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ON THE ROMAN DOMINATION POLYNOMIAL OF THE COMMUTING AND NON-COMMUTING GRAPHS ASSOCIATED TO THE DIHEDRAL GROUPS

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Abstract. A graph associated to a finite group is a way to analyze some properties of a group graphically. Many graphs of groups have been constructed according to the properties of the groups such as the commuting and non-commuting graphs. Besides, a Roman dominating function (in brief RDF) of a graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, defined as a function f from the set $V(\Gamma)$ to the set of numbers $\{0, 1, 2\}$ in which for any vertex $u \in V(\Gamma)$ with $f(u) = 0$ there is at least a vertex $v \in V(\Gamma)$ with $f(v) = 2$ is adjacent to u. The sum of the values $f(u)$ for all the vertices of Γ is defined by the weight of the RDF f. Meanwhile, the Roman domination number (RDN) of Γ, $\gamma_R(\Gamma)$, is the minimum weight of an RDF defined in Γ. Based on that, the Roman domination polynomial (RDP) of a graph Γ on p vertices is defined by $R(\Gamma, x) = \sum_{j=\gamma_R(G)}^{2p} r(\Gamma, j)x^j$, where $r(\Gamma, j)$ is the number of RDFs of Γ with weight j. In this paper, the RDPs of the commuting and non-commuting graphs associated to dihedral groups with order $2n$ are computed and some examples are given to illustrate the results.

Keywords. Roman domination, Roman domination polynomial, commuting graph, non-commuting graph, dihedral groups
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1. Introduction and preliminaries

All the graphs considered throughout this paper are finite and simple, namely finite, undirected and have no self-loops or multiple edges. Let $\Gamma = (V, E)$ be a graph. The order and size of Γ are denoted respectively by $|V(\Gamma)| = p$ and $|E(\Gamma)| = q$. As usual the complete graph, null graph, cycles and paths on p vertices are denoted in this paper as K_p, $\overline{K_p}$, C_p and P_p, respectively. The following expressions $\lceil x\rceil$ and $\lfloor x\rfloor$ of the positive real number x are denoted respectively to the least positive integer bigger than or equal to x and the largest positive

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integer less than or equal to x.

The join $\Gamma_1 \vee \Gamma_2$ of two graphs Γ_1 and Γ_2, with different vertex sets $|V(\Gamma_1)| = p_1$ and $|V(\Gamma_2)| = p_2$, contains the original graphs Γ_1 and Γ_2 and every vertex of Γ_1 is linked to each vertex of Γ_2 by an edge.

The RDN of a graph $\Gamma = (V, E)$, $\gamma_R(\Gamma)$, is defined in [3] as the smallest weight, $W(f(V)) = \sum_{u \in V(\Gamma)} f(u)$, of an RDF $f : V(\Gamma) \to \{0, 1, 2\}$ in which for any vertex u in Γ with $f(u) = 0$ there is at least a vertex, say v, from the neighbors of u satisfied that $f(v) = 2$. For details, we refer the readers to [1].

In 2022, Deepak *et al.* [2] defined the RDP of a graph Γ as $R(\Gamma, x) = \sum_{j=\gamma_R(\Gamma)}^{2p} r(\Gamma, j)x^j$, where $r(\Gamma, j)$ is the number of RDFs of Γ with weight j and studied some of its properties. In addition, the RDPs of paths and cycles are studied in details in [4] and [5], respectively. Moreover, in the next year Ramane and Patil [18] defined the A-polynomial for some family of caveman graphs. The following lemmas present some properties of the RDPs of graphs that are needed in this paper.

Lemma 1.1 gives the RDP of the null graph $\overline{K_p}$.

Lemma 1.1. [2] Consider Γ to be the null graph $\overline{K_p}$ on p vertices. Then

$$R(\Gamma, x) = \sum_{j=0}^{p} \binom{p}{j} x^{p+j} = x^p \big(1 + x\big)^p.$$

Next, the RDP of a complete graph K_p is given in the following lemma.

