Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 27 (2025) 63-68 https://doi.org/10.56947/amcs.v27.454

ON PICARD AND MANN ITERATION CONVERGENCE FOR A CLASS OF QUASI-CONTRACTIVE OPERATORS

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Abstract. This paper observes that a definition adopted by some authors for comparing the rates of convergence of two iteration schemes is not correct. The paper further proves that the Picard iteration scheme converges faster than the Mann iteration scheme for contraction mappings and quasi-contraction mappings satisfying Zamfirescu conditions, in an arbitrary Banach space.

Keywords. Contraction mapping, Picard iteration scheme, Mann iteration scheme
2020 Mathematics Subject Classification. Primary 47H10, 54H25; Secondary 40A05

1. Introduction and preliminaries

Let ρ be a metric on a set X. A self-mapping on X, $T : X \longrightarrow X$ is called a (strict) *contraction* if there exists a constant $k \in [0, 1)$ such that $\rho(Tx, Ty) \leq k\rho(x, y)$ for all $x, y \in X$. The well-known Banach Contraction Principle is given by:

Theorem B (Banach Contraction Principle). Let (X, ρ) be a complete metric space and let $T : X \longrightarrow X$ be a contraction mapping with contraction constant $k \in [0, 1)$. Then there exists a unique $p \in X$ such that $p = Tp$.

The Banach Contraction Principle has played, and continues to play significant roles in the proofs of several existence results in Ordinary and Partial Differential Equations. Popular proofs of Theorem B rely on the application of the *Picard iteration scheme* defined by: $x_0 \in X$,

$$x_{n+1} = Tx_n, \ \ n = 0, 1, 2, ... \tag{1.1}$$

W.R. Mann [6] observed that if E denotes the circle plus its interior and T stands for a rotation of $\frac{\pi}{4}$ radians about the center of the circle then the Picard iteration scheme cannot be applied to approximate the unique fixed point of T. To overcome the limitations of the Picard iteration scheme for mappings such as

Date: Received: Dec 30, 2024; Accepted: Jan 20, 2025.
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a rotation of $\frac{\pi}{4}$ radians about the center of the circle, Mann [6] introduced the following iteration scheme: Let E be a real Banach space, let K be a nonempty convex subset of E. Let $x_1 \in K$,

$$x_{n+1} = T(\frac{1}{n}\sum_{j=1}^{n} x_j), \;\; n = 1, 2, 3, ... \tag{1.2}$$

G.G. Johnson [5], following the technique of [6], employed the following iteration scheme in his work: $z_0 \in K$,

$$z_{n+1} = \frac{1}{n+1}Tz_n + \frac{n}{n+1}z_n, \;\; n = 0, 1, 2, ... \tag{1.3}$$

The Mann-Johnson iterative scheme (1.3) is a special case of the iterative scheme: $z_0 \in K$,

$$z_{n+1} = \alpha_n Tz_n + (1-\alpha_n)z_n, \;\; n = 0, 1, 2, ..., \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1)$, which is now widely referred to as the *Mann iterative scheme*. Much research efforts have been devoted to the approximation of fixed points and solutions of nonlinear mappings through iterative schemes, from the time of Banach [1] to the present, see e.g., [8].

In [2], the following concept is adopted:

Definition 2.7 of [2]. "Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy

$$\|u_n - p\| \le a_n, \;\; \|v_n - p\| \le b_n, \;\;\; n = 0, 1, 2, ...,$$

respectively. If $\{a_n\}_{n=0}^{\infty}$ converges (to zero) faster than $\{b_n\}_{n=0}^{\infty}$, then it can be said that $\{u_n\}_{n=0}^{\infty}$ *converges faster* than $\{v_n\}_{n=0}^{\infty}$ to p."

Definition 2.7 of [2] is also adopted by the authors of [3]. One aim of this paper is to show that Definition 2.7 of [2] is not correct. Another aim of this paper is to provide proofs that the Picard iteration scheme converges faster than the Mann iteration scheme for contraction mappings and quasi-contraction mappings satisfying Zamfirescu conditions, in arbitrary Banach spaces.

Zamfirescu [9] introduced a class of mappings slightly more general than the contraction mappings and proved the following theorem.

Theorem Z [9]. Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a mapping for which there exist the real numbers λ, μ and γ satisfying $0 < \lambda < 1$, $0 < \mu, \gamma < \frac{1}{2}$, such that for each pair x, y in X, at least one of the following is true:

(z_1) $d(Tx, Ty) \le \lambda d(x, y)$;
(z_2) $d(Tx, Ty) \le \mu[d(x, Tx) + d(y, Ty)]$;
(z_3) $d(Tx, Ty) \le \gamma[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \;\; n = 0, 1, 2, ...,$$

converges to p, for any $x_0 \in X$.

