Y be two continuous random variables with copula C. The Spearman's rho of X and Y is given by :

$$\rho_C = 12 \int_0^1 \int_0^1 C(u,v)\, du\, dv - 3 \tag{2.4}$$

an the medial correlation coefficient or Blomqvist beta of X and Y is given by :

$$\beta_C = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1 \tag{2.5}$$

Definition 2.3. [7] Let X and Y be two continuous random variables whose copula is C. The tail dependence coefficients, lower left (LL), lower right (LR), upper left (UL) and upper right (UR), are defined in terms of limits as follows

$$\lambda_{LL} = \lim_{u \to 0^+} \frac{C(u,u)}{u}, \qquad \lambda_{LR} = \lim_{u \to 0^+} \frac{u - C(1-u,u)}{u} \tag{2.6}$$

and

$$\lambda_{UL} = \lim_{u \to 0^+} \frac{u - C(u,1-u)}{u}, \qquad \lambda_{UR} = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u} \tag{2.7}$$

Proposition 2.4. [12] *Let A and B be two bivariate copulas, we have*

$$d(A,B)^2 = ||A||^2 + ||B||^2 - 2{<}A,B{>}$$

where $<,>$, $||.||$ and d respectively define the Sobolev scalar product, the Sobolev norm and the Sobolev distance.

2.2. **On the copula with horizontal or vertical section of a homographic function.** According to [3], the family of copulas with horizontal section of a homograph function is defined by

$$C(u,v) = \frac{a(v)u + b(v)}{r(v)u + s(v)} \tag{2.8}$$

while the family of copulas with vertical section of a homographic function is defined by

$$C(u,v) = \frac{a(u)v + b(u)}{r(u)v + s(u)} \tag{2.9}$$

where $(u,v) \in I^2$ and a, b, r and s appropriate functions defined on I.

As we explained in [3], the horizontal section of the copulas of this class can be considered as a homographic function of variable u whose constants are functions of v (Formula 2.8) and the vertical section can be considered as a homographic function of variable v whose constants are functions of u (Formula 2.9). This class can be also considered as a generalization of Ali-Mikhail-Haq family copulas.

To ensure that C defined in (2.8) to satisfy the conditions (2.1), it must be the case that (see [3], Theorem 2):

- $b(v) = 0$, $s(v) \neq 0$ and $a(v) = v(r(v) + s(v))$, $\forall v \in I$;
- $a(0) = r(1) = 0$.

Consequently, it is shown the class of copulas with horizontal section of a homographic function is written as follow :

$$C_{rs}(u, v) = \frac{uv[r(v) + s(v)]}{r(v)u + s(v)} \tag{2.10}$$

If C_{rs} defines a copula then we have (see [3], Theorem 3):

$$s(v) > 0, \; r(v) + s(v) > 0, \quad \forall (u, v) \in (0; 1)^2 \tag{2.11}$$

and

$$s(v) \geq 0, \; r(v) + s(v) \geq 0, \quad \forall (u, v) \in \{0, 1\}^2 \tag{2.12}$$

For more details on the functions r and s for which C_{rs} in (2.10) defines a copula, the reader can see [3].

Results have been established for the copula family with horizontal section of a homographic function, but we have stressed that the results are the same for the copula family with vertical section of a homographic function. Both types (Formula (2.8) and Formula (2.9)) can be considered as the same copula in terms of dependence properties.

We recall below the lower bound (Equation 2.13) and the upper bound (Equation 2.14) of the class of copulas with a horizontal or vertical section of a homographic function :

The form of expression (2.8) of the first modified Ali-Mikhail-Haq copula presented in [6] for $a = 1$

$$C_1(u, v) = \frac{uv^2}{1 - u + uv} \tag{2.13}$$

and the bivariate Gumbel logistic copula (see [10])

$$C_{LG}(u, v) = \frac{uv}{u + v - uv} \tag{2.14}$$

3. Main results

In this section, we enumerate a number of dependence properties for any copula with horizontal section of a homographic function and study two examples in details.

3.1. Some dependence metrics and tail cœfficients.

Theorem 3.1. Let C_{rs} be the copula defined by (2.10), C_1 and C_{LG} be the copulas defined by (2.13) and (2.14) respectively. Then, we have

$$C_1(u, v) \leq C_{rs}(u, v) \leq C_{LG}(u, v), \quad (u, v) \in I^2 \tag{3.1}$$

Before proving Theorem 3.1, we need the following Lemma.

Lemma 3.2. *Let $C_{rs}(u,v)$ be the copula defined by (2.10) for all $(u,v) \in I^2$. Then, we have the follow :*

$$(v-1)s(v) \leq r(v) \leq \frac{1-v}{v}s(v) \tag{3.2}$$

Proof. Since C_{rs} is a copula, then

$$\max(u+v-1,0) \leq C_{rs}(u,v) \leq \min(u,v), \text{ for all } (u,v) \in I^2.$$

- If $u \geq v$, $C_{rs}(u,v) - \min(u,v) = \dfrac{v(u-1)s(v)}{ur(v)+s(v)} \leq 0$
- If $u < v$, $C_{rs}(u,v) - \min(u,v) = \dfrac{u[(v-u)r(v)+(v-1)s(v)]}{r(v)u+s(v)}$.

