

SOME NEW RESULTS ON INTEGRAL DIVERGENCE

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ABSTRACT. This paper studies the divergence of certain simple and double integrals. In particular, a divergence result is established for simple integrals of submultiplicative functions. Several divergence integral results for double integrals involving functions of the sum and product of variables are also proved. One of these allows for the reconsideration of some existing theorems. Overall, the results provide novel techniques for evaluating divergent integrals that may be useful beyond the framework considered.

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1. INTRODUCTION

The basic goal of integral analysis is to determine whether integrals converge to a finite value, called integral convergence, or diverge to $\pm\infty$, called integral divergence. The classical approach is to study the behavior of the corresponding integrands over the integration domain using well-established methods. The most famous of these is the use of Riemann integrals, which state that

- the integral $\int_1^{+\infty} x^{-a} dx$ converges when $a > 1$, and diverges when $a \leq 1$,
- the integral $\int_0^1 x^{-a} dx$ converges when $a < 1$, and diverges when $a \geq 1$.

Applications of these convergence rules span several areas of mathematics and physics, including series summation, operator theory, probability theory, signal processing, engineering, and mechanics.

In this context, general results on integral divergence are crucial for defining the boundaries of certain mathematical techniques. In particular, identifying the conditions under which integrals diverge provides a better understanding of the behavior of integral inequalities and their limitations. In practice, understanding integral divergence is essential for designing robust algorithms, ensuring accurate computations, and maintaining the reliability of results. This knowledge also helps to assess the stability and convergence of numerical methods, which is

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particularly important when dealing with improper integrals or functions with singularities.

Although much has been achieved, the establishment of new integral divergence results remains a challenge which is addressed in this paper. It consists of two main parts. The first part is motivated by a particular functional assumption that appears in some integral inequality results: the submultiplicativity assumption. Some related references include [5], [6], [7] and [4]. However, there seems to be a gap in the knowledge of the integrability of submultiplicative functions. We partially fill this gap by establishing a divergence result for simple integrals of such functions. The second part is devoted to two divergence results for double integrals, which involve several functions of sums and products of variables. The notion of submultiplicativity is also considered in this setting. This allows to relativize the convergence of certain integrals whose forms are similar to those considered in Hilbert-type integral inequalities, as follows:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{h(x,y)} dx dy,$$

where f and g represent the functions of interest, and h serves as a kernel governing the interaction between the two variables (see [8]). In particular, one of our results re-discusses several theorems in [5] and [6], establishing Hilbert-type integral inequalities with "a priori finite upper bounds" for double integrals, which we prove to be divergent. The proofs are given in full detail, using original techniques that may be useful beyond the current framework.

The rest of the paper contains the following sections: Section 2 is devoted to a simple integral divergence result for submultiplicative functions. Section 3 deals with general double integral divergence results. The last section, Section 4, gives a conclusion, together with an extensive discussion of possible applications. An appendix contains the statement of two intermediate results.

2. A SIMPLE INTEGRAL DIVERGENCE RESULT

2.1. Submultiplicative functions. Let us now briefly introduce the notion of submultiplicativity applied to a nonzero function $f : [0, +\infty) \mapsto [0, +\infty)$. We say that f is submultiplicative if and only if, for any $x, y \in [0, +\infty)$, we have

$$f(xy) \leq f(x)f(y).$$

Some examples of submultiplicative functions are $f(x) = x^\alpha$ with $\alpha \in \mathbb{R}$, and, less trivially, $f(x) = \log(\beta + x)$ with $\beta \geq e$, $f(x) = 1/\tanh(\gamma x)$ with $\gamma > 0$, and $f(x) = x^\eta[1 + |\log(x)|]$ with $\eta \geq 1$.

