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ADDITIVITY OF MULTIPLICATIVE JORDAN HIGHER SEMI-DERIVATIONS ON RINGS AND STANDARD OPERATOR ALGEBRAS

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ABSTRACT. In this paper we study the additivity of multiplicative Jordan higher semi-derivations on rings and standard operator algebras.

1. INTRODUCTION

Recently, many mathematicians have devoted themselves to studying the additivity of maps. The first result about the additivity of multiplicative isomorphisms on rings was given by Martindale III [12]. Daif [6] introduced the definition of a multiplicative derivation of a ring and studied the additivity of such map. In the context of derivations, Lu [11] studied the additivity of multiplicative Jordan derivations on rings and Jing and Lu [10] generalized the results of Lu to a larger class of rings.

The concept of derivation was extended to higher derivation by Schmidt and Hasse [14]. Thus, in analogy to the concepts of multiplicative derivation and multiplicative Jordan derivation, on rings, defined in the literature [6, 10, 11], Ashraf and Parveen [2] introduced the notions of multiplicative higher derivation (originally called higher derivable map) and multiplicative Jordan higher derivation (originally called Jordan higher derivable map), of a ring, and they studied the additivity of the second class of these maps.

Bergen [3] introduced the notion of a semi-derivation of a ring, based on the concept of derivation, and Siddeeque and Khan [15] studied the additivity of multiplicative semi-derivation of a ring (a concept based on the notions of a semi-derivation and multiplicative derivation). Filippis et al. [8] introduced the notion of a Jordan semi-derivation of a ring, based on the concept of Jordan derivation, and Ferreira and Marietto [7] studied the additivity of multiplicative Jordan semi-derivation of a ring (a concept based on the notions of a Jordan semi-derivation and multiplicative Jordan derivation).

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Motivated by the facts mentioned above, in this paper we introduce the notions of a Jordan higher semi-derivation of a ring and multiplicative Jordan higher semi-derivation of a ring, based on the notions of a Jordan semi-derivation, Jordan (σ, τ)-higher derivations (introduced by Alkenani et al. [1]) and multiplicative Jordan higher derivation, and present a study on the additivity of the last class. We end the paper by presenting an application of obtained results on the standard operator algebras.

2. Multiplicative Jordan higher semi-derivations on rings

Let $\mathscr{R}$ be a ring and $g : \mathscr{R} \to \mathscr{R}$ an arbitrary map. We denote by $g^0 = id_{\mathscr{R}}$ (the identity map on $\mathscr{R}$) and $g^{n+1} = g^n \circ g$, for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\circ$ is the map composition operation.

A family $D = \{d_n\}_{n\in\mathbb{N}}$ of maps $d_n : \mathscr{R} \to \mathscr{R}$ is said to be a *multiplicative higher semi-derivation of $\mathscr{R}$ (associated with g)* (resp., *multiplicative Jordan higher semi-derivation of $\mathscr{R}$ (associated with g)*) if $d_0 = id_{\mathscr{R}}$,

$$d_n(ab) = \sum_{p+q=n} d_p(g^{n-p}(a))d_q(b) = \sum_{p+q=n} d_p(a)d_q(g^{n-q}(b)),$$

for all elements $a, b \in \mathscr{R}$ and for each $n \in \mathbb{N}$ (resp.,

$$\begin{aligned} d_n(ab + ba) &= \sum_{p+q=n} d_p(g^{n-p}(a))d_q(b) + d_p(g^{n-p}(b))d_q(a) \\ &= \sum_{p+q=n} d_p(a)d_q(g^{n-q}(b)) + d_p(b)d_q(g^{n-q}(a)), \end{aligned}$$

for all elements $a, b \in \mathscr{R}$ and for each $n \in \mathbb{N}$), and $d_n(g(a)) = g(d_n(a))$, for all element $a \in \mathscr{R}$ and for each integer $n \in \mathbb{N}$.

A multiplicative higher semi-derivation of $\mathscr{R}$ (associated with g) (resp., multiplicative Jordan higher semi-derivation of $\mathscr{R}$ (associated with g)) $D = \{d_n\}_{n\in\mathbb{N}}$ is said to be *additive* if d_n is an additive map, for each integer $n \in \mathbb{N}$. An additive multiplicative higher semi-derivation of $\mathscr{R}$ (associated with g) (resp., additive multiplicative Jordan higher semi-derivation of $\mathscr{R}$ (associated with g)) is called a *higher semi-derivation of $\mathscr{R}$ (associated with g)* (resp., *Jordan higher semi-derivation of $\mathscr{R}$ (associated with g)*).

The main result of this section reads as follows.