We have :
$C_{rs}(u,v) - \min(u,v) \leq 0 \Longrightarrow r(v) \leq \dfrac{1-v}{v-u}s(v)$ for all $u,v \in I$
So, for all $v \in I$,

$$r(v) \leq \frac{1-v}{v}s(v) \tag{3.3}$$

- If $u \leq 1-v$, $C_{rs}(u,v) - \max(u+v-1,0) = C(u,v) \geq 0$
- If $u > 1-v$, $C_{rs}(u,v) - \max(u+v-1,0) = \dfrac{(1-u)[r(v)u+(1-v)s(v)]}{r(v)u+s(v)}$

We have :
$C_{rs}(u,v) - \max(u+v-1,0) \geq 0 \Longrightarrow r(v) \geq \dfrac{v-1}{u}s(v)$ for all $u,v \in I$
Thus, for all $v \in I$,

$$r(v) \geq (v-1)s(v) \tag{3.4}$$

From (3.3) and (3.4) we get (3.2) □

Proof. (**of the Theorem 3.1**)
Let's study the sign of $C_{rs}(u,v) - C_1(u,v)$ and $C_{rs}(u,v) - C_{LG}(u,v)$ separately.

- $$\begin{aligned} C_{rs}(u,v) - C_1(u,v) &= \frac{uv(r(v)+s(v))}{r(v)u+s(v)} - \frac{uv^2}{1-u+uv} \\ &= \frac{uv[(1-u)r(v)+(1+uv-u-v)s(v)]}{[ur(v)+s(v)][1-u+uv]} \\ &= \frac{uv(1-u)[r(v)+(1-v)s(v)]}{[ur(v)+s(v)][1-u+uv]} \end{aligned}$$

By using (2.11) and Lemma 3.2, we obtain

$$\frac{uv(1-u)[r(v)+(1-v)s(v)]}{[ur(v)+s(v)][1-u+uv]} \geq 0$$

Then, for all $u,v \in I$,

$$C_{rs}(u,v) \geq C_1(u,v) \tag{3.5}$$

- $$\begin{aligned} C_{rs}(u,v) - C_{LG}(u,v) &= \frac{uv(r(v)+s(v))}{r(v)u+s(v)} - \frac{uv}{u+v-uv} \\ &= \frac{uv[v(1-u)r(v) - (1+uv-u-v)s(v)]}{[ur(v)+s(v)][u+v-uv]} \\ &= \frac{uv(1-u)[vr(v)-(1-v)s(v)]}{[ur(v)+s(v)][u+v-uv]} \end{aligned}$$

By using (2.11) and Lemma3.2, we have

$$\frac{uv(1-u)[vr(v)-(1-v)s(v)]}{[ur(v)+s(v)][u+v-uv]} \leq 0$$

Then, for all $u, v \in I$,

$$C_{rs}(u,v) \leq C_{LG}(u,v) \tag{3.6}$$

From (3.5) and (3.6) we have the result. □

Theorem 3.3. Let C_{rs} be the copula defined by (2.10), C_1 and C_{LG} be the copulas defined by (2.13) and (2.14) respectively. Then we have the following properties:

(1) If $r(v) = (v-1)s(v)$, then $C_{rs}(u,v) = C_1(u,v)$
(2) If $r(v) = \dfrac{1-v}{v}s(v)$, then $C_{rs}(u,v) = C_{LG}(u,v)$

Proof. We have $C_{rs}(u,v) = \dfrac{uv[r(v)+s(v)]}{r(v)u+s(v)}$ for all $u, v \in I$.

(1) If $r(v) = (v-1)s(v)$, then

$$\begin{aligned} C_{rs}(u,v) &= \frac{uv[(v-1)s(v)+s(v)]}{u(v-1)s(v)+s(v)} \\ &= \frac{uv^2}{1-u+uv} \\ &= C_1(u,v) \end{aligned}$$

(2) If $r(v) = \dfrac{1-v}{v}s(v)$, then

$$\begin{aligned} C_{rs}(u,v) &= \frac{uv\left[\dfrac{1-v}{v}s(v)+s(v)\right]}{u\dfrac{1-v}{v}s(v)+s(v)} \\ &= \frac{uv(1-v)+uv^2}{u(1-v)+v} \\ &= \frac{uv}{u+v-uv} = C_{LG}(u,v) \end{aligned}$$

This completes the demonstration of Theorem 3.3 □

Corollary 3.4. Let ρ_{rs} and β_{rs} be the Spearman rho and the medial correlation of the copula C_{rs}, respectively, we have

(1) The range of ρ_{rs} is $[-0.4784; 0.4784]$
(2) The range of β_{rs} is $\left[-\frac{1}{3}; \frac{1}{3}\right]$

Proof. Let ρ_1, and ρ_{LG} be the Spearman rhos of the copulas C_1 and C_{LG} respectively.