A key property of submultiplicative functions is that the product of submultiplicative functions is itself submultiplicative. Consequently, the following functions, which are the products of two submultiplicative functions, are also submultiplicative: $f(x) = x^\alpha \log(\beta + x)$ with $\alpha \in \mathbb{R}$ and $\beta \geq e$, and $f(x) = x^\alpha / \tanh(\gamma x)$ with $\alpha \in \mathbb{R}$ and $\gamma > 0$, among many other possible examples. For more information on this notion, see [2], [1] and [3].

2.2. A divergence integral result. After reviewing the literature, it appears that the integral convergence behavior of submultiplicative functions is an understudied topic. We fill a gap with the proposition below, which establishes a divergence result for submultiplicative functions defined on $[0, +\infty)$.

Proposition 2.1. *Let $f : [0, +\infty) \mapsto [0, +\infty)$ be a nonzero function. If f is submultiplicative, then f is not integrable over $[0, +\infty)$, i.e., the following integral divergence holds:*

$$\int_0^{+\infty} f(x)dx = +\infty.$$

Proof of Proposition 2.1. We proceed by contradiction, assuming that f is integrable over $[0, +\infty)$, i.e., $\int_0^{+\infty} f(x)dx < +\infty$. Then, by doing the change of variables $x = \alpha y$ according to y with $\alpha \geq 0$, we have

$$\int_0^{+\infty} f(x)dx = \int_0^{+\infty} f(\alpha y)\alpha dy = \alpha \int_0^{+\infty} f(\alpha y)dy. \quad (2.1)$$

The submultiplicative property of f and $\alpha \geq 0$ give

$$\alpha \int_0^{+\infty} f(\alpha y)dy \leq \alpha \int_0^{+\infty} f(\alpha)f(y)dy = \alpha f(\alpha) \int_0^{+\infty} f(y)dy. \quad (2.2)$$

Combining Equations (2.1) and (2.2), and uniformizing the notation, we obtain

$$\int_0^{+\infty} f(x)dx \leq \alpha f(\alpha) \int_0^{+\infty} f(x)dx.$$

By simplification with the positive finite integral $\int_0^{+\infty} f(x)dx$, we get $\alpha f(\alpha) \geq 1$, so that

$$f(\alpha) \geq \frac{1}{\alpha}.$$

Integrating with respect to α and referring to a key divergent integral, we get

$$\int_0^{+\infty} f(\alpha)d\alpha \geq \int_0^{+\infty} \frac{1}{\alpha}d\alpha = +\infty,$$

which gives $\int_0^{+\infty} f(\alpha)d\alpha = +\infty$. This contradicts the integrability assumption made on f . So we conclude that $\int_0^{+\infty} f(x)dx = +\infty$. This completes the proof. $\square$

To the best of our knowledge, this is a new result in the literature. It allows a more thorough use of submultiplicative functions in various integral manipulations that can lead to divergent integrals. It can also be used to construct counterexamples that highlight the limitations of mathematical tools involving integrals.

Remark 2.2. As a consequence of Proposition 2.1, since the product of submultiplicative functions is itself submultiplicative, if $f, g : [0, +\infty) \mapsto [0, +\infty)$ are nonzero and submultiplicative, then we have

$$\int_0^{+\infty} f(x)g(x)dx = +\infty.$$

In particular, choosing the submultiplicative function $g(x) = x^\alpha$ with $\alpha \in \mathbb{R}$ leads to

$$\int_0^{+\infty} x^\alpha f(x) dx = +\infty.$$

The case $\alpha = 0$ corresponds to the statement of Proposition 2.1. In this sense, thanks to the flexibility of the notion of submultiplicativity, Proposition 2.1 has the ability to extend itself.

An interesting extension of this work can be to consider submultiplicative functions defined on a bounded interval, such as $[0, a]$ with $a > 0$, and to establish divergence integral results.

Some new general double integral divergence results are given in the rest of the paper.

3. SOME DOUBLE INTEGRAL DIVERGENCE RESULTS

3.1. A first general result. The proposition below deals with a divergence result of a double integral involving the composition of two functions defined on $[0, +\infty)$, one of which is not integrable.