Lemma 1.2. [2] For the complete graph K_p, with $p \geq 2$,

$$R(K_p, x) = x^p + \sum_{j=2}^{2p} \left[\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \binom{p}{r} \binom{p-r}{j-2r} \right] x^j.$$

Now, Lemma 1.3 gives the RDP of the join $\Gamma_1 \vee \Gamma_2$.

Lemma 1.3. [2] Let Γ_1, Γ_2 be any two graphs with $|\Gamma_1| = p_1$, $|\Gamma_2| = p_2$, respectively. Then

$$R(\Gamma_1 \vee \Gamma_2, x) = \sum_{r=1}^{p_1} \binom{p_1}{r} \left[\sum_{t=0}^{p_1-r} \binom{p_1 - r}{t} \left(\sum_{s=1}^{p_2} \binom{p_2}{s} \sum_{l=0}^{p_2-s} \binom{p_2 - s}{l} \right) \right] x^{2r+t+2s+l}$$
$$+ \big(1 + x\big)^{p_2} \Big(R(G_1, x) - x^{p_1} \Big) + \big(1 + x\big)^{p_1} \Big(R(G_2, x) - x^{p_2} \Big) + x^{p_1+p_2}.$$

Recall that, the following properties of the combination $\binom{n}{r}$ are considered in the above lemmas.

(1) If $r = 0$ or $r = n$, then $\binom{n}{r} = 1$.

(2) If $r > n$, then $\binom{n}{r} = 0$.

(3) If $r = 1$, then $\binom{n}{1} = n$.

The following lemma presents the RDP of disconnected graphs.

Lemma 1.4. [2] Assume the graph $\Gamma = \cup_{i=1}^{m} \Gamma_i$, where $\Gamma_1, \ldots, \Gamma_m$ are the components of Γ. Then

$$R(\Gamma, x) = R(\Gamma_1, x) \cdots R(\Gamma_m, x).$$

Besides, a group consists of a set of elements and an operation combines the elements of the set to form an element also must be in the set. This operation on the set satisfies four conditions called the group axioms [6]. A group is called finite if its set is finite. The dihedral group is an example of a finite non-abelian group. In the following, the definitions of a center of group, a dihedral group and some properties of a dihedral group are provided.

Definition 1.5. [7] Consider G to be a finite group. The subset $Z(G)$ of G containing all the elements of G which are commute with every element in G is called the center of the group G i.e. $Z(G) = \{\alpha \in G |\, \alpha y = y\alpha,\, \forall y \in G\}$.

Next, the definition of a dihedral group and some of its properties are given.

Definition 1.6. [8] A dihedral group of order $2n$ where $n \geq 3$, denoted by D_{2n}, is a group of symmetries of a regular polygon, which includes rotations and reflections. A dihedral group can be presented also as:

$$D_{2n} \cong \left\langle a, b :\ a^n = b^2 = e,\ baba = e \right\rangle.$$

Some properties of a dihedral group D_{2n} are presented in the following lemma.

Lemma 1.7. [8] Consider $D_{2n} = \{e, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$ to be a dihedral group of order $2n$. Then D_{2n} has the following properties:

(1) For odd n,
  (a) the center $Z(D_{2n}) = \{e\}$,
  (b) the elements of the form a^i, where $i = 1, 2, \ldots, n-1$, are commute one to each other and a^i are not commute with the elements of the form a^jb, where $j = 0, 1, 2, \ldots, n-1$,
  (c) the elements of the form a^jb, where $j = 0, 1, 2, \ldots, n-1$, are not commute with any element in D_{2n} except the identity element.

(2) For even n,
  (a) the center $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$,
  (b) the non-central elements of the form a^i i.e. $i = 1, 2, \ldots, n-1$ and $i \neq \dfrac{n}{2}$, are commute one to each other and a^i are not commute with the elements of the form a^jb, where $j = 0, 1, 2, \ldots, n-1$,

(c) each element of the form $a^j b$, where $j = 0, 1, 2, \ldots, n-1$, is not commute with any element in D_{2n} except the identity element and the element of the form $a^{\frac{n}{2}j}b$.