(1) By using (2.4), we can write :

$$\begin{aligned}\rho_1 &= 12\int_0^1\int_0^1 C_1(u,v)du\,dv - 3\\ &= 12\left(\frac{7}{2}-\frac{\pi^2}{3}\right) - 3 \approx -0,4784\end{aligned}$$

and

$$\begin{aligned}\rho_{LG} &= 12\int_0^1\int_0^1 C_1(u,v)du\,dv - 3\\ &= 4\pi^2 - 39 \approx 0,4784\end{aligned}$$

Thus, according to Theorem 3.1,

$$\rho_{rs} \in [\rho_1; \rho_{LG}] \approx [-0.4784; 0.4784]$$

(2) Let β_1 and β_{LG} be the medial correlations of the copulas C_1 and C_{LG} respectively.
Based on the formula (2.5), we have

$$\begin{aligned}\beta_1 =& 4C_1\left(\frac{1}{2},\frac{1}{2}\right) - 1 \quad \text{and} \quad & \beta_{LG} =& 4C_{LG}\left(\frac{1}{2},\frac{1}{2}\right) - 1\\ =& -\frac{1}{3} & =& \frac{1}{3}\end{aligned}$$

So, using Theorem 3.1, we deduce that $\beta_{rs} \in [\beta_1; \beta_{LG}] = \left[-\frac{1}{3}; \frac{1}{3}\right]$ □

Theorem 3.5. Let $C_{rs}(u,v)$ be the copula defined in (2.10) for all $(u,v) \in I^2$. X and Y be two continuous random variables of copula C_{rs}. We have the following:

(i) If $r(v) < 0$, then X and Y are negative quadrant dependent.
(ii) If $r(v) = 0$, then X and Y are independent.
(iii) If $r(v) > 0$, then X and Y are positive quadrant dependent.

Proof. Let $P(u,v) = uv$ be the independent copula.

$$C_{rs}(u,v) - P(u,v) = \frac{uv(r(v)+s(v))}{r(v)u+s(v)} - uv = \frac{(1-u)r(v)}{r(v)u+s(v)}$$

. Using (2.11) we have, $r(v)+s(v) > 0 \Longrightarrow r(v)u+s(v) > 0$. Thus :

(i) If $r(v) < 0$, then $C_{rs}(u,v) - P(u,v) < 0 \Longrightarrow C_{rs}(u,v) < uv$
(ii) If $r(v) = 0$, then $C_{rs}(u,v) - P(u,v) = 0 \Longrightarrow C_{rs}(u,v) = uv$

(iii) If $r(v) > 0$, then $C_{rs}(u,v) - P(u,v) > 0 \Longrightarrow C_{rs}(u,v) > uv$

(i), **(ii)** and **(iii)** imply negative quadrant dependence, independence and positive quadrant dependence respectively □

Proposition 3.6. Let C_{rs} be the copula defined in (2.10). Then the followings hold.

(1) $d_{max}(C_{rs}, C^-) \approx 0.74$ and $d_{min}(C_{rs}, C^-) \approx 0.42$
(2) $d_{max}(C_{rs}, C^+) \approx 0.74$ and $d_{min}(C_{rs}, C^+) \approx 0.42$

where d_{max} denotes the maximal distance and d_{min} denotes the minimal distance.

Proof. Using Theorem 3.1, we have

$$\begin{aligned} d_{max}(C_{rs}, C^-) =& d(C_{LG}, C^-) \\ d_{min}(C_{rs}, C^-) =& d(C_1, C^-) \\ d_{max}(C_{rs}, C^+) =& d(C_1, C^+) \\ d_{min}(C_{rs}, C^+) =& d(C_{LG}, C^+) \end{aligned}$$

By using Proposition 2.4 and numerical calculation with **R** environment, we obtain (1) and (2) □

Theorem 3.7. *Let C_{rs} be the copula defined by (2.10). If $s(0) > 0$, $s(1) > 0$ and $s(0) + r(0) > 0$, then the copula C_{rs} does not capture tail dependencies.*

Proof. Let show that the tail cœfficients, lower left (LL), lower right (LR), upper left (UL) and upper right (UR), which describe the dependence of random variables in extreme cases, thus copula is C_{rs}, are zero.
Let C_{rs} the copula defined by (2.10) with $s(0) > 0$, $s(1) > 0$ and $s(0) + r(0) > 0$.
According to the Definition 2.3, we have :
the lower left (LL) tail cœfficient is

$$\begin{aligned} \lambda_{LL} &= \lim_{u \to 0^+} \frac{C_{rs}(u,u)}{u} \\ &= \lim_{u \to 0^+} \frac{u^2[r(u) + s(u)]}{u[ur(u) + s(u)]} \\ &= \lim_{u \to 0^+} \frac{u[r(u) + s(u)]}{ur(u) + s(u)} \\ &= \frac{0}{s(0)} \\ &= 0 \end{aligned}$$