Proposition 3.1. *Let $f, g : [0, +\infty) \mapsto [0, +\infty)$ be nonzero functions. If g is not integrable over $[0, +\infty)$, i.e.,*

$$\int_0^{+\infty} g(x) dx = +\infty,$$

then we have

$$\int_0^{+\infty} \int_0^{+\infty} f\left[\frac{x}{g(y)}\right] dx dy = +\infty.$$

Proof of Proposition 3.1. The change of variables $x = zg(y)$ according to z gives

$$\begin{aligned} \int_0^{+\infty} \int_0^{+\infty} f\left[\frac{x}{g(y)}\right] dx dy &= \int_0^{+\infty} \left\{ \int_0^{+\infty} f\left[\frac{x}{g(y)}\right] dx \right\} dy \\ &= \int_0^{+\infty} \left\{ \int_0^{+\infty} f(z)g(y) dz \right\} dy. \end{aligned} \quad (3.1)$$

Applying the Fubini-Tonelli integral theorem (recalled in the appendix), which is possible because of the positivity of the integrand, and using $\int_0^{+\infty} g(y) dy = +\infty$, we get

$$\int_0^{+\infty} \left\{ \int_0^{+\infty} f(z)g(y) dz \right\} dy = \left[\int_0^{+\infty} f(z) dz \right] \left[\int_0^{+\infty} g(y) dy \right] = +\infty. \quad (3.2)$$

It follows from Equations (3.1) and (3.2) that

$$\int_0^{+\infty} \int_0^{+\infty} f\left[\frac{x}{g(y)}\right] dx dy = +\infty.$$

This ends the proof. □

Proposition 3.1 thus establishes a simple divergence criterion for double integrals involving an integrand of a certain composed form. Since the class of simple integral divergent functions is infinite, this proposition can be used in many mathematical scenarios. Let us give just one example in relation to a previous result. Let $f, g : [0, +\infty) \mapsto [0, +\infty)$ be nonzero functions. If g is submultiplicative, then Proposition 2.1 implies that $\int_0^{+\infty} g(x)dx = +\infty$, and it follows from Proposition 3.1 that

$$\int_0^{+\infty} \int_0^{+\infty} f\left[\frac{x}{g(y)}\right] dxdy = +\infty.$$

To the best of our knowledge, this result is new.

3.2. Discussion on some existing theorems. In the context of Proposition 3.1, if we take $g(x) = x^\alpha$ with $\alpha \in \mathbb{R}$, then we have

$$\int_0^{+\infty} \int_0^{+\infty} f\left(\frac{x}{y^\alpha}\right) dxdy = +\infty.$$

In particular, if we choose $\alpha = -1$, we get

$$\int_0^{+\infty} \int_0^{+\infty} f(xy) dxdy = +\infty.$$

This simple consequence challenges several published theorems on technical upper bounds for double integrals, including those described below.

- We mention [5, Theorem 2.5]. This result proposes an upper bound for a double integral of the form $\int_0^{+\infty} \int_0^{+\infty} f(xy) dxdy$ with a given function f .
- We also identify several results in [6], and especially [6, Theorems 2.1, 2.2, 2.6, 2.7]. All these theorems give bounds for double integrals of the form $\int_0^{+\infty} \int_0^{+\infty} f(xy) dxdy$ with certain functions f .

With the claim that $\int_0^{+\infty} \int_0^{+\infty} f(xy) dxdy = +\infty$, the interest of these theorems seems limited. Given this, we suspect that there are gaps in the corresponding proofs, or that the upper bounds are not finite under the assumptions considered. For [5, Theorem 2.5], we can see that there are several gaps in its statement, including the fact that " g is non-increasing" is combined with the assumption " $g'(y) \geq 0$ for any $y \in [0, +\infty)$ ", which are incompatible, and also incompatible with " $g(0) = 0$ and $\lim_{z \rightarrow +\infty} g(z) = +\infty$ ". This may be the beginning of an explanation of the integral inequality obtained in this theorem.