Meanwhile, the idea of studying algebraic structures by using properties of graphs is one of the interesting topics in research because of its impressive results and questions. One of these studies is the study of graphs of groups. Many researches are presented on the commuting graph (or shortly CG) and the non-commuting graph (shortly NCG) associated to finite groups. For instance, Vahidi and Talebi [9], Segev and Seitz [10], Raza and Faizi [11] and Parker [12] studied the CGs related to some specific groups; while Abdollahi *et al.* [13], Moradipour *et al.* [14] and Ahanjideh and Iranmanesh [15] have studied the NCGs. In the following, the definitions of the CG and the NCG associated to finite non-abelian groups are provided. Firstly, the definition of the CG of non-abelian finite groups is given.

Definition 1.8. [9] Consider G to be a non-abelian finite group. The CG of G, denoted by Γ_G^{com} is a simple undirected graph with vertex set $G - Z(G)$ and two vertices are adjacent if they are commute.

In [16], the CG of dihedral groups, D_{2n}, is stated as follows:

Lemma 1.9. [16] Let $G = D_{2n}$ be a dihedral group of order $2n$, where $D_{2n} \cong \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ and $n \geq 3$. Then the CG of D_{2n}, $\Gamma_{D_{2n}}^{com}$ is given as

$$\Gamma_{D_{2n}}^{com} = \begin{cases} K_{n-1} \cup nK_1, & \text{if } n \text{ is odd;} \\ \\ K_{n-2} \cup \frac{n}{2}K_2, & \text{if } n \text{ is even.} \end{cases}$$

Next, the definition of the NCG of non-abelian finite groups is presented.

Definition 1.10. [13] Let G be a non-abelian finite group. The NCG of G, denoted by Γ_G^{ncom} is a simple graph with vertex set $G - Z(G)$ and two vertices are adjacent if and only if they are not commute.

The following lemma gives the NCG of D_{2n}.

Lemma 1.11. [17] Let $D_{2n} = \{e, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$ be the dihedral group of order $2n$. Then the NCG of D_{2n}, $\Gamma_{D_{2n}}^{ncom}$, is one of the following forms.

(1) If n is odd, then $\Gamma_{D_{2n}}^{ncom} = K_n \vee \overline{K_{n-1}}$.
(2) If n is even, then $\Gamma_{D_{2n}}^{ncom} = U_n \vee \overline{K_{n-2}}$, where U_n is an $(n-2)$-regular graph with n vertices.

In this paper, the RDPs of the CG and the NCG associated to the dihedral groups are found and illustrated by some examples.

2. Main results

First, the RDP of the CG associated to the dihedral groups, D_{2n}, is obtained as follows.

Theorem 2.1. *Let $D_{2n} = \langle a, b : a^n = b^2 = baba = e\rangle$ be the dihedral group of order $2n$, where $n \geq 3$ and $\Gamma^{com}_{D_{2n}}$ is its CG. Then the RDP of $\Gamma^{com}_{D_{2n}}$ is given as:*

(1) *If n is odd, then*

$$R(\Gamma^{com}_{D_{2n}}, x) = x^n\big(1+x\big)^n\Bigg[x^{n-1} + \sum_{j=2}^{2n-2}\Bigg[\sum_{r=1}^{\lfloor \frac{j}{2}\rfloor}\binom{n-1}{r}\binom{n-r-1}{j-2r}\Bigg]x^j\Bigg].$$

(2) *If n is even, then*

$$R(\Gamma^{com}_{D_{2n}}, x) = x^n\big(x^2+2x+3\big)^{\frac{n}{2}}\Bigg[x^{n-2} + \sum_{j=2}^{2n-4}\Bigg[\sum_{r=1}^{\lfloor \frac{j}{2}\rfloor}\binom{n-2}{r}\binom{n-r-2}{j-2r}\Bigg]x^j\Bigg].$$