the lower right (LR) tail cœfficient is

$$\begin{aligned}
\lambda_{LR} &= \lim_{u\to 0^+} \frac{u - C_{rs}(1-u,u)}{u} \\
&= \lim_{u\to 0^+} \frac{u[r(u)(1-u)+s(u)] - u(1-u)[r(u)+s(u)]}{u[r(u)(1-u)+s(u)]} \\
&= \lim_{u\to 0^+} \frac{us(u)}{(1-u)r(u)+s(u)} \\
&= \frac{0}{r(0)+s(0)} \\
&= 0
\end{aligned}$$

the upper left (UL) tail cœfficient is

$$\begin{aligned}
\lambda_{UL} &= \lim_{u\to 0^+} \frac{u - C_{rs}(u,1-u)}{u} \\
&= \lim_{u\to 0^+} \frac{u[r(1-u)u+s(1-u)] - u(1-u)[r(1-u)+s(1-u)]}{u[r(1-u)u+s(1-u)]} \\
&= \lim_{u\to 0^+} \frac{r(1-u)(2u-1)+s(1-u)u}{r(1-u)u+s(1-u)} \\
&= \frac{0}{s(1)} \\
&= 0
\end{aligned}$$

and the upper right (UR) tail cœfficient is

$$\begin{aligned}
\lambda_{UR} &= \lim_{u\to 1^-} \frac{1-2u+C_{rs}(u,u)}{1-u} \\
&= \lim_{u\to 1^-} \frac{u(1-u)r(u)+(1-u)^2s(u)}{(1-u)[ur(u)+s(u)]} \\
&= \lim_{u\to 1^-} \frac{ur(u)+(1-u)s(u)}{ur(u)+s(u)} \\
&= \frac{r(1)}{r(1)+s(1)} = 0 \quad \text{because } r(1) \\
&= 0
\end{aligned}$$

Hence the proof of Theorem 3.7 □

Remark 3.8. If $s(0), s(1)$ and $s(0)+r(0)$ are zero, then it is possible that C_{rs} have a tail dependency.

3.2. **First copula with horizontal section of homographic function.**

Proposition 3.9. *The function* $C_\delta(u,v) = \dfrac{uv}{u+(1-u)v^{-\delta}}$ *for all* $(u,v) \in I^2$, *is a copula with horizontal section of a homographic function for* $\delta \in [-1;1]$.

Proof. For all $v \in [0;1]$, we have :

$$\begin{aligned}C_\delta(0,v) =& \frac{0 \times v}{0 + (1-0)v^{-\delta}}\\ =& 0\\ C_\delta(1,v) =& \frac{1 \times v}{1 + (1-1)v^{-\delta}}\\ =& v\end{aligned}$$

For all $u \in [0;1]$, we have :

$$\begin{aligned}C_\delta(u,0) =& \frac{u \times 0}{u + (1-u) \times 0^{-\delta}} \text{ if } \delta \in [-1;0]\\ =& 0\\ C_\delta(u,0) =& \frac{u \times 0^{1+\delta}}{u \times 0^{\delta} + (1-u)} \text{ if } \delta \in [0;1]\\ =& 0\\ C_\delta(u,1) =& \frac{u \times 1}{u + (1-u) \times 1^{-\delta}}\\ =& u\end{aligned}$$

Thus, C_δ satisfies the boundary conditions of a copula given in (2.1)
Furthermore, the density function of C_δ is given by :

$$\frac{\partial^2 C_\delta(u,v)}{\partial u \partial v} = \frac{(1-\delta)uv^{-\delta} + (1+\delta)(1-u)v^{-2\delta}}{[u + (1-u)v^{-\delta}]^3}$$

For $\delta \in [-1;1]$,

$$\frac{\partial^2 C_\delta(u,v)}{\partial u \partial v} \geq 0, \ \forall (u,v) \in I^2.$$

Thus the function C_δ verifies the 2-increasingness property given by (2.3).
To show that C_δ is a copula with horizontal section of a homographic function, simply note that after some algebraic operations, the copula C_δ can be written as

$$C_\delta(u,v) = \frac{uv[r(v) + s(v)]}{r(v)u + s(v)}$$

where $r(v) = \alpha\left((\sqrt{v})^{\delta} - (\sqrt{v})^{-\delta}\right)$ and $s(v) = \alpha(\sqrt{v})^{-\delta}; \alpha > 0$ □

Proposition 3.10. *The range of the Spearman rho of the copula C_δ is given by $[-0.4784; 0.4784]$ and the medial correlation cœfficient is $\dfrac{1-2^\delta}{1+2^\delta}$.*

Proof. Let ρ_δ be the Spearman rho of the copula C_δ.
Let's put,

$$f(\delta) = \frac{uv}{u + (1-u)v^{-\delta}} = C_\delta(u,v) \text{ for all } (u,v) \in I^2 \text{ with } \delta \in [-1;1]$$

.