We end this subsection by mentioning that Proposition 3.1 can be extended to higher dimension with little mathematical effort, the key point being a change of variables on only one variable depending on a non-integrable function.

3.3. Specific results. The proposition below gives a divergence result for double integrals involving functions of product and sum of variables. Such special forms are classical in Hilbert-type integral inequalities. We refer again to [8] on this topic.

Proposition 3.2. *Let $\mu \in (0, 1)$, $\nu = 1 - \mu$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be nonzero functions.*

(1) If g is non-increasing, then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu y^\nu)}{g(\mu x + \nu y)} dx dy = +\infty.$$

(2) If f is submultiplicative and g is non-increasing, then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu) f(y^\nu)}{g(\mu x + \nu y)} dx dy = +\infty.$$

Proof of Proposition 3.2. Let us prove the two points in turn. We will see that the point 2 is actually a consequence of the point 1.

(1) The standard Young inequality gives, for any $x, y \in [0, +\infty)$,

$$\mu x + \nu y \geq x^\mu y^\nu.$$

See [9] and also the appendix. The non-increasing assumption of g implies that

$$g(\mu x + \nu y) \leq g(x^\mu y^\nu).$$

Using this, the positivity of f and g , and integrating with respect to x and y , we get

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu y^\nu)}{g(\mu x + \nu y)} dx dy \geq \int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu y^\nu)}{g(x^\mu y^\nu)} dx dy. \quad (3.3)$$

We can write this last integral as

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu y^\nu)}{g(x^\mu y^\nu)} dx dy = \int_0^{+\infty} \int_0^{+\infty} h(xy^{\nu/\mu}) dx dy, \quad (3.4)$$

where

$$h(z) = \frac{f(z^\mu)}{g(z^\mu)}.$$

It follows from Proposition 3.1, applied with " h instead of f " and " $g(x) = x^{-\nu/\mu}$ satisfying $\int_0^{+\infty} g(x) dx = +\infty$ ", that

$$\int_0^{+\infty} \int_0^{+\infty} h(xy^{\nu/\mu}) dx dy = +\infty. \quad (3.5)$$

Combining Equations (3.3), (3.4) and (3.5), we get

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu y^\nu)}{g(\mu x + \nu y)} dx dy = +\infty.$$

(2) The submultiplicativity of f implies that, for any $x, y \in [0, +\infty)$,

$$f(x^\mu) f(y^\nu) \geq f(x^\mu y^\nu).$$

Using this, the positivity of f and g , and integrating with respect to x and y , we get

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu) f(y^\nu)}{g(\mu x + \nu y)} dx dy \geq \int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu y^\nu)}{g(\mu x + \nu y)} dx dy.$$

This, combined with the result of the point 1, i.e.,

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu y^\nu)}{g(\mu x + \nu y)} dx dy = +\infty,$$

gives

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu) f(y^\nu)}{g(\mu x + \nu y)} dx dy = +\infty.$$

The proof ends. $\square$

With this proposition, we have identified a large class of bivariate functions defined on $[0, +\infty)^2$ that are not integrable. This result can simplify or complement some studies aimed at bounding double integrals by directly showing their divergence for some specific integrands. It can also help in finding counterexamples to various double integral problems that arise in many mathematical scenarios. The use of the submultiplicativity assumption in this divergence integral context also adds a degree of originality.

Some examples of the use of Proposition 3.2 are given below.

- If we take $f(x) = e^{-x}$ and $g(x) = e^{-\theta\sqrt{x}}$ with $\theta > 0$, which is non-increasing, then the point 1 gives

$$\int_0^{+\infty} \int_0^{+\infty} e^{-x^\mu y^\nu + \theta\sqrt{\mu x + \nu y}} dx dy = \int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu y^\nu)}{g(\mu x + \nu y)} dx dy = +\infty.$$

- If we take $f(x) = e^{-x}$ and $g(x) = 1/\sqrt{x}$, which is non-increasing, then the point 1 gives

$$\int_0^{+\infty} \int_0^{+\infty} e^{-x^\mu y^\nu} \sqrt{\mu x + \nu y} dx dy = \int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu y^\nu)}{g(\mu x + \nu y)} dx dy = +\infty.$$

In particular, we have

$$\int_0^{+\infty} \int_0^{+\infty} e^{-\sqrt{xy}} \sqrt{x+y} dx dy = +\infty.$$

This divergence does not seem to be immediately obvious for computational software such as Maple or Mathematica, which supports the interest of our result.