The following correct definition is also recalled in [2].

Definition 1.1. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to real numbers a and b, respectively. Assume that $b_n = b$ for only a finite number of terms. Then if

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0,$$

the sequence $\{a_n\}_{n=0}^{\infty}$ is said to *converge faster* to a than $\{b_n\}_{n=0}^{\infty}$ is to b.

In the proofs of our results in Section 2, we shall employ Definition 1.1.

2. Main Results

2.1. **Observations on Definition 2.7 of [2]; A Counterexample.** Let $\{u_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be two sequences in a normed linear space X which converge to the same point $p \in X$. The following are possibilities for Definition 2.7 of [2]. Let $\|u_n - p\| = \dfrac{1}{n^{7/3}}$, $\|w_n - p\| = \dfrac{1}{n^{5/2}}$ and $a_n = \dfrac{1}{n^2}$, $b_n = \dfrac{1}{n^{4/3}}$ for each $n \in \mathbb{N}$. Then $\|u_n - p\| < a_n$ and $\|w_n - p\| < b_n$ for each $n \in \mathbb{N}$.

$$\frac{a_n}{b_n} = \frac{1}{n^{2/3}} \to 0 \text{ as } n \to \infty,$$

showing that the sequence of numbers, $\{a_n\}_{n=1}^{\infty}$ converges faster to zero than the sequence of numbers, $\{b_n\}_{n=1}^{\infty}$. However,

$$\frac{\|u_n - p\|}{\|w_n - p\|} = n^{1/6} \text{ for each } n \in \mathbb{N},$$

showing that the sequence $\{u_n\}_{n=1}^{\infty}$ does not converge faster to the point p than the sequence $\{w_n\}_{n=1}^{\infty}$. Thus, Definition 2.7 of [2] is not correct. The procedure which Rhoades [7] employed to compare the rates of convergence of two iteration schemes makes a better sense.

2.2. **Picard Iterates Converge Faster than Mann iterates.**

Theorem 2.1. *Let E be an arbitrary Banach space, let K be a closed convex subset of E and let $T : K \longrightarrow K$ be a contraction mapping with a contraction constant $k \in [0, 1)$. Let $x_0 \in K$, $\{x_n\}_{n=0}^{\infty}$ be the Picard iterates defined by scheme* (1.1) *and let $z_0 \in K$, $\{z_n\}_{n=0}^{\infty}$ be the Mann iterates defined by scheme* (1.4)*, where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0, 1)$ satisfying*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$.

Then

(a) *both the Picard and the Mann iterative schemes converge to the unique fixed point p of T;*

(b) *the Picard sequence* $\{x_n\}_{n=0}^{\infty}$ *converges faster to* p *than the Mann sequence* $\{z_n\}_{n=0}^{\infty}$.

Proof. Let E be an arbitrary Banach space, let K be a closed convex subset of E and let $T : K \longrightarrow K$ be a contraction mapping with a contraction constant $k \in [0, 1)$. By the Banach Contraction Principle, Theorem B, T has a unique fixed point, $p \in K$, say.

(a) The proof that the Picard sequence $\{x_n\}_{n=0}^{\infty}$ converges to p is well known, see e.g., [1] and [4]. We prove that the Mann sequence $\{z_n\}_{n=0}^{\infty}$ converges to p, as well. The proof that $\{z_n\}_{n=0}^{\infty}$ converges to p only requires the fact that $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1)$ and Condition (i). From the Mann scheme (1.4) we see that

$$\|z_{n+1} - p\| \leq \alpha_n \|Tz_n - p\| + (1 - \alpha_n)\|z_n - p\| \leq [1 - (1 - k)\alpha_n]\|z_n - p\|, \quad (2.1)$$

for all $n = 0, 1, 2, ...$, which yields that $\|z_{n+1} - p\| \leq \|z_n - p\|$. It follows that $\lim\limits_{n\to\infty} \|z_n - p\|$ exists. From inequality (2.1), we see that

$$(1 - k)\sum_{j=0}^{n-1} \alpha_j \|z_j - p\| \leq \|z_0 - p\| - \|z_n - p\| \leq \|z_0 - p\|$$

for all natural numbers, $n \geq 1$. Since by Condition (i), $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, it can be easily shown that $\liminf\limits_{n\to\infty} \|z_n - p\| = 0$. Hence, the Mann sequence $\{z_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point p, of T.