We have

$$f'(\delta) = \frac{u(1-u)v^{1-\delta}\ln v}{[u+(1-u)v^{-\delta}]^2} \leq 0.$$

Then f is decreasing on $[-1;1]$. Consequently, the copula C_δ is bounded as follows:

$$C_1(u,v) \leq C_\delta(u,v) \leq C_{-1}(u,v) = C_{LG}(u,v),\ \forall\ (u,v) \in I^2$$

where C_1 and C_{LG} are the copulas defined by (2.13) and (2.14) respectively. Hence, we deduce that

$$\rho_\delta \in [-0.4784; 0.4784].$$

Based on the formula (2.5), the Blomqvist beta of the copula C_δ is given by :

$$\begin{aligned}
\beta_\delta &= 4C(\frac{1}{2},\frac{1}{2}) - 1 \\
&= 4 \times \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}+(1-\frac{1}{2})\times\frac{1}{2^{-\delta}}} - 1 \\
&= \frac{1-2^\delta}{1+2^\delta}
\end{aligned}$$

□

Proposition 3.11. *The lower tail and the upper tail cœfficients of the copula C_δ are respectively given by*

$$\lambda_{LL} = \begin{cases} \frac{1}{2} & \text{if } \delta = -1 \\ 0 & \text{otherwise} \end{cases}, \quad \lambda_{LR} = \begin{cases} \frac{1}{2} & \text{if } \delta = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \lambda_{UL} = \lambda_{UR} = 0$$

Proof. By using formula (2.6) and through a simple calculation, lower left (LL) and lower right (LR) tail cœfficients are derived.
Since $s(1) > 0$, then using Theorem 3.7, we deduce the upper left and right tail cœfficients □

In table 1, we present distance of C_δ from Fréchet bounds and independent copula respectively, for fixed values of δ.

Table 1. Variations of the distances as a function of δ

$\delta \to$	-1	-0.77	-0.55	-0.33	-0.11	0	0.11	0.33	0.55	0.77	1
$d(C_\delta, C^-)$	0.74	0.70	0.67	0.63	0.59	0.57	0.55	0.52	0.48	0.45	0.42
$d(C_\delta, P)$	0.23	0.18	0.13	0.07	0.026	0	0.026	0.08	0.13	0.18	0.23
$d(C_\delta, C^+)$	0.42	0.45	0.48	0.52	0.55	0.57	0.59	0.63	0.67	0.70	0.74

The results of the table 1 tell us about the degree of dependence as well as the quadrant dependencies of the copula C_δ. Generally when δ travels from -1 to

1, C_δ approaches C^- while moving away from C^+. For $\delta \in [-1;0[$, $d(C_\delta, C^-) > d(C^-, P)$ or equivalently, $d(C_\delta, C^+) < d(C^+, P)$. This shows that $C_\delta \geq P$. Conversely, for $\delta \in]0;1]$, $d(C_\delta, C^-) < d(C^-, P)$ and $d(C_\delta, C^+) > d(C^+, P)$ implying that $C_\delta \leq P$. Finally, when $\delta = 0$, then $d(C_\delta, P) = 0$ i.e. $C_\delta = P$.

To illustrate the theoretical results on the dependence of the copula C_δ, we present below, the contour plots of the copula C_δ and the scatterplots of observations generated from the copula C_δ for $\delta = -1$ and $\delta = 1$. Note that all figures are generated using the statistical software **R**.

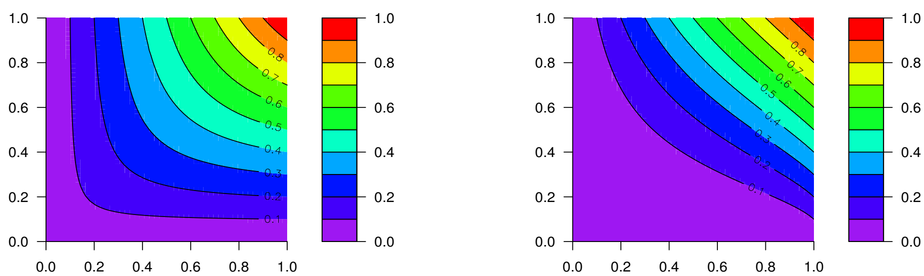

FIGURE 1. Contour plot of C_{-1} (left) and of C_1 (right)

Let (U, V) be a uniform random vector of joint distribution C_δ. The conditional distribution of the variable V given $U = u$ is :

$$
\begin{aligned}
c_u(v) =& \frac{\partial C_\delta(u,v)}{\partial u} \\
=& \frac{v^{1-\delta}}{\left[u + (1-u)v^{-\delta}\right]^2}
\end{aligned}
$$

The function c_u is increasing almost for all $v \in I$ (see [10]). So, its inverse function c_u^{-1} exists. Using the conditional distribution method and by using **R** environment, we generate 500 observations of the pair uniform variables (U, V) according to the following steps:

- Generate 500 observations of two independent uniform $(0;1)$ variables U and Λ.

Let be $U = (u_1, ..., u_{500})$ and $\Lambda = (\lambda_1, ..., \lambda_{500})$.