- If we take $f(x) = 1/\tanh(x)$, which is submultiplicative and $g(x) = 1/\log(1+x)$, which is non-increasing, then the point 2 gives

$$\int_0^{+\infty} \int_0^{+\infty} \frac{\log(1 + \mu x + \nu y)}{\tanh(x^\mu) \tanh(y^\nu)} dx dy = \int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu) f(y^\nu)}{g(\mu x + \nu y)} dx dy = +\infty.$$

- If we take $f(x) = x^\alpha$ with $\alpha \in \mathbb{R}$, which is submultiplicative and $g(x) = 1/(1+x)$, which is non-increasing, then the point 2 gives

$$\int_0^{+\infty} \int_0^{+\infty} x^{\alpha\mu} y^{\alpha\nu} (1 + \mu x + \nu y) dx dy = \int_0^{+\infty} \int_0^{+\infty} \frac{f(x^\mu) f(y^\nu)}{g(\mu x + \nu y)} dx dy = +\infty.$$

Of course, in a precise setting such as these examples, the divergence can be demonstrated using other techniques. Proposition 3.2 has the merit of proposing a comprehensive and general framework.

An interesting extension of this proposition is to consider functions defined on a bounded interval, such as $[0, a]$ with $a > 0$.

4. CONCLUSION

The contributions of this paper were to give general and tractable divergence theorems for certain types of simple and double integrals. We also filled a gap by showing how the notion of submultiplicativity of a function can be used in this framework. We have also relativized the interest of some established theorems on integral inequalities by discussing that technical upper bounds are established for double integrals which are in fact infinite.

As outlined in the introduction, the potential applications of these results are vast. In functional analysis, they provide insights into the behavior of functions in integral inequalities that are fundamental to the study of operator theory and approximation methods. In numerical analysis, understanding the divergence of certain integrals helps to refine algorithms, ensuring more stable computations when dealing with singularities or improper integrals. Applications extend to mathematical physics, in particular quantum and statistical mechanics, where integrals of similar forms arise in the calculation of expectation values and propagators. Furthermore, the results are relevant to the study of partial differential equations, where Hilbert-type integral inequalities often appear in the solutions of boundary value problems. The techniques developed in this paper may also be applicable in signal processing, where understanding the divergence or convergence of integrals can provide information about the stability of signal reconstruction methods. All these applications will require further work and the necessary collaboration with experts.

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APPENDIX

In this appendix, we present the general statements of two fundamental results used in this paper, namely, the Fubini-Tonelli integral theorem and the standard Young inequality.

- **Fubini-Tonelli integral theorem.** Let $(X, \mathcal{A}, \phi)$ and $(Y, \mathcal{B}, \psi)$ be σ -finite measure spaces and $f : X \times Y \mapsto [0, +\infty)$ be a measurable function. Then the following equalities hold:

$$\begin{aligned} \int_{X \times Y} f(x, y) d(\phi \times \psi)(x, y) &= \int_X \left[\int_Y f(x, y) d\psi(y) \right] d\phi(x) \\ &= \int_Y \left[\int_X f(x, y) d\phi(x) \right] d\psi(y). \end{aligned}$$

The crux of this result is that the exchange of the order of integration holds true even if the integrals are divergent.

- **Standard Young inequality.** Let $a, b \geq 0$ and $p, q > 1$ be such that $1/p + 1/q = 1$. Then the following inequality holds:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The equality occurs if and only if $b = a^{p-1}$.

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