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FIXED POINT THEOREMS FOR GENERALIZED $\theta-\phi-$CONTRACTION MAPPINGS IN RECTANGULAR QUASI B-METRIC SPACES

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This paper is dedicated to Professor Abdelhamid Bourass

Abstract. A generalized version of both rectangular metric spaces and rectangular quasi-metric spaces is known as rectangular quasi-b-metric spaces (RQB-MS). In the current work, we define generalized (θ,ϕ)-contraction mappings and study fixed point (FP) results for the maps introduced in the setting of rectangular quasi-b-metric spaces. Our results generalize many existing results. We also provide examples in support of our main findings.

Keywords. Fixed point, rectangular quasi b-metric spaces, $\theta-\phi-$contraction
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1. Introduction

The Banach contraction principle is a basic result in fixed point theory (FPT) [2]. Due to its importance, various mathematicians studied many interesting extensions and generalizations (see [5, 12, 16, 18, 22]).

Numerous generalizations of the concept of metric spaces (MS) are defined, and some FPTs have been proved in these spaces. For instance, asymmetric MS was introduced by Wilson [23] as a generalization of MS. Many mathematicians worked on this interesting space (see also [20]). Branciari ([4]) seems to be the first to generalize MS in 2000. In the generalization, the triangle inequality is replaced by the quadrilateral inequality $d(x,y)\leq d(x,z)+d(z,u)+d(u,y)$ for all pairwise distinct points x,y,z and u. Any MS is a generalized MS, but in general, a generalized MS might not be a MS. Various FP results were established on such spaces (see [11, 13, 15] for more details).

The notion of b-rectangular MS has been introduced by the authors in [10], and many authors have investigated many existing FPTs in such spaces (see, e.g., [14, 19]). Bontu Nasir et al. in [9] introduced the notions of quasi-b-generalized MS. Any generalized MS but in general, RQB-MS might not be a generalized MS. The concept of $\theta-\phi-$contraction has been introduced by Zheng et al. in

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[24]. They also established some FP results for such mappings in complete MS and generalized the results of Kannan and Brower.

In the current paper, we introduce a new notion of generalized $\theta-\phi-$contraction and establish some results of FP for such mappings in complete rectangular quasi-b-metric. The results presented in the paper extend the corresponding results of Zheng et al.[24] and Banach [12] on RQB-MS. Also, we derive some useful corollaries of these results.

2. PRELIMINARIES

In this section, we give basic notions concerning a $\theta-\phi-$contraction in the setting of b-MS.

Definition 2.1. [9] Suppose a non-empty set $\mathcal{X}$ and $\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a mapping such that $\forall x_1, x_2 \in \mathcal{X}$ and $\forall$ distinct points $u, v \in \mathcal{X}$, each of them different from x_1 and x_2, one has
(i) $\rho(x_1, x_2) = 0$ if and only if $x_1 = x_2$;
(ii) $\rho(x_1, x_2) \leq s\left[\rho(x_1, u) + \rho(u, v) + \rho(v, x_2)\right]$. (b-rectangular inequality)
Then $(\mathcal{X}, \rho)$ is called an QRB-MS.

Definition 2.2. [9] Suppose a QRB-MS $(\mathcal{X}, \rho)$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}$, and $x \in \mathcal{X}$. Then
(i) The sequence $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) converges to x if and only if

$$\lim_{n\to+\infty} \rho\left(x, x_n\right) = \lim_{n\to+\infty} \rho\left(x_n, x\right) = 0.$$

(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) Cauchy if

$$\lim_{n,m\to+\infty} \rho\left(x_n, x_m\right) = \lim_{n,m\to+\infty} \rho\left(x_m, x_n\right) = 0.$$

Example 2.3. Define $\mathcal{X} := A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}, 2 \leq n \leq 7\}$ and $B = [1, 2]$. Define $\rho : \mathcal{X} \times \mathcal{X} \to [0, +\infty[$ as follows:

$$\left\{\rho(a, b) = \rho(b, a) \; for \; all \; a, b \in \mathcal{X}.\right.$$

and

$$\begin{cases} \rho\left(\frac{1}{2},\frac{1}{3}\right)=\rho\left(\frac{1}{4},\frac{1}{5}\right)=\rho\left(\frac{1}{6},\frac{1}{7}\right)=0,05 \\ \rho\left(\frac{1}{3},\frac{1}{2}\right)=\rho\left(\frac{1}{5},\frac{1}{4}\right)=\rho\left(\frac{1}{7},\frac{1}{6}\right)=0,04 \\ \rho\left(\frac{1}{2},\frac{1}{4}\right)=\rho\left(\frac{1}{3},\frac{1}{7}\right)=\rho\left(\frac{1}{5},\frac{1}{6}\right)=0,08 \\ \rho\left(\frac{1}{4},\frac{1}{2}\right)=\rho\left(\frac{1}{7},\frac{1}{3}\right)=\rho\left(\frac{1}{6},\frac{1}{5}\right)=0,05 \\ \rho\left(\frac{1}{2},\frac{1}{6}\right)=\rho\left(\frac{1}{3},\frac{1}{4}\right)=\rho\left(\frac{1}{5},\frac{1}{7}\right)=0,4 \\ \rho\left(\frac{1}{2},\frac{1}{5}\right)=\rho\left(\frac{1}{3},\frac{1}{6}\right)=\rho\left(\frac{1}{4},\frac{1}{7}\right)=0,24 \\ \rho\left(\frac{1}{2},\frac{1}{7}\right)=\rho\left(\frac{1}{3},\frac{1}{5}\right)=\rho\left(\frac{1}{4},\frac{1}{6}\right)=0,15 \\ \qquad \rho\left(a,b\right)=\left(\left|a-b\right|\right)^{2} \ \textit{otherwise.} \end{cases}$$

Then $(\mathcal{X},\rho)$ is a QRB-MS with coefficient $s=3$.

The following notion was introduced in [11].

Definition 2.4. [11]. Suppose Θ be the family of all increasing and continuous functions $\theta:]0,+\infty[\to]1,+\infty[$: For each sequence $(x_n)\subset]0,+\infty[$;

$$\lim_{n\to 0} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty}\theta\left(x_n\right)=1;$$

In [24]. Zheng et al. presented the concept of $\theta-\phi-$contraction on MS.