- Numerically invert the function c_u.
- Set $v_i = c_{u_i}^{-1}(\lambda_i)$ for $i = 1, ..., 500$, where c_u^{-1} is the inverse function of c_u.
- The desired pair is $(U, V) = \{(u_1, v_1); ...; (u_{500}, v_{500})\}$.

Figure 2 shows scatterplots of the pair (U, V) for fixed values of δ.

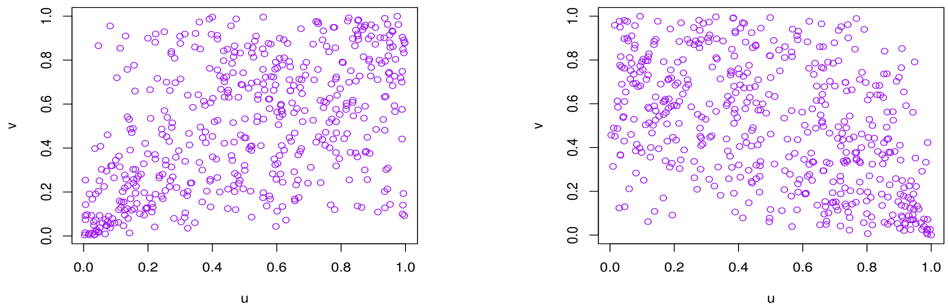

FIGURE 2. Scatterplot of data with copula C_{-1} (left) and copula C_1 (right)

The Figure 2 shows clearly the existence of the lower left tail dependence when $\delta = -1$ and lower right tail dependence when $\delta = 1$ as given in Proposition 3.11.

4.2. Second copula with horizontal section of homographic function.

Proposition 3.12. *For all $(u, v) \in I^2$, the function*

$$C_\gamma(u, v) = \frac{uve^{\gamma \sinh(v)}}{ue^{\gamma \sinh(v)} + (1-u)e^{\gamma[\cosh(v) - \mathrm{e}^{-1}]}}$$

defines a copula with horizontal section of a homographic function for $\gamma \in [-\mathrm{e}; \mathrm{e}]$.

Proof.

- Verification of boundary conditions :

For all $v \in I$, we have :

$$\begin{aligned} C_\gamma(0, v) &= \frac{0 \times ve^{\gamma \sinh(v)}}{0 \times e^{\gamma \sinh(v)} + (1-0)e^{\gamma(\cosh(v) - e^{-1})}} \\ &= \frac{0}{e^{\gamma(\cosh(v) - e^{-1})}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} C_\gamma(1, v) &= \frac{1 \times ve^{\gamma \sinh(v)}}{1 \times e^{\gamma \sinh(v)} + (1-1)e^{\gamma(\cosh(v) - e^{-1})}} \\ &= \frac{ve^{\gamma \sinh(v)}}{e^{\gamma \sinh(v)}} \\ &= v \end{aligned}$$

Similarly, for all $u \in I$, we have :

$$\begin{aligned} C_\gamma(u, 0) &= \frac{u \times 0 \times e^{\gamma \sinh(0)}}{u \times e^{\gamma \sinh(0)} + (1-u)e^{\gamma(\cosh(0) - e^{-1})}} \\ &= \frac{0}{u + (1-u)e^{\gamma[1 - e^{-1}]}} \\ &= 0 \end{aligned}$$

$$\begin{aligned}
C_\gamma(u,1) &= \frac{u \times 1 \times e^{\gamma\sinh(1)}}{u \times e^{\gamma\sinh(1)} + (1-u)e^{\gamma(\cosh(1)-e^{-1})}}\\
&= \frac{ue^{\gamma\sinh(1)}}{e^{\gamma\sinh(1)}}\\
&= u
\end{aligned}$$

- Verification of the 2-growth condition :

The partial second derivative of the copula C_γ can be given for all $(u,v) \in I$ as

$$\frac{\partial^2 C_\gamma(u,v)}{\partial v \partial u} = \frac{\left[u(1-\gamma ve^{-v})e^{\gamma\sinh(v)} + (1-u)(1+\gamma ve^{-v})e^{\gamma(\cosh(v)-e^{-1})}\right]e^{\gamma(e^v-e^{-1})}}{\left[ue^{\gamma\sinh(v)} + (1-u)e^{\gamma(\cosh(v)-e^{-1})}\right]^3}$$

For all $\gamma \in [-e;e]$ and $v \in I$, we have

$$\begin{aligned}
1-\gamma ve^{-v} \geq& 1-ve^{-v}\\
1+\gamma ve^{-v} \geq& 1-ve^{-v}
\end{aligned}$$

.

We show easily that $1-ve^{-v} \geq 0, \forall v \in I$.

Therefore, we have

$$\frac{\partial^2 C_\gamma(u,v)}{\partial v \partial u} \geq 0, \forall (u,v) \in I^2$$

.