Proof. By Lemma 1.9, the CG of the dihedral groups D_{2n} is given as

$$\Gamma^{com}_{D_{2n}} = \begin{cases} K_{n-1} \cup nK_1, & \text{if } n \text{ is odd;} \\ K_{n-2} \cup \frac{n}{2}K_2, & \text{if } n \text{ is even.} \end{cases}$$

Therefore by using Lemma 1.4, the RDP of $\Gamma^{com}_{D_{2n}}$ is given as

$$R(\Gamma^{com}_{D_{2n}}, x) = \begin{cases} R(K_{n-1}) \cdot \big(R(K_1)\big)^n, & \text{if } n \text{ is odd;} \\ R(K_{n-2}) \cdot \big(R(K_2)\big)^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

Also by Lemma 1.1, $R(K_1) = x(1+x)$ and by Lemma 1.2, $R(K_2) = x^2(x^2 + 2x+3)$. Hence by using Lemma 1.2 again, the result is obtained. □

Now, the following example illustrates the result in Theorem 2.1 part (1).

Example 2.2. Let $D_{10} = \{e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$ be the dihedral group of order 10, where $n = 5$. By Lemma 1.9, the CG of the dihedral groups D_{10} is $\Gamma^{com}_{D_{10}} = K_4 \cup 5K_1$. By using Definition 1.8 and Lemma 1.7, the vertex set and the edge set of $\Gamma^{com}_{D_{10}}$ are given respectively by $V(\Gamma^{com}_{D_{10}}) = \{a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$ and

$$E(\Gamma^{com}_{D_{10}}) = \big\{(a, a^2), (a, a^3), (a, a^4), (a^2, a^3), (a^2, a^4), (a^3, a^4)\big\}.$$

Therefore, $\Gamma^{com}_{D_{10}}$ is shown as in the following figure.

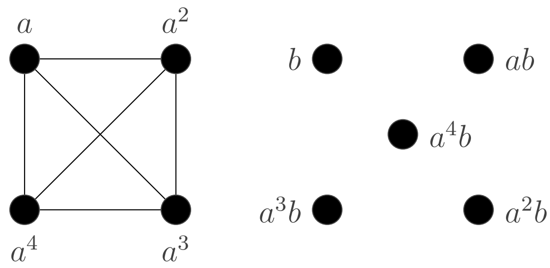

FIGURE 1. The CG of D_{10}

Now by Theorem 2.1, the RDP of $\Gamma^{com}_{D_{10}}$ is obtained by

$$
\begin{aligned}
R(\Gamma^{com}_{D_{10}}, x) =& x^5(1+x)^5\left[x^4 + \sum_{j=2}^{8}\left[\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor}\binom{4}{r}\binom{4-r}{j-2r}\right]x^j\right] \\
=& x^5(1+x)^5\left[x^4 + x^2\sum_{r=1}^{1}\binom{4}{r}\binom{4-r}{2-2r} + x^3\sum_{r=1}^{1}\binom{4}{r}\binom{4-r}{3-2r}\right. \\
& + x^4\sum_{r=1}^{2}\binom{4}{r}\binom{4-r}{4-2r} + x^5\sum_{r=1}^{2}\binom{4}{r}\binom{4-r}{5-2r} + x^6\sum_{r=1}^{3}\binom{4}{r}\binom{4-r}{6-2r} \\
& \left. + x^7\sum_{r=1}^{3}\binom{4}{r}\binom{4-r}{7-2r} + x^8\sum_{r=1}^{4}\binom{4}{r}\binom{4-r}{8-2r}\right] \\
=& x^5(1+x)^5\left[x^4 + 4x^2 + 12x^3 + (12+6)x^4 + (4+12)x^5 + (0+6+4)x^6\right. \\
& \left. + (0+0+4)x^7 + x^8\right] \\
=& x^7(1+x)^5\left[x^6 + 4x^5 + 10x^4 + 16x^3 + 19x^2 + 12x + 4\right] \\
=& \left(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5\right)\left(x^{13} + 4x^{12} + 10x^{11} + 16x^{10} + 19x^9\right. \\
& \left. + 12x^8 + 4x^7\right) \\
=& x^{18} + 9x^{17} + 40x^{16} + 116x^{15} + 244x^{14} + 388x^{13} + 468x^{12} + 420x^{11} + 271x^{10} \\
& + 119x^9 + 32x^8 + 4x^7.
\end{aligned}
$$

The polynomial is represented graphically as in Figure 2.