Definition 2.5. [24] Let Φ be the family of all nondecreasing and continuous functions ϕ: $[1,+\infty[\to [1,+\infty[$: For each $t\in]1,+\infty[$, $lim_{n\to\infty}\phi^n(t)=1$.

It should be remarked also that the authors in [24] proved the following nice results.

Lemma 2.6. [24] *If $\phi\in\Phi$. Then $\phi(t)<t$ for all $t\in]1,\infty[$ and $\phi(1)$=1.*

Definition 2.7. [24]. For a MS $(\mathcal{X},\rho)$ and a mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$.
$\mathcal{T}$ is called a $\theta-\phi-$contraction if there exist $\theta\in\Theta$ and $\phi\in\Phi$: for any $a,b\in\mathcal{X}$,

$$\rho\left(\mathcal{T}a,\mathcal{T}b\right)>0\Rightarrow\theta\left[\rho\left(\mathcal{T}a,\mathcal{T}b\right)\right]\leq\phi\left(\theta\left[N\left(a,b\right)\right]\right),$$

where

$$N\left(a,b\right)=\max\left\{\rho\left(a,b\right),\rho\left(a,\mathcal{T}a\right),\rho\left(b,\mathcal{T}b\right)\right\}.$$

Theorem 2.8. [24]. *For a complete MS $(\mathcal{X},d)$ and a $\theta-\phi-$contraction $\mathcal{T}:\mathcal{X}\to\mathcal{X}$. Then $\mathcal{X}$ has a unique FP.*

3. Main result

Lemma 3.1. *Suppose a QRB-MS $(\mathcal{X}, \eta)$. Let sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$. Then*

(a) *If x_n forward convergent to x and y_n backward convergent to y as $n \to \infty$, with $x \neq y$, $x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then we have*
$$\frac{1}{s}\eta\left(x, y\right) \leq \lim_{n\to\infty} \inf \eta\left(x_n, y_n\right) \leq \lim_{n\to\infty} \sup \eta\left(x_n, y_n\right).$$

(b) *If x_n backward convergent to x and y_n forward convergent to y as $n \to \infty$, with $x \neq y$, $x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then we have*
$$\frac{1}{s}\eta\left(y, x\right) \leq \lim_{n\to\infty} \inf \eta\left(y_n, x_n\right) \leq \lim_{n\to\infty} \sup \eta\left(y_n, x_n\right).$$

(c) *If $y \in \mathcal{X}$ and $\{x_n\}$ is a Cauchy sequence (CS) in $\mathcal{X}$ with $x_n \neq x_m$ for any $m, n \in \mathbb{N}$, $m \neq n$, converging to $x \neq y$, then*
$$\frac{1}{s}\eta\left(x, y\right) \leq \lim_{n\to\infty} \inf \eta\left(x_n, y\right) \leq \lim_{n\to\infty} \sup \eta\left(x_n, y\right).$$
for all $x \in \mathcal{X}$.

(d) *If $y \in \mathcal{X}$ and $\{x_n\}$ is a CS in $\mathcal{X}$ with $x_n \neq x_m$ for any $m, n \in \mathbb{N}$, $m \neq n$, converging to $x \neq y$, then*
$$\frac{1}{s}\eta\left(x, y\right) \leq \lim_{n\to\infty} \inf \eta\left(y, x_n\right) \leq \lim_{n\to\infty} \sup \eta\left(y, x_n\right).$$
for all $x \in \mathcal{X}$.

Proof. Using the b-rectangular inequality, it is easy to see that
$$\eta(x, y) \leq s\left[\eta(x, x_n) + s\eta(x_n; y_n) + s\eta(y_n; y)\right]$$
$$\eta(y, x) \leq s\left[\eta(y, y_n) + s\eta(y_n; x_n) + s\eta(x_n; x)\right],$$
taking the lower limit as $n \to \infty$ and the upper limit as $n \to \infty$, we obtain

(a)
$$\frac{1}{s}\eta\left(x, y\right) \leq \lim_{n\to\infty} \inf \eta\left(x_n, y_n\right) \leq \lim_{n\to\infty} \sup \eta\left(x_n, y_n\right),$$
and

(b)
$$\frac{1}{s}\eta\left(y, x\right) \leq \lim_{n\to\infty} \inf \eta\left(y_n, x_n\right) \leq \lim_{n\to\infty} \sup \eta\left(y_n, x_n\right).$$

If $y \in \mathcal{X}$, then, for infinitely many $m, n \in \mathbb{N}$,
$$\eta(x, y) \leq s\eta(x, x_n) + s\eta(x_n, x_m) + s\eta(x_m, y),$$
and
$$\eta(y, x) \leq s\eta(x_n, x) + s\eta(x_m, x_n) + s\eta(x_n, y).$$
Taking the lower limit as $n \to \infty$ and the upper limit as $n \to \infty$, we obtain

(c)
$$\frac{1}{s}\eta\left(x,y\right)\leq\lim_{n\to\infty}\inf\eta\left(x_n,y\right)\leq\lim_{n\to\infty}\sup\eta\left(x_n,y\right),$$
and

(d)
$$\frac{1}{s}\eta\left(x,y\right)\leq\lim_{n\to\infty}\inf\eta\left(y,x_n\right)\leq\lim_{n\to\infty}\sup\eta\left(y,x_n\right).$$

□

Lemma 3.2. *Suppose an QRB-MS $(\mathcal{X},\eta)$ and $\{x_n\}_n$ be a forward (or backward) CS with pairwise disjoint elements in $\mathcal{X}$. If $\{x_n\}_n$ forward converges to $x\in\mathcal{X}$ and backward converges to $y\in\mathcal{X}$, then $x=y$.*

Proof. Fix $\varepsilon>0$. First assume that $\{x_n\}_n$ is a forward CS, so there exists $n_1\in\mathbb{N}$: $\eta(x_n,x_m)<\frac{\varepsilon}{3s}$ for all $m\geq n\geq n_1$. Since x_n forward converges to x so there exists $n_2\in\mathbb{N}$: $\eta(x,x_n\leq\frac{\varepsilon}{3s}$. Also x_n backward converges to y there exists $n_3\in\mathbb{N}$: $\eta(x_n,x_m)<\frac{\varepsilon}{3s}$ for all $m\geq n\geq n_3$. Then for all $l\geq\max\{n_1,n_2,n_3\}$, $\eta(x,y)\leq s\left[\eta(x,x_n)+\eta(x_n,x_m)+\eta(x_m,y)\right]<s\frac{\varepsilon}{3s}+s\frac{\varepsilon}{3s}+s\frac{\varepsilon}{3s}=\varepsilon$. As $\varepsilon>0$ was arbitrary, we deduce that $\eta(x,y)=0$, which implies $x=y$. When $\{x_n\}_n$ is a backward CS, the proof is in similar fashion. □