Theorem 2.1. *Let $\mathscr{R}$ be a ring containing a non-trivial idempotent e_1, $\mathscr{R} = \bigoplus_{i,j=1,2}\mathscr{R}_{ij}$ the Peirce decomposition of $\mathscr{R}$, relative to e_1, $g : \mathscr{R} \to \mathscr{R}$ an arbitrary endomorphism of $\mathscr{R}$ and t an element of $\mathscr{R}$. For each integer $n \in \mathbb{N}$, assume that the following property holds:*

$t = 0$ is the only element of $\mathscr{R}$ satisfying the identities

$$tr_{ij} + g^n(r_{ij})t = 0 \text{ and } tg^n(r_{ij}) + r_{ij}t = 0, \tag{♧}$$

for all $r_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$ resulting from the use of at least one of the five following cases, where each of them is composed of two subcases:

(1) *(a) $(i, j) = (1, 1)$ and (b) $(i, j) = (1, 2)$,*
(2) *(a) $(i, j) = (1, 1)$ and (b) $(i, j) = (2, 1)$,*
(3) *(a) $(i, j) = (2, 2)$ and (b) $(i, j) = (1, 2)$,*
(4) *(a) $(i, j) = (2, 2)$ and (b) $(i, j) = (2, 1)$,*
(5) *(a) $(i, j) = (1, 2)$ and (b) $(i, j) = (2, 1)$, provided the conditions:*

$$t = te_1 + g^n(e_1)t \text{ and } t = tg^n(e_1) + e_1t, \tag{2.1}$$

are satisfied.

Then every multiplicative Jordan higher semi-derivation of $\mathscr{R}$ (associated with g) is additive.

In addition, if $\mathscr{R}$ is a prime ring of characteristic different from 2 and $g : \mathscr{R} \to \mathscr{R}$ is a ring isomorphism of $\mathscr{R}$, then every multiplicative Jordan higher semi-derivation of $\mathscr{R}$ (associated with g) is a higher semi-derivation of $\mathscr{R}$ (associated with g).

We will prove the Theorem 2.1 using the Second Principle of Mathematical Induction.

Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a multiplicative Jordan higher semi-derivation of $\mathscr{R}$ (associated with g). First of all, we note that $\{g^n\}_{n\in\mathbb{N}}$ is a family of endomorphism of $\mathscr{R}$ (this basic fact will be used throughout this paper without explicit mention) and that d_0 and d_1 are also additive maps, by definition and [7, Theorem 1], respectively. Hence, suppose that for each positive integer n, each map d_k $(0 \leq k < n)$ belonging to D is additive. Based on the techniques used by Ferreira and Marietto [7], we shall organize the proof of Theorem 2.1 in a series of claims.

Claim 2.2. $d_n(0) = 0$.

Proof. By induction hypothesis on n, we have $d_k(0) = 0$, for each integer $0 \leq k < n$. Since $d_p(g^{n-p}(0))d_q(0)+d_p(g^{n-p}(0))d_q(0) = 0$ (resp., $d_p(0)d_q(g^{n-q}(0))+d_p(0)d_q(g^{n-q}(0)) = 0$), for each integers $p, q \in \mathbb{N}$ satisfying $p + q = n$, then

$$\begin{aligned} &d_n(0) = d_n(00 + 00) = \textstyle\sum_{p+q=n} d_p(g^{n-p}(0))d_q(0) + d_p(g^{n-p}(0))d_q(0) \\ &(\text{resp.}, \ = \textstyle\sum_{p+q=n} d_p(0)d_q(g^{n-q}(0)) + d_p(0)d_q(g^{n-q}(0)) = 0. \end{aligned}$$

□

The following well known result will be used throughout this paper: Let e_1 be a nontrivial idempotent of $\mathscr{R}$ and formally set $e_2 = 1_{\mathscr{R}} - e_1$ ($\mathscr{R}$ need not have an identity element $1_{\mathscr{R}}$). Then $\mathscr{R}$ has a Peirce decomposition $\mathscr{R} = \mathscr{R}_{11} \oplus \mathscr{R}_{12} \oplus \mathscr{R}_{21} \oplus \mathscr{R}_{22}$, relative to e_1, where $\mathscr{R}_{ij} = e_i\mathscr{R}e_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: $\mathscr{R}_{ij}\mathscr{R}_{kl} \subseteq \delta_{jk}\mathscr{R}_{il}$, where δ_{jk} is the *Kronecker delta function*. More details about the Peirce decomposition and its properties, can be found in references [9] and [13].