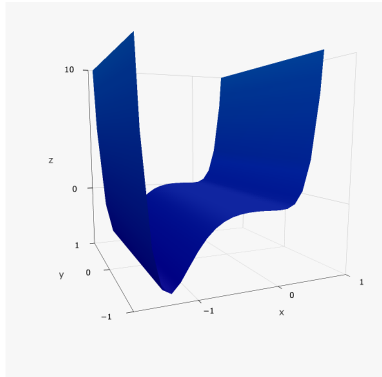

FIGURE 2. The graph of $R\big(\Gamma^{com}_{D_{10}}\big)$

The following theorem presents the RDP of the NCG associated to the dihedral groups D_{2n} when n is an odd integer.

Theorem 2.3. *Let $D_{2n} = \langle a, b : a^n = b^2 = baba = e\rangle$ be the dihedral group of order $2n$, where $n \geq 3$ is an odd integer, and $\Gamma^{ncom}_{D_{2n}}$ be its NCG. Then the RDP of $\Gamma^{ncom}_{D_{2n}}$ is given as:*

$$\begin{aligned}R(\Gamma^{ncom}_{D_{2n}}, x) = &\sum_{r=1}^{n}\binom{n}{r}\left[\sum_{t=0}^{n-r}\binom{n-r}{t}\left(\sum_{s=1}^{n-1}\binom{n-1}{s}\sum_{l=0}^{n-s-1}\binom{n-s-1}{l}\right)\right]x^{2r+t+2s+l}\\ &+ \big(1+x\big)^{n-1}\left[\sum_{j=2}^{2n}\left[\sum_{r=1}^{\lfloor \frac{j}{2}\rfloor}\binom{n}{r}\binom{n-r}{j-2r}\right]x^j\right]\\ &+ x^{n-1}\big(1+x\big)^n\left[\big(1+x\big)^{n-1}-1\right] + x^{2n-1}.\end{aligned}$$

Proof. By Lemma 1.11 part (1), the NCG of the dihedral groups D_{2n}, $\Gamma^{ncom}_{D_{2n}}$, is given as $\Gamma^{ncom}_{D_{2n}} = K_n \vee \overline{K_{n-1}}$. Therefore by Lemma 1.3, the RDP of $\Gamma^{ncom}_{D_{2n}}$ is given by:

$$\begin{aligned}R(\Gamma^{ncom}_{D_{2n}}, x) = &\sum_{r=1}^{n}\binom{n}{r}\left[\sum_{t=0}^{n-r}\binom{n-r}{t}\left(\sum_{s=1}^{n-1}\binom{n-1}{s}\sum_{l=0}^{n-s-1}\binom{n-s-1}{l}\right)\right]x^{2r+t+2s+l}\\ &+ \big(1+x\big)^{n-1}\Big(R(K_n, x) - x^n\Big) + \big(1+x\big)^n\Big(R(\overline{K_{n-1}}, x) - x^{n-1}\Big) + x^{n+n-1}.\end{aligned}$$

Hence by using Lemma 1.1 and Lemma 1.2, the RDP of $\Gamma^{ncom}_{D_{2n}}$ is presented as

$$
\begin{aligned}
R(\Gamma^{ncom}_{D_{2n}}, x) = &\sum_{r=1}^{n} \binom{n}{r} \left[\sum_{t=0}^{n-r} \binom{n-r}{t} \left(\sum_{s=1}^{n-1} \binom{n-1}{s} \sum_{l=0}^{n-s-1} \binom{n-s-1}{l} \right) \right] x^{2r+t+2s+l} \\
&+ (1+x)^{n-1} \left[\sum_{j=2}^{2n} \left[\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \binom{n}{r} \binom{n-r}{j-2r} \right] x^j \right] \\
&+ x^{n-1}(1+x)^n \left[(1+x)^{n-1} - 1 \right] + x^{2n-1},
\end{aligned}
$$

which completes the proof. □

Next, Example 2.4 illustrates Theorem 2.3.