Lemma 3.3. *Let $(\mathcal{X},\eta)$ be a RQB-MS and let $\{x_n\}$ be a sequence in $\mathcal{X}$:*
$$\lim_{n\to\infty}\eta\left(x_n,x_{n+1}\right)=\lim_{n\to\infty}\ \eta\left(x_n,x_{n+2}\right)=0, \tag{3.1}$$
and
$$\lim_{n\to\infty}\eta\left(x_{n+1},x_n\right)=\lim_{n\to\infty}\ \eta\left(x_{n+2},x_n\right)=0. \tag{3.2}$$
If $\{x_n\}$ is not a CS, then there exist $\varepsilon>0$ and two sequences $\{m_{(k)}\}$ and $\{n_{(k)}\}$ of positive integers:
$$\varepsilon\leq\lim_{k\to\infty}\inf\eta\left(x_{m_{(k)}},x_{n_{(k)}}\right)\leq\lim_{k\to\infty}\sup\eta\left(x_{m_{(k)}},x_{n_{(k)}}\right)\leq s\varepsilon,$$
$$\varepsilon\leq\lim_{k\to\infty}\inf\eta\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup\eta\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq s\varepsilon,$$
$$\varepsilon\leq\lim_{k\to\infty}\inf\eta\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup\eta\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\leq s\varepsilon,$$
$$\frac{\varepsilon}{s}\leq\lim_{k\to\infty}\inf\eta\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup\eta\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\leq s^2\varepsilon$$
$$\varepsilon\leq\lim_{k\to\infty}\inf\eta\left(x_{n_{(k)}},x_{m_{(k)}}\right)\leq\lim_{k\to\infty}\sup\eta\left(x_{n_{(k)}},x_{m_{(k)}}\right)\leq s\varepsilon,$$
$$\varepsilon\leq\lim_{k\to\infty}\inf\eta\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup\eta\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\leq s\varepsilon,$$
$$\varepsilon\leq\lim_{k\to\infty}\inf\eta\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup\eta\left(x_{n_{(k)}},x_{m_{(k)+1}}\right)\leq s\varepsilon,$$
$$\frac{\varepsilon}{s}\leq\lim_{k\to\infty}\inf\eta\left(x_{n_{(k)+1}},x_{m_{(k)+1}}\right)\leq\lim_{k\to\infty}\sup\eta\left(x_{n_{(k)+1}},x_{m_{(k)+1}}\right)\leq s^2\varepsilon.$$

Proof. If $\{x_n\}$ s not a CS, then there exist $\varepsilon > 0$ and two sequences $\{m_{(k)}\}$ and $\{n_{(k)}\}$ of positive integers:

$$m(k) > n(k) > k,\ \varepsilon \leq \eta\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\ ,\ \eta\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) < \varepsilon, \tag{3.3}$$

and

$$\varepsilon \leq \eta\left(x_{n_{(k)}}, x_{m_{(k)}}\right)\ ,\ \eta\left(x_{n_{(k)-1}}, x_{m_{(k)}}\right) < \varepsilon, \tag{3.4}$$

for all positive integers k. By the b-rectangular inequality, we have

$$\varepsilon \leq \eta\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\left[\eta\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + \eta\left(x_{m_{(k)+1}}, x_{m_{(k)-1}}\right) + \eta\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right)\right]. \tag{3.5}$$

Taking the upper and lower limits as $k \to \infty$ in (3.5), we obtain

$$\varepsilon \leq \lim_{k\to\infty} \inf \eta\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq \lim_{k\to\infty} \sup \eta\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\varepsilon. \tag{3.6}$$

Using the b-rectangular inequality again, we have

$$\varepsilon \leq \eta\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq s\left[\eta\left(x_{n_{(k)}}, x_{m_{(k)-1}}\right) + \eta\left(x_{m_{(k)-1}}, x_{m_{(k)}}\right) + \eta\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)\right]. \tag{3.7}$$

Taking the upper and lower limits as $k \to \infty$ in (3.7), we obtain

$$\varepsilon \leq \lim_{k\to\infty} \inf \eta\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup \eta\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq s\varepsilon. \tag{3.8}$$

Going the same way, we can easily prove the rest of the inequalities. $\square$

Definition 3.4. Let $(\mathcal{X}, \eta)$ be a QRB-MS with parameter $s > 1$ space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping. $\mathcal{T}$ is called a $\theta-$contraction if there exist $\theta \in \Theta$ and $r \in]0, 1[$:

$$\eta(\mathcal{T}x, \mathcal{T}y) > 0 \Rightarrow \theta\left[s^2\eta(\mathcal{T}x, \mathcal{T}y)\right] \leq \theta\left[\eta(x, y)\right]^r.$$

Theorem 3.5. *Let $(\mathcal{X}, \eta)$ be a complete QRB-MS and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be an θ-contraction, i.e, there exist $\theta \in \Theta$ and $r \in]0, 1[$: for any $x, y \in \mathcal{X}$, we have*

$$\eta(\mathcal{T}x, \mathcal{T}y) > 0 \Rightarrow \theta\left[s^2\eta(\mathcal{T}x, \mathcal{T}y)\right] \leq \theta\left[\eta(x, y)\right]^r. \tag{3.9}$$

Then $\mathcal{T}$ has a unique FP.