Claim 2.3. For all $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$, write $t = d_n(a_{11} + a_{12} + a_{21} + a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22})$. If

$$\begin{aligned} &d_n((a_{11} + a_{12} + a_{21} + a_{22})r_{ij} + r_{ij}(a_{11} + a_{12} + a_{21} + a_{22})) \\ &= d_n(a_{11}r_{ij} + r_{ij}a_{11}) + d_n(a_{12}r_{ij} + r_{ij}a_{12}) + d_n(a_{21}r_{ij} + r_{ij}a_{21}) \\ &+ d_n(a_{22}r_{ij} + r_{ij}a_{22}), \end{aligned} \tag{2.2}$$

for all $r_{ij} \in \mathscr{R}_{ij}$ resulting from the use of at least one of the five cases, then

$$tr_{ij} + g^n(r_{ij})t = 0 \text{ and } tg^n(r_{ij}) + r_{ij}t = 0,$$

for all $r_{ij} \in \mathscr{R}_{ij}$ satisfying this same case.

Proof. Let $r_{ij} \in \mathscr{R}_{ij}$ satisfying at least one of the five cases of Theorem 2.1. Note that by the definition of d_n, we have

$$d_n((a_{11} + a_{12} + a_{21} + a_{22})r_{ij} + r_{ij}(a_{11} + a_{12} + a_{21} + a_{22}))$$

$$= \sum_{p+q=n} d_p(g^{n-p}(a_{11}+a_{12}+a_{21}+a_{22}))d_q(r_{ij}) + d_p(g^{n-p}(r_{ij}))d_q(a_{11}+a_{12}+a_{21}+a_{22}) \tag{2.3}$$
$$= \sum_{p+q=n} d_p(a_{11}+a_{12}+a_{21}+a_{22})d_q(g^{n-q}(r_{ij})) + d_p(r_{ij})d_q(g^{n-q}(a_{11}+a_{12}+a_{21}+a_{22})) \tag{2.4}$$

and that

$$\begin{aligned} &d_n(a_{11}r_{ij}+r_{ij}a_{11}) + d_n(a_{12}r_{ij}+r_{ij}a_{12}) + d_n(a_{21}r_{ij}+r_{ij}a_{21}) \\ &+ d_n(a_{22}r_{ij}+r_{ij}a_{22}) \\ &= \sum_{p+q=n} d_p(g^{n-p}(a_{11}))d_q(r_{ij}) + d_p(g^{n-p}(r_{ij}))d_q(a_{11}) \\ &+ \sum_{p+q=n} d_p(g^{n-p}(a_{12}))d_q(r_{ij}) + d_p(g^{n-p}(r_{ij}))d_q(a_{12}) \\ &+ \sum_{p+q=n} d_p(g^{n-p}(a_{21}))d_q(r_{ij}) + d_p(g^{n-p}(r_{ij}))d_q(a_{21}) \\ &+ \sum_{p+q=n} d_p(g^{n-p}(a_{22}))d_q(r_{ij}) + d_p(g^{n-p}(r_{ij}))d_q(a_{22}) \qquad (2.5) \\ &= \sum_{p+q=n} d_p(a_{11})d_q(g^{n-q}(r_{ij})) + d_p(r_{ij})d_q(g^{n-q}(a_{11})) \\ &+ \sum_{p+q=n} d_p(a_{12})d_q(g^{n-q}(r_{ij})) + d_p(r_{ij})d_q(g^{n-q}(a_{12})) \\ &+ \sum_{p+q=n} d_p(a_{21})d_q(g^{n-q}(r_{ij})) + d_p(r_{ij})d_q(g^{n-q}(a_{21})) \\ &+ \sum_{p+q=n} d_p(a_{22})d_q(g^{n-q}(r_{ij})) + d_p(r_{ij})d_q(g^{n-q}(a_{22})). \qquad (2.6) \end{aligned}$$

Taking into account the hypothesis of the claim and subtracting (2.5) from (2.3) and (2.6) from (2.4), we get

$$\begin{aligned} &\sum_{p+q=n}(d_p(g^{n-p}(a_{11}+a_{12}+a_{21}+a_{22})) - d_p(g^{n-p}(a_{11})) - d_p(g^{n-p}(a_{12})) \\ &- d_p(g^{n-p}(a_{21})) - d_p(g^{n-p}(a_{22})))d_q(r_{ij}) + d_p(g^{n-p}(r_{ij}))(d_q(a_{11}+a_{12} \\ &+ a_{21}+a_{22}) - d_q(a_{11}) - d_q(a_{12}) - d_q(a_{21}) - d_q(a_{22})) = 0 \end{aligned}$$

and

$$\begin{aligned} &\sum_{p+q=n}(d_p(a_{11}+a_{12}+a_{21}+a_{22}) - d_p(a_{11}) - d_p(a_{12}) - d_p(a_{21}) \\ &- d_p(a_{22}))d_q(g^{n-q}(r_{ij})) + d_p(r_{ij})(d_q(g^{n-q}(a_{11}+a_{12}+a_{21}+a_{22})) \\ &- d_q(g^{n-q}(a_{11})) - d_q(g^{n-q}(a_{12})) - d_q(g^{n-q}(a_{21})) - d_q(