Example 2.4. Let $D_6 = \{e, a, a^2, b, ab, a^2b\}$ be the dihedral group of order 6, where $n = 3$. By Lemma 1.11, the NCG of the dihedral groups D_6 is $\Gamma^{ncom}_{D_6} = K_3 \vee \overline{K_2}$. By using Definition 1.10 and Lemma 1.7, the set of vertices and the set of edges of $\Gamma^{ncom}_{D_6}$ are given respectively by $V(\Gamma^{ncom}_{D_6}) = \{a, a^2, b, ab, a^2b\}$ and

$$
E(\Gamma^{ncom}_{D_6}) = \big\{(b, ab), (b, a^2b), (b, a), (b, a^2), (ab, a^2b), (ab, a), (ab, a^2), (a^2b, a), (a^2b, a^2)\big\}.
$$

Therefore, $\Gamma^{ncom}_{D_6}$ is shown as in the following figure.

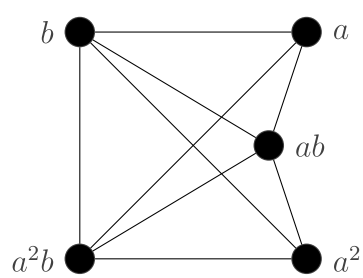

FIGURE 3. The NCG of D_6

Now by Theorem 2.3, the RDP of $\Gamma^{ncom}_{D_6}$ is obtained by

$$\begin{aligned}
R(\Gamma^{ncom}_{D_6},x) =& \sum_{r=1}^{3}\binom{3}{r}\Bigg[\sum_{t=0}^{3-r}\binom{3-r}{t}\Bigg(\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}\Bigg)\Bigg]x^{2r+t+2s+l}\\
&+(1+x)^2\Bigg[\sum_{j=2}^{6}\Bigg[\sum_{r=1}^{\lfloor\frac{j}{2}\rfloor}\binom{3}{r}\binom{3-r}{j-2r}\Bigg]x^j\Bigg]+x^2(1+x)^3\Big[(1+x)^2-1\Big]+x^5\\
=&3\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}x^{2+2s+l}+6\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}x^{3+2s+l}\\
&+6\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}x^{4+2s+l}+3\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}x^{5+2s+l}\\
&+\sum_{s=1}^{2}\binom{2}{s}\sum_{l=0}^{2-s}\binom{2-s}{l}x^{6+2s+l}+x^3(1+x)^3(x+2)\\
&+(1+x)^2\big[x^6+3x^5+6x^4+7x^3+3x^2\big]+x^5\\
=&6x^4+6x^5+3x^6+12x^5+12x^6+6x^7+12x^6+12x^7+6x^8\\
&+6x^7+6x^8+3x^9+2x^8+2x^9+x^{10}+(1+x)^2(x^5+3x^4+2x^3)\\
&+(1+x)^2(x^6+3x^5+6x^4+7x^3+3x^2)+x^5\\
=&x^{10}+5x^9+15x^8+30x^7+45x^6+50x^5+34x^4+15x^3+3x^2.
\end{aligned}$$

The polynomial is represented graphically as in Figure 4.

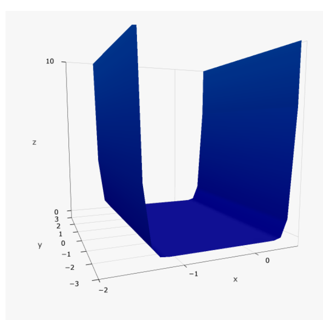

FIGURE 4. The graph of $R\big(\Gamma^{com}_{D_6}\big)$

Next, Theorem 2.5 presents the RDP of the NCG associated to the dihedral groups D_{2n} when n is an even integer.