With a simple algebraic transformation, we deduce that for all $(u,v) \in I^2$,

$$C_\gamma(u,v) = \frac{uv[r(v)+s(v)]}{r(v)u+s(v)}$$

where

$$\begin{aligned}
r(v) =& e^{\gamma\sinh(v)} - e^{\gamma(\cosh(v)-e^{-1})}\\
s(v) =& e^{\gamma(\cosh(v)-e^{-1})}
\end{aligned}$$

Therefore C_γ is a copula with horizontal section of a homographic function □

Proposition 3.13. Let ρ_γ and β_γ be Spearman's rho and the medial correlation cœcoefficient of the copula C_γ, then we have :

(1) $\rho_\gamma \in [-0.4270; 0.4270]$
(2) $\beta_\gamma \in [-0.3134; 0.3134]$

Proof. For all $(u,v) \in I^2$ and $\gamma \in [-e;e]$, let pose :

$$h(\gamma) = C_\gamma(u,v) = \frac{uve^{\gamma\sinh(v)}}{ue^{\gamma\sinh(v)} + (1-u)e^{\gamma[\cosh(v)-\mathrm{e}^{-1}]}}$$

h is derivable for all $\gamma \in [-e,e]$ and we have :

$$h'(\gamma) = -\frac{uv(1-u)e^{-v+\gamma(e^v-e^{-1})}}{\left[ue^{\gamma\sinh(v)} + (1-u)e^{\gamma(\cosh(v)-e^{-1})}\right]^2} \leq 0$$

So, for all $\gamma \in [-e; e]$, C_γ is decreasing with respect to γ. Thus, for all $(u, v) \in I^2$, we have

$$C_e(u, v) \leq C_\gamma(u, v) \leq C_{-e}(u, v) \tag{3.7}$$

(1) By referring to the formula (2.4) and using some operations, we have

$$\rho_{-e} = 12 \int_0^1 \frac{v\left[e^{-2e\sinh(v)} - e^{\cosh(1+v)+\sinh(1+v)-v+2}\right]}{\left[e^{-e\sinh(v)} - e^{-e\cosh(v)+1}\right]^2}\, dv - 3$$

and

$$\rho_e = 12 \int_0^1 \frac{v\left[e^{-2e\sinh(v)} + (e^{1-v} - 1)e^{\cosh(1+v)+\sinh(1+v)-1}\right]}{\left[e^{e\sinh(v)} - e^{e\cosh(v)11}\right]^2}\, dv - 3$$

By using numeric calculation, we obtain : $\rho_e \approx -0.4270$ and $\rho_{-e} \approx 0.4270$ Based on the formula (3.7), we have

$$\rho_\gamma \in [\rho_e; \rho_{-e}] \approx [-0.4270; 0.4270]$$

(2) We have

$$\begin{aligned}\beta_\gamma =& 4 \times C_\gamma\left(\frac{1}{2}, \frac{1}{2}\right) - 1\\ =& \frac{e^{\gamma\sinh\left(\frac{1}{2}\right)} - e^{\gamma\left(\cosh\left(\frac{1}{2}\right) - e^{-1}\right)}}{e^{\gamma\sinh\left(\frac{1}{2}\right)} + e^{\gamma\left(\cosh\left(\frac{1}{2}\right) - e^{-1}\right)}}\end{aligned}$$

In particular we have,

$$\beta_e \approx -0.3134 \text{ and } \beta_{-e} \approx 0.3134$$

By using formula (3.7) we obtain that

$$\beta_\gamma \in [\beta_e; \beta_{-e}] \approx [-0.3134; 0.3134]$$

Thus, Proposition 3.13 is proved □

Proposition 3.14. *For $\gamma \in [-e; e]$, the copula C_γ does not capture any tail dependency i.e. $\lambda_{LL} = \lambda_{LR} = \lambda_{UL} = \lambda_{UR} = 0$*

Proof. For all $(u, v) \in I^2$ and $\gamma \in [-e; e]$, the copula C_γ can be written by

$$C_\gamma(u, v) = \frac{uv[r(v) + s(v)]}{ur(v) + s(v)}$$

where $r(v) = e^{\gamma\sinh(v)} - e^{\gamma(\cosh(v)-e^{-1})}$ and $s(v) = e^{\gamma(\cosh(v)-e^{-1})}$.
With simple calculation, we have :

$$\begin{aligned} s(0) =& e^{\gamma(1-e^{-1})} \neq 0\\ r(0) + s(0) =& 1 \neq 0\\ s(1) =& e^{\gamma\sinh(1)} \neq 0.\end{aligned}$$

So, using Theorem 3.7, we conclude. □

Using a numerical technique, we obtain distance of C_γ from Fréchet bounds and independent copula, respectively, recorded in the table 2.

TABLE 2. Variation of the distances as a function of γ.