Proof. Let $x_0 \in \mathcal{X}$ be an arbitrary point in $\mathcal{X}$ and define a sequence $\{x_n\}$ by

$$x_{n+1} = \mathcal{T}x_n = \mathcal{T}^{n+1}x_0,$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\eta(x_{n_0}, x_{n_0+1}) = 0$. Then the proof is complete.
We can suppose that $\eta(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Substituting $x = x_{n-1}$ and $y = x_n$, from (3.9), for all $n \in \mathbb{N}$, we have

$$\theta\left[\eta(x_n, x_{n+1})\right] \leq \theta\left[s^3\eta(x_n, x_{n+1})\right] \leq \left[\theta(\eta(x_{n-1}, x_n))\right]^r, \forall n \in \mathbb{N} \tag{3.10}$$

Repeating this step, we conclude that

$$\theta(\eta(x_n, x_{n+1})) \leq (\theta(\eta(x_{n-1}, x_n)))^r \leq (\theta(\eta(x_{n-2}, x_{n-1})))^{r^2} \leq ... \leq \theta(\eta(x_0, x_1))^{r^n}.$$

From (3.3) and using (θ_1) we get

$$\eta(x_n, x_{n+1}) < \eta(x_{n-1}, x_n). \tag{3.11}$$

Therefore, $\eta\left(x_{n,}x_{n+1}\right)_{n\in\mathbb{N}}$ is monotone strictly decreasing sequence of non negative real numbers. Consequently, there exists $\alpha\geq 0$:

$$\lim_{n\to\infty}\eta\left(x_{n+1,}x_{n}\right)\quad=\alpha.$$

Now, we claim that $\alpha=0$. Arguing by contradiction, we assume that $\alpha>0$. Since $\eta\left(x_{n,}x_{n+1}\right)_{n\in\mathbb{N}}$ is a non negative decreasing sequence, then we have

$$\eta\left(x_{n,}x_{n+1}\right)\geq\alpha\quad\forall n\in\mathbb{N}.$$

By property of θ we get,

$$1<\theta\left(\alpha\right)\leq\theta\left(\eta\left(x_{0},x_{1}\right)\right)^{r^{n}}. \tag{3.12}$$

By letting $n\to\infty$ in inequality (3.12), we obtain

$$1<\theta\left(\alpha\right)\leq 1.$$

It is a contradiction. Therefore,

$$\lim_{n\to\infty}\eta\left(x_{n,}x_{n+1}\right)=0. \tag{3.13}$$

Substituting $x=x_{n-1}$ and $y=x_{n+1}$, from (3.9), for all $n\in\mathbb{N}$, we have

$$\theta\left[\eta\left(x_{n},x_{n+2}\right)\right]\leq\theta\left[s^{2}\eta\left(x_{n},x_{n+2}\right)\right]\leq\left[\theta\left(\eta\left(x_{n-1},x_{n}\right)\right)\right]^{r},\forall n\in\mathbb{N}. \tag{3.14}$$

Repeating this step, we conclude that

$$\theta\left(\eta\left(x_{n},x_{n+2}\right)\right)\leq\left(\theta\left(\eta\left(x_{n-1},x_{n+1}\right)\right)\right)^{r}\leq\left(\theta\left(\eta\left(x_{n-2},x_{n}\right)\right)\right)^{r^{2}}\leq...\leq\theta\left(\eta\left(x_{0},x_{2}\right)\right)^{r^{n}}.$$

From (3.14) and using (θ_{1}) we get

$$\eta\left(x_{n},x_{n+2}\right)<\eta\left(x_{n-1},x_{n+1}\right). \tag{3.15}$$

Therefore, $\eta\left(x_{n,}x_{n+2}\right)_{n\in\mathbb{N}}$ is monotone strictly decreasing sequence of non negative real

numbers. Consequently, there exists $\delta\geq 0$:

$$\lim_{n\to\infty}\eta\left(x_{n+1,}x_{n}\right)\quad=\delta.$$

Now, we claim that $\alpha=0$. Arguing by contradiction, we assume that $\delta>0$. Since $\eta\left(x_{n,}x_{n+2}\right)_{n\in\mathbb{N}}$ is a non negative decreasing sequence, then we have

$$\eta\left(x_{n,}x_{n+2}\right)\geq\delta\quad\forall n\in\mathbb{N}.$$

By property of θ we get,

$$1<\theta\left(\delta\right)\leq\theta\left(\eta\left(x_{0},x_{2}\right)\right)^{r^{n}}. \tag{3.16}$$

By letting $n\to\infty$ in inequality (3.5), we obtain

$$1<\theta\left(\delta\right)\leq 1.$$

It is a contradiction. Therefore,

$$\lim_{n\to\infty}\eta\left(x_{n,}x_{n+2}\right)=0. \tag{3.17}$$

Substituting $x=x_{n}$ and $y=x_{n-1}$, from (3.9), for all $n\in\mathbb{N}$, we have

$$\theta\left[\eta\left(x_{n+1},x_{n}\right)\right]\leq\theta\left[s^{2}\eta\left(x_{n+1},x_{n}\right)\right]\leq\left[\theta\left(\eta\left(x_{n},x_{n-1}\right)\right)\right]^{r},\forall n\in\mathbb{N}. \tag{3.18}$$

Repeating this step, we conclude that

$$\theta\left(\eta\left(x_n,x_{n+1}\right)\right)\leq\left(\theta\left(\eta\left(x_{n-1},x_n\right)\right)\right)^r\leq\left(\theta\left(\eta\left(x_{n-2},x_{n-1}\right)\right)\right)^{r^2}\leq...\leq\theta\left(\eta\left(x_0,x_1\right)\right)^{r^n}.$$

From (3.18) and using (θ_1) we get

$$\eta\left(x_{n+1},x_n\right)<\eta\left(x_n,x_{n-1}\right). \tag{3.19}$$

Therefore, $d\left(x_{n+1,}x_n\right)_{n\in\mathbb{N}}$ is monotone strictly decreasing sequence of non negative real numbers. Consequently, there exists $\lambda\geq 0$:

$$\lim_{n\to\infty}\eta\left(x_{n+1,}x_n\right) \quad =\lambda.$$

Now, we claim that $\lambda=0$. Arguing by contradiction, we assume that $\alpha>0$. Since $\eta\left(x_{n+1,}x_n\right)_{n\in\mathbb{N}}$ is a non negative decreasing sequence, then we have

$$\eta\left(x_{n+1,}x_n\right)\geq\alpha\quad\forall n\in\mathbb{N}.$$

By property of θ we get,

$$1<\theta\left(\lambda\right)\leq\theta\left(\eta\left(x_1,x_0\right)\right)^{r^n}. \tag{3.20}$$