(b) Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $(0, 1)$ satisfying $\lim\limits_{n\to\infty} \alpha_n = 0$. From the Mann scheme (1.4),

$$\begin{aligned} \|z_{n+1} - p\| &= \|(1 - \alpha_n)(z_n - p) + \alpha_n(Tz_n - p)\| \\ &\geq (1 - \alpha_n)\|z_n - p\| - \alpha_n\|Tz_n - p\| \\ &\geq [1 - (1 + k)\alpha_n]\|z_n - p\| \qquad (2.2) \end{aligned}$$

for all $n = 0, 1, 2, ...$. There exists a natural number N_1 such that $1 - (1 + k)\alpha_n > 0$ for all $n \geq N_1$. Now, we compare the Picard iterates and the Mann iterates. From the Picard scheme (1.1), inequality (2.2) and the fact that T is a contraction map,

$$\frac{\|x_{n+1} - p\|}{\|z_{n+1} - p\|} \leq \frac{k}{1 - (1 + k)\alpha_n} \frac{\|x_n - p\|}{\|z_n - p\|} \; \forall\, n \geq N_1.$$

Choose a natural number $N_2 > N_1$ such that $0 < \alpha_n < \frac{1-k}{(1+k)^2}$ for all $n \geq N_2$. Then

$$\frac{\|x_{n+1} - p\|}{\|z_{n+1} - p\|} \leq \frac{k}{1 - (1 + k)\alpha_n} \frac{\|x_n - p\|}{\|z_n - p\|} \leq \frac{1 + k}{2} \frac{\|x_n - p\|}{\|z_n - p\|}$$

for all $n \geq N_2$. We observe that $l = \frac{1+k}{2} \in (0, 1)$ and that

$$\frac{\|x_{n+1} - p\|}{\|z_{n+1} - p\|} \leq l \frac{\|x_n - p\|}{\|z_n - p\|} \leq l^{n+1} \frac{\|x_0 - p\|}{\|z_0 - p\|}$$

for all $n \geq N_2$. Thus, $\lim_{n\to\infty} \frac{\|x_n - p\|}{\|z_n - p\|} = 0$. Hence, the Picard iteration scheme converges faster than the Mann iteration scheme. □

Theorem 2.2. *Let E be an arbitrary Banach space, let K be a closed convex subset of E and let $T : K \longrightarrow K$ be a Zamfirescu mapping, i.e., T satisfies at least one of the Conditions: $(z_1), (z_2)$ and (z_3) of Theorem Z. Let $x_0 \in K$, $\{x_n\}_{n=0}^{\infty}$ be the Picard iterates defined by scheme (1.1) and let $z_0 \in K$, $\{z_n\}_{n=0}^{\infty}$ be the Mann iterates defined by scheme (1.4), where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0,1)$ satisfying*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$.

Then

(a) *both the Picard and the Mann iterative schemes converge to the unique fixed point p of T;*

(b) *the Picard sequence $\{x_n\}_{n=0}^{\infty}$ converges faster to p than the Mann sequence $\{z_n\}_{n=0}^{\infty}$.*

Proof. Let $T : K \longrightarrow K$ be a Zamfirescu mapping. Then T satisfies at least one of the Conditions: (z_1), (z_2) and (z_3) of Theorem Z. Let $p \in K$ be the unique fixed point of T. For any $x \in K$, the inequality

$$\|Tx - p\| \leq \max\{\lambda, \frac{\mu}{1-\mu}, \frac{\gamma}{1-\gamma}\}\|x - p\|$$

holds and $k = \max\{\lambda, \frac{\mu}{1-\mu}, \frac{\gamma}{1-\gamma}\} \in (0,1)$. From this point on, the proof follows as in the proof of Theorem 2.1. □

3. Conclusion

In this paper, we have shown that the procedure adopted by Berinde [2] and some other authors, for comparing the rates of convergence of two iteration schemes is not adequate. We have also proved that the Picard iteration scheme converges faster than the Mann iteration scheme for contraction mappings and quasi-contraction mappings satisfying Zamfirescu conditions, in an arbitrary Banach space. We note that the sequence of original parameters, $\left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}$ in the Mann-Johnson iterative scheme (1.3) is a special case of the sequence of parameters $\{\alpha_n\}_{n=0}^{\infty}$ in our Theorems 2.1 and 2.2.

Acknowledgement. The research for this paper was concluded after the author's arrival at the University of Eswatini, Eswatini. The author is therefore grateful to the Vice Chancellor and staff of the University of Eswatini for the hospitality accorded him.

The author is also grateful to the anonymous referee, who read through the work thoroughly and made some helpful comments.

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