$\gamma \to$	$-e$	-2.11	-1.51	-0.90	-0.30	0	0.30	0.90	1.51	2.11	e
$d(C_\gamma, C^-)$	0.72	0.69	0.66	0.627	0.59	0.57	0.56	0.52	0.49	0.46	0.43
$d(C_\gamma, P)$	0.19	0.15	0.10	0.06	0.02	0	0.02	0.06	0.10	0.15	0.19
$d(C_\gamma, C^+)$	0.43	0.46	0.49	0.52	0.56	0.57	0.59	0.62	0.66	0.69	0.72

For $\gamma \in [-e, 0[, d(C_\gamma, C^-) > d(P, C^-)$ and $d(C_\gamma, C^+) < d(P, C^+)$ showing that $C_\gamma > P$ i.e. that C_γ captures positive dependencies. Conversely, for $\gamma \in]0, e], d(C_\gamma, C^-) < d(P, C^-)$ and $d(C_\gamma, C^+) > d(P, C^+)$ showing that $C_\gamma < P$ i.e that C_γ captures negative dependencies. For $\gamma = 0$, $d(C_\gamma, P) = 0$. This implies that C_γ is the independent copula.

As with the first copula, we present below the level curves of the copula C_γ as well as scatter plots of the 500 observations generated from the copula C_γ for $\gamma = -e$ and $\gamma = e$.

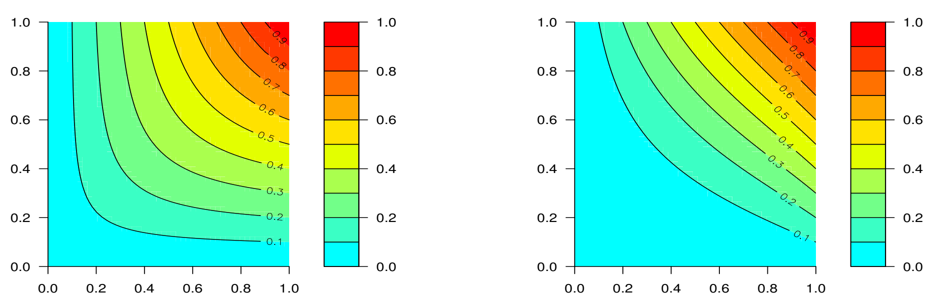

FIGURE 3. Contour plot of $C_{-\mathrm{e}}$ (left) and of C_{e} (right)

Let (U, V) be a uniform random vector of joint distribution C_γ. The conditional distribution of the variable V given $U = u$ is

$$
\begin{aligned}
c_u(v) =& \frac{\partial C_\gamma(u, v)}{\partial u} \\
=& \frac{v \exp\{\gamma(\mathrm{e}^v - \mathrm{e}^{-1})\}}{[u \exp\{\gamma \sinh(v)\} + (1 - u) \exp\{\gamma(\cosh(v) - \mathrm{e}^{-1})\}]^2}
\end{aligned}
$$

The function c_u is increasing almost for all $v \in I$ (see [10]). So its inverse function c_u^{-1} exists. Using the conditional distribution method and by using **R** environment, we generate 500 observations of the random vector (U, V) as follows:

- Generate 500 observations of two independent uniform $(0; 1)$ variables U and T.
- Numerically invert the function c_u.
- Set $v_i = c_{u_i}^{-1}(t_i)$ for $i = 1, ..., 500$, where c_u^{-1} is the inverse function of c_u.
- The desired pair is $(U, V) = \{(u_1, v_1); ...; (u_{500}, v_{500})\}$.

Figure 4 shows the scatterplot of the random vector (U, V) for fixed values of γ.

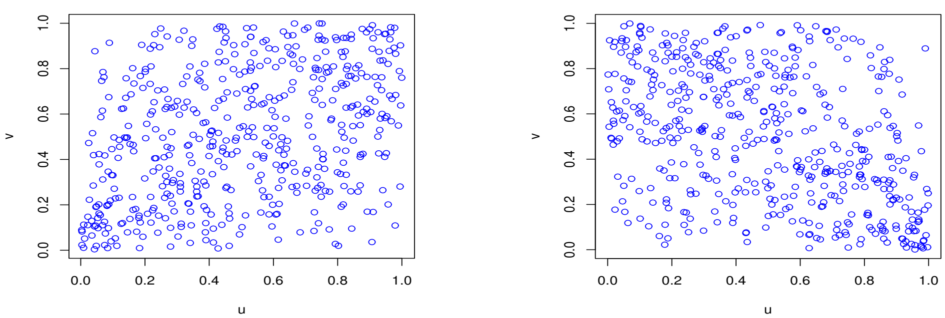

FIGURE 4. Scatterplot of data with copula $C_{-\mathrm{e}}$ (left) and copula C_{e} (right)

Unlike the first copula, the second copula does not capture tail dependencies as indicated by Proposition 3.14.

4. Conclusion

In this article, we have presented some metrics of dependencies, such as Spearman's measure of association (or the Spearman rho), the medial correlation coefficient (or the Blomqvist beta) and tail cœfficients of the class of copulas defined in [3]. We have also presented the distances of this copula family from the Fréchet bounds and the product copula. It is showed that the range of the Spearman rho is $[-0.4784; 0.4784]$ and that the Blomqvist beta is $[-0.3333; 0.3333]$. This shows that this class of copulas can capture relatively high negative and positive dependencies. Two examples were presented and studied in more detail.

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