By letting $n\to\infty$ in inequality (3.20), we obtain

$$1<\theta\left(\lambda\right)\leq 1.$$

It is a contradiction. Therefore,

$$\lim_{n\to\infty}\eta\left(x_{n+1,}x_n\right)=0. \tag{3.21}$$

Substituting $x=x_{n+1}$ and $y=x_{n-1}$, from (3.9), for all $n\in\mathbb{N}$, we have

$$\theta\left[\eta\left(x_{n+2},x_n\right)\right]\leq\theta\left[s^2\eta\left(x_{n+2},x_n\right)\right]\leq\left[\theta\left(\eta\left(x_n,x_{n-1}\right)\right)\right]^r,\forall n\in\mathbb{N}. \tag{3.22}$$

Repeating this step, we conclude that

$$\theta\left(\eta\left(x_{n+2},x_n\right)\right)\leq\left(\theta\left(\eta\left(x_{n+1},x_{n-1}\right)\right)\right)^r\leq\left(\theta\left(\eta\left(x_n,x_{n-2}\right)\right)\right)^{r^2}\leq...\leq\theta\left(\eta\left(x_2,x_0\right)\right)^{r^n}.$$

By (3.36) and using (θ_1) we get

$$\eta\left(x_{n+2},x_n\right)<\eta\left(x_{n+1},x_{n-1}\right). \tag{3.23}$$

Therefore, $d\left(x_{n+2,}x_n\right)_{n\in\mathbb{N}}$ is monotone strictly decreasing sequence of non negative real numbers. Consequently, there exists $\beta\geq 0$:

$$\lim_{n\to\infty}\eta\left(x_{n+2,}x_n\right) \quad =\beta.$$

Now, we claim that $\beta=0$. Arguing by contradiction, we assume that $\delta>0$. Since $\eta\left(x_{n+2,}x_n\right)_{n\in\mathbb{N}}$ is a non negative decreasing sequence, then we have

$$\eta\left(x_{n+2,}x_n\right)\geq\beta\quad\forall n\in\mathbb{N}.$$

By property of θ we get,

$$1<\theta\left(\beta\right)\leq\theta\left(\eta\left(x_2,x_0\right)\right)^{r^n}. \tag{3.24}$$

By letting $n\to\infty$ in inequality (3.24), we obtain

$$1<\theta\left(\beta\right)\leq 1.$$

It is a contradiction. Therefore,

$$\lim_{n\to\infty}\eta\left(x_{n+2},x_n\right)=0. \tag{3.25}$$

Firstly we show $\{x_n\}_{n\in\mathbb{N}}$ is a right-CS, if otherwise there exists an $\varepsilon>0$ for which we can find sequences of positive integers $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that, for all positive integers $k, n_k>m_k>k$. By Lemma (3.3), we have

$$\varepsilon \leq \lim_{k\to\infty}\inf \eta\left(x_{m_{(k)}},x_{n_{(k)}}\right) \leq \lim_{k\to\infty}\sup \eta\left(x_{m_{(k)}},x_{n_{(k)}}\right) \leq s\varepsilon,$$

$$\varepsilon \leq \lim_{k\to\infty}\inf \eta\left(x_{n_{(k)}},x_{m_{(k)+1}}\right) \leq \lim_{k\to\infty}\sup \eta\left(x_{n_{(k)}},x_{m_{(k)+1}}\right) \leq s\varepsilon,$$

$$\varepsilon \leq \lim_{k\to\infty}\inf \eta\left(x_{m_{(k)}},x_{n_{(k)+1}}\right) \leq \lim_{k\to\infty}\sup \eta\left(x_{m_{(k)}},x_{n_{(k)+1}}\right) \leq s\varepsilon,$$

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty}\inf \eta\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right) \leq \lim_{k\to\infty}\sup \eta\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right) \leq s^2\varepsilon$$

Applying (3.9) with $x=x_{m_{(k)}}$ and $y=x_{n_{(k)}}$, we obtain

$$\theta\left[s^2\eta\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\right] \leq \left[\theta\left(M\left(x_{m_{(k)}},x_{n_{(k)}}\right)\right)\right]^r. \tag{3.26}$$

Letting $k\to\infty$ the above inequality, applying the continuity of θ, we obtain

$$\theta\left(\frac{\varepsilon}{s}s^2\right)=\theta(\varepsilon s)\leq \theta\left(s^2\lim_{k\to\infty}\eta\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\right)\leq\left[\theta\left(\lim_{k\to\infty}\eta\left(x_{m_{(k)}},x_{n_{(k)}}\right)\right)\right]^r.$$

Therefore,

$$\theta(s\varepsilon)\leq[\theta(s\varepsilon)]^r<\theta(s\varepsilon).$$

Since θ is increasing, and continuous function, we get

$$s\varepsilon<s\varepsilon,$$

which is a contradiction. Then

$$\lim_{n,m\to\infty}\eta\left(x_m,x_n\right)=0.$$

Consequently $\{x_n\}$ is a right-Cauchy in $\mathcal{X}$. By completeness of $(\mathcal{X},\eta)$, there exists $z\in\mathcal{X}$:

$$\lim_{n\to\infty}\eta\left(x_n,z\right)=0.$$

Secondly we show $\{x_n\}_{n\in\mathbb{N}}$ is a left-CS, if otherwise there exists an $\varepsilon>0$ for which we can find sequences of positive integers $\{m_{(k)}\}$ and $\{n_{(k)}\}$ such that, for all positive integers $k, m_k>n_k>k$. By Lemma (3.3), we have

$$\varepsilon \leq \lim_{k\to\infty}\inf \eta\left(x_{n_{(k)}},x_{m_{(k)}}\right) \leq \lim_{k\to\infty}\sup \eta\left(x_{n_{(k)}},x_{m_{(k)}}\right) \leq s\varepsilon,$$

$$\varepsilon \leq \lim_{k\to\infty}\inf \eta\left(x_{m_{(k)}},x_{n_{(k)+1}}\right) \leq \lim_{k\to\infty}\sup \eta\left(x_{m_{(k)}},x_{n_{(k)+1}}\right) \leq s\varepsilon,$$

$$\varepsilon \leq \lim_{k\to\infty}\inf \eta\left(x_{n_{(k)}},x_{m_{(k)+1}}\right) \leq \lim_{k\to\infty}\sup \eta\left(x_{n_{(k)}},x_{m_{(k)+1}}\right) \leq s\varepsilon,$$