Theorem 2.5. *Let $D_{2n}=\langle a,b:\ a^n=b^2=baba=e\rangle$ be the dihedral group of order $2n$, where $n\geq 3$ is an even integer, and $\Gamma^{ncom}_{D_{2n}}$ be its NCG. Then the RDP* *of* $\Gamma^{ncom}_{D_{2n}}$ *is given as:*

$$R(\Gamma^{ncom}_{D_{2n}},x)=\sum_{r=1}^{n}\binom{n}{r}\left[\sum_{t=0}^{n-r}\binom{n-r}{t}\left(\sum_{s=1}^{n-2}\binom{n-2}{s}\sum_{l=0}^{n-s-2}\binom{n-s-2}{l}\right)\right]x^{2r+t+2s+l}$$
$$+\left(1+x\right)^{n-2}\left[\sum_{j=3}^{2n}\left[\sum_{r=1}^{\lfloor\frac{j}{2}\rfloor}\binom{n}{r}\binom{n-r}{j-2r}\right]x^{j}\right]$$
$$+x^{n-2}\left(1+x\right)^{n}\left[\left(1+x\right)^{n-2}-1\right]+x^{2n-2}.$$

Proof. By Lemma 1.11 part (2), the NCG of the dihedral groups D_{2n}, $\Gamma^{ncom}_{D_{2n}}$, is given as $\Gamma^{ncom}_{D_{2n}}=U_n\vee\overline{K_{n-2}}$, where U_n is an $(n-2)$-regular graph on n vertices. Thus by Lemma 1.3, the RDP of $\Gamma^{ncom}_{D_{2n}}$ is given by:

$$R(\Gamma^{ncom}_{D_{2n}},x)=\sum_{r=1}^{n}\binom{n}{r}\left[\sum_{t=0}^{n-r}\binom{n-r}{t}\left(\sum_{s=1}^{n-2}\binom{n-2}{s}\sum_{l=0}^{n-s-2}\binom{n-s-2}{l}\right)\right]x^{2r+t+2s+l}$$
$$+\left(1+x\right)^{n-2}\left(R(U_n,x)-x^{n}\right)+\left(1+x\right)^{n}\left(R(\overline{K_{n-2}},x)-x^{n-2}\right)+x^{2n-2}.$$

By Lemma 1.1, $R(\overline{K_{n-2}},x)=x^{n-2}\left(1+x\right)^{n-2}$. Now, the remaining is the RDP of the graph U_n, $R(U_n,x)$. By Lemma 1.11 part (2), the graph U_n is an $(n-2)$-regular graph on n vertices. Therefore, $\gamma_R(U_n)=3$. Thus by the definition of the RDP of graph

$$R(\Gamma,x)=\sum_{j=\gamma_R(\Gamma)}^{2p}r(\Gamma,j)x^{j}$$

(recall that p is the number of vertices of a graph Γ), for any $3\leq j\leq 2n$ there are $\sum_{r=1}^{\lfloor\frac{j}{2}\rfloor}\binom{n}{r}\binom{n-r}{j-2r}$ Roman dominating function of the graph U_n with weight j, where r is the number of vertices of U_n which taking the value 2 under a Roman dominating function of U_n. Also, there is one more Roman dominating function of U_n with weight n when all the vertices of U_n taking the value 1. Hence,

$$R(U_n,x)=x^{n}+\sum_{j=3}^{2n}\left[\sum_{r=1}^{\lfloor\frac{j}{2}\rfloor}\binom{n}{r}\binom{n-r}{j-2r}\right]x^{j},$$

and the proof is completed. □

3. Conclusion

In this paper, the RDP of the CG and the NCG associated to the dihedral groups are obtained. Furthermore, some examples are given to illustrate the results with three dimension representation for the polynomials.

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