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty}\inf \eta\left(x_{n_{(k)+1}},x_{m_{(k)+1}}\right) \leq \lim_{k\to\infty}\sup \eta\left(x_{n_{(k)+1}},x_{m_{(k)+1}}\right) \leq s^2\varepsilon.$$

Applying (3.9) with $x=x_{n_{(k)}}$ and $y=x_{m_{(k)}}$, we obtain

$$\theta\left[s^2\eta\left(x_{n_{(k)+1}},x_{m_{(k)+1}}\right)\right] \leq \left[\theta\left(\eta\left(x_{n_{(k)}},x_{m_{(k)}}\right)\right)\right]^r. \tag{3.27}$$

Letting $k \to \infty$ the above inequality, applying the continuity of θ and using (3.27), we obtain

$$\theta\left(\frac{\varepsilon}{s}s^2\right) = \theta(\varepsilon s) \le \theta\left(s^2 \lim_{k\to\infty} \eta\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right)\right) \le \left[\theta\left(\lim_{k\to\infty} \eta\left(x_{n_{(k)}}, x_{m_{(k)}}\right)\right)\right]^r.$$

Therefore,

$$\theta(s\varepsilon) \le [\theta(s\varepsilon)]^r < \theta(s\varepsilon).$$

Since θ is increasing, we get

$$s\varepsilon < s\varepsilon,$$

which is a contradiction. Then

$$\lim_{n,m\to\infty} \eta(x_n, x_m) = 0.$$

Consequently $\{x_n\}$ is a left-Cauchy in $\mathcal{X}$. By completeness of $(\mathcal{X}, \eta)$, there exists $u \in \mathcal{X}$:

$$\lim_{n\to\infty} \eta(u, x_n) = 0.$$

By the lemma 3.2, we conclude that $z = u$. Now, we show that $\eta(\mathcal{T}z, z) = 0$ or $\eta(z, \mathcal{T}z) = 0$ arguing by contradiction, we assume that

$$\eta(\mathcal{T}z, z) > 0.$$

Since x_n forward converges to x as $n \to \infty$ for all $n \in \mathbf{N}$, then from Lemma (3.1), we conclude that

$$\frac{1}{s}\eta(z, \mathcal{T}z) \le \lim_{n\to\infty} \sup \eta(\mathcal{T}x_n, \mathcal{T}z) \le s\eta(z, \mathcal{T}z). \tag{3.28}$$

Now, applying (3.9) with $x = x_n$ and $y = z$, we have

$$\theta\left(s^2\eta(\mathcal{T}x_n, \mathcal{T}z)\right) \le [\theta(\eta(x_n, z))]^r, \ \forall n \in \mathbb{N}. \tag{3.29}$$

By letting $n \to \infty$ in inequality (3.29), using (3.28) and θ_3 we obtain

$$\begin{aligned}
\theta\left[s^2\frac{1}{s}\eta(z, \mathcal{T}z)\right] &= \theta[s\eta(z, \mathcal{T}z)] \\
&\le \theta\left[s^2 \lim_{n\to\infty} \eta(\mathcal{T}x_n, \mathcal{T}z)\right] \\
&\le [\theta(\eta(z, \mathcal{T}z))]^r \\
&< \theta(\eta(z, \mathcal{T}z)).
\end{aligned}$$

By (θ_1) and (θ_3), we get

$$s\eta(z, \mathcal{T}z) < \eta(z, \mathcal{T}z),$$

therefore

$$\eta(z, \mathcal{T}z)(s-1) < 0 \Rightarrow s < 1.$$

Which is a contradiction. Hence $\eta(\mathcal{T}z, z) = 0$. Next applying (3.9) with $x = z$ and $y = x_n$, we have

$$\theta\left(s^2\eta(\mathcal{T}z, \mathcal{T}x_n)\right) \le [\theta(\eta(z, x_n))]^r, \ \forall n \in \mathbb{N}, \tag{3.30}$$

Since x_n backward converges to x as $n\to\infty$ for all $n\in\mathbf{N}$, then from Lemma (3.1), we conclude that

$$\frac{1}{s}d\left(\mathcal{T}z,z\right)\leq\limsup_{n\to\infty}\eta\left(\mathcal{T}z,\mathcal{T}x_n\right)\leq s\eta\left(\mathcal{T}z,z\right). \tag{3.31}$$

By letting $n\to\infty$ in inequality (3.31), using (3.30) and θ_3 we obtain

$$\begin{aligned}\theta\left[s^2\frac{1}{s}\eta\left(\mathcal{T}z,z\right)\right]&=\theta\left[s\eta\left(\mathcal{T}z,z\right)\right]\\&\leq\theta\left[s^2\lim_{n\to\infty}\eta\left(\mathcal{T}z,\mathcal{T}x_n\right)\right]\\&\leq\left[\theta\left(\eta\left(\mathcal{T}z,z\right)\right)\right]^r\\&<\theta\left(\eta\left(\mathcal{T}z,z\right)\right).\end{aligned}$$

By (θ_1) and (θ_3), we get

$$s\eta(\mathcal{T}z,z)<\eta(\mathcal{T}z,z),$$

therefore

$$\eta(\mathcal{T}z,z)(s-1)<0\Rightarrow s<1.$$

Which is a contradiction. Hence $\eta\left(\mathcal{T}z,z\right)=0$.
To prove the uniqueness. Now, suppose that $z,u\in\mathcal{X}$ are two FPs of $\mathcal{T}$ such that $u\neq z$. Therefore, we have

$$\eta\left(z,u\right)=\eta\left(\mathcal{T}z,\mathcal{T}u\right)>0.$$

Applying (3.9) with $x=z$ and $y=u$, we have

$$\theta\left(\eta\left(z,u\right)\right)=\theta\left(\eta\left(\mathcal{T}u,\mathcal{T}z\right)\right)\leq\theta\left(s^2\eta\left(\mathcal{T}u,\mathcal{T}z\right)\right)\leq\left[\theta\left(\eta\left(z,u\right)\right)\right]^r$$
$$\theta\left(\eta\left(z,u\right)\right)\leq\left[\theta\left(\eta\left(z,u\right)\right)\right]^r<\theta\left(\eta\left(z,u\right)\right)$$

which implies that

$$\eta\left(z,u\right)<\eta\left(z,u\right),$$

which is a contradiction. Therefore $u=z$. □

Corollary 3.6. *Let $(\mathcal{X},\eta)$ be a complete QRB-MS and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be given mapping. Suppose that there exist $\theta\in\Theta$ and $k\in]0,1[$ such that for any $x,y\in\mathcal{X}$, we have*

$$\eta\left(\mathcal{T}x,\mathcal{T}y\right)>0\Rightarrow\left[s^2\eta\left(\mathcal{T}x,\mathcal{T}y\right)\right]\leq k\left[\left(\eta\left(x,y\right)\right)\right].$$

Then $\mathcal{T}$ has a unique FP.

Example 3.7. Let $\mathcal{X}=[1,2]$.
Define $\eta:\mathcal{X}\times\mathcal{X}\to[0,+\infty[$ by

$$\eta(x,y)=\begin{cases}(x-y)^2\ if\ x\geq y\\ \frac{1}{2}(y-x)^2\ if\ x<y.\end{cases}$$

Then $(\mathcal{X},\eta)$ is a QRB-MS with coefficient s=2. Define mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ by

$$\mathcal{T}(x)=\sqrt{x}.$$

Evidently, $\mathcal{T}(x) \in \mathcal{X}$. Let $\theta(t) = e^{\sqrt{t}}$, $r = \frac{1}{2}$. It obvious that $\theta \in \Theta$ and $r \in]0,1[$. Consider the following possibilities:

$1: x \geq y$. Then

$$\mathcal{T}(x) = x^{\frac{1}{4}},\ \mathcal{T}(y) = y^{\frac{1}{4}},\ \eta(\mathcal{T}x, \mathcal{T}y) = (x^{\frac{1}{4}} - y^{\frac{1}{4}})^2.$$

On the other hand

$$\theta\left[s^2\eta(\mathcal{T}x, \mathcal{T}y)\right] = e^{2(x^{\frac{1}{4}} - y^{\frac{1}{4}})}$$

and

$$\eta(x, y) = (x - y)^2.$$

On the other hand

$$[\theta(\eta(x, y))]^r = e^{\frac{1}{2}(x-y)}.$$

Since

$$\frac{1}{2}(x - y) = \frac{1}{2}(x^{\frac{1}{4}} - y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})$$

Since $x \geq 1$, so

$$2(x^{\frac{1}{4}} - y^{\frac{1}{4}}) \leq \frac{1}{2}(x^{\frac{1}{4}} - y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}}),$$

therefore

$$\theta(s^2\eta(\mathcal{T}x, \mathcal{T}y) \leq [\theta(\eta(x, \mathcal{T}x))]^r.$$

$2: x < y$. Then

$$\mathcal{T}(x) = x^{\frac{1}{4}},\ \mathcal{T}(y) = y^{\frac{1}{4}},\ \eta(\mathcal{T}x, \mathcal{T}y) = \frac{1}{2}(x^{\frac{1}{4}} - y^{\frac{1}{4}})^2.$$

On the other hand

$$\theta\left[s^2\eta(\mathcal{T}x, \mathcal{T}y)\right] = e^{\sqrt{2}(x^{\frac{1}{4}} - y^{\frac{1}{4}})},$$

and

$$\eta(x, y) = \frac{1}{2}(x - y)^2.$$

On the other hand

$$[\theta(\eta(x, y))]^r = e^{\frac{1}{2\sqrt{2}}(x-y)}.$$

Since

$$\frac{1}{2}(x - y) = \frac{1}{2}(x^{\frac{1}{4}} - y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})$$

Since $x \geq 1$, so

$$2(x^{\frac{1}{4}} - y^{\frac{1}{4}}) \leq \frac{1}{2}(x^{\frac{1}{4}} - y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}}),$$

therefore

$$\theta(s^2\eta(\mathcal{T}x, \mathcal{T}y) \leq [\theta(\eta(x, \mathcal{T}x))]^r.$$

Hence, the condition (3.9) is satisfied. Therefore, $\mathcal{T}$ has a unique FP $z = \frac{1}{3}$.

Theorem 3.8. *Let $(\mathcal{X},\eta)$ be a complete b-MS and let $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be an $5\theta-\phi$-contraction, i.e, there exist $\theta\in\Theta$ and $\phi\in\Phi$ such that for any $x,y\in\mathcal{X}$, we have*

$$\eta(\mathcal{T}x,\mathcal{T}y)>0\Rightarrow\theta\left[s^2\eta(\mathcal{T}x,\mathcal{T}y)\right]\le\phi\left[\theta\left(\eta(x,y)\right)\right]. \tag{3.32}$$

Then $\mathcal{T}$ has a unique FP.

Proof. The proof can easily sketched following the same scenario as in the proof of Theorem 3.5. ☐

Corollary 3.9. *Let $(\mathcal{X},\eta)$ be a complete QRB-MS and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be given mapping. Suppose that there exist $\theta\in\Theta$ and $k\in\,]0,1[$ such that for any $a_1,a_2\in\mathcal{X}$,*
we have

$$\eta(\mathcal{T}a_1,\mathcal{T}a_2)>0\Rightarrow\theta\left[s^2\eta(\mathcal{T}a_1,\mathcal{T}a_2)\right]\le\theta\left[(\eta(a_1,a_2))\right]^k.$$

Then $\mathcal{T}$ has a unique FP.

Example 3.10. Let $\mathcal{X}=A\cup B$, where $A=\{\frac{1}{n}:n\in\{3,4,5,6\}\}$ and $B=\left[\frac{1}{2},\frac{3}{2}\right]$. Define $\eta:\mathcal{X}\times\mathcal{X}\to[0,+\infty[$ as follows:

$$\begin{cases}\eta(a_1,a_2)=\eta(a_2,a_1)\ for\ all\ a_1,a_2\in\mathcal{X};\\ \eta(a_1,a_2)=0\Leftrightarrow a_2=a_1.\end{cases}$$

and

$$\begin{cases}\eta\left(\frac{1}{3},\frac{1}{4}\right)=\eta\left(\frac{1}{4},\frac{1}{5}\right)=0,1\\ \eta\left(\frac{1}{4},\frac{1}{3}\right)=\eta\left(\frac{1}{5},\frac{1}{4}\right)=0,05\\ \eta\left(\frac{1}{3},\frac{1}{5}\right)=\eta\left(\frac{1}{4},\frac{1}{6}\right)=0,05\\ \eta\left(\frac{1}{5},\frac{1}{3}\right)=\eta\left(\frac{1}{6},\frac{1}{4}\right)=0,1\\ \eta\left(\frac{1}{3},\frac{1}{6}\right)=\eta\left(\frac{1}{5},\frac{1}{6}\right)=0,5\\ \eta(a_1,a_2)=(|a_1-a_2|)^2\ otherwise.\end{cases}$$

Then $(\mathcal{X},\eta)$ is a b-rectangular MS with coefficient s=3. However we have the following:
1) $(\mathcal{X},\eta)$ is not a MS, as $\eta\left(\frac{1}{5},\frac{1}{6}\right)=0.5>0.15=\eta\left(\frac{1}{5},\frac{1}{4}\right)+\eta\left(\frac{1}{4},\frac{1}{6}\right)$.
2) $(\mathcal{X},\eta)$ is not a b-MS for s=3, as $\eta\left(\frac{1}{5},\frac{1}{6}\right)=0.5>0.45=3\left[\eta\left(\frac{1}{5},\frac{1}{4}\right)+\eta\left(\frac{1}{4},\frac{1}{6}\right)\right]$.
3) $(\mathcal{X},\eta)$ is not a rectangular MS, as $\eta\left(\frac{1}{5},\frac{1}{6}\right)=0.5>0.2=\eta\left(\frac{1}{5},\frac{1}{3}\right)+\eta\left(\frac{1}{3},\frac{1}{4}\right)+\eta\left(\frac{1}{4},\frac{1}{6}\right)$.

4) $(\mathcal{X},\eta)$ is not a b-rectangular MS, as $\eta\left(\frac{1}{3},\frac{1}{4}\right)\neq\eta\left(\frac{1}{4},\frac{1}{3}\right)$. Define mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ by

$$\mathcal{T}(a_1)=\begin{cases}\dfrac{\sqrt{a_1}+3}{4}\ if\ a_1\in\left[\dfrac{1}{2},\dfrac{3}{2}\right]\\ 1\ if\ a_1\in A.\end{cases}$$

Then, $\mathcal{T}(a_1)\in\left[\frac{1}{2},\frac{3}{2}\right]$. Let $\theta(t)=\sqrt{t}+1$, $\phi(t)=\frac{t+1}{2}$. It obvious that $\theta\in\Theta$ and $\phi\in\Phi$.
Consider the following possibilities:
case 1: $a_1,a_2\in\left[\frac{1}{2},\frac{3}{2}\right]$ with $a_1\neq a_2$, assume that $a_1>a_2$.

$$\eta(\mathcal{T}a_1,\mathcal{T}a_2)=\left(\frac{\sqrt{a_1}-\sqrt{a_2}}{4}\right)^2.$$

Therefore,

$$\theta(s^2\eta(\mathcal{T}a_1,\mathcal{T}a_2)=\frac{3}{4}\left(\sqrt{a_1}-\sqrt{a_2}\right)+1$$

and

$$\phi\left[\theta(\eta(a_1,a_2))\right]=\frac{a_1-a_2}{2}+1.$$

On the other hand

$$\begin{aligned}\theta(s^2\eta(\mathcal{T}a_1,\mathcal{T}a_2)-\phi\left[\theta(\eta(a_1,a_2))\right]&=\frac{3\left(\sqrt{a_1}-\sqrt{a_2}\right)-2(a_1-a_2)}{4}\\&=\frac{1}{4}\left(\left(\sqrt{a_1}-\sqrt{a_2}\right)\right)\left[3-2\left(\sqrt{a_1}+\sqrt{a_2}\right)\right].\end{aligned}$$

Since $a_1,a_2\in\left[\frac{1}{2},\frac{3}{2}\right]$, then

$$\frac{1}{4}\left(\left(\sqrt{a_1}-\sqrt{a_2}\right)\right)\left[3-2\left(\sqrt{a_1}+\sqrt{a_2}\right)\right]\leq 0.$$

Hence

$$\theta(s^2\eta(\mathcal{T}a_1,\mathcal{T}a_2)\leq\phi\left[\theta(\eta(a_1,a_2))\right].$$

case 2: $a_1\in\left[\frac{1}{2},\frac{3}{2}\right],a_2\in A$ or $a_2\in\left[\frac{1}{2},\frac{3}{2}\right],a_1\in A$.
Therefore, $\mathcal{T}(a_1)=\frac{\sqrt{a_1}+3}{4}$, $\mathcal{T}(a_2)=1$, then $\eta(\mathcal{T}a_1,\mathcal{T}a_2)=\left(\left|\frac{\sqrt{a_1}-1}{4}\right|\right)^2$
In this case consider two possibilities:
$i)$ $a_1\geq 1$: Then $\sqrt{a_1}\geq 1$. Therefore,

$$\eta(\mathcal{T}a_1,\mathcal{T}a_2)=\left(\frac{\sqrt{a_1}-1}{4}\right)^2.$$

So, we have

$$\theta(s^2\eta(\mathcal{T}a_1,\mathcal{T}a_2)=\frac{3}{4}\left(\sqrt{a_1}-1\right)+1.$$

Hence

$$\theta(s^2\eta(\mathcal{T}a_1,\mathcal{T}a_2)\leq\phi\left[\theta(\eta(a_1,a_2))\right].$$

$ii)$ $a_1 < 1$: Then $\sqrt{a_1} < 1$. Therefore,

$$\eta(\mathcal{T}a_1, \mathcal{T}a_2) = \left(|\frac{1-\sqrt{a_1}}{4}|\right)^2 = \left(\frac{1-\sqrt{a_1}}{4}\right)^2.$$

Since, $a_1 \in \left[\frac{1}{2}, 1\right]$, then

$$\theta(s^2\eta(\mathcal{T}a_1, \mathcal{T}a_2) \leq \phi\left[\theta(\eta(a_1, a_2))\right].$$

Hence, the condition (3.32) is satisfied. Therefore, $\mathcal{T}$ has a unique FP $z = 1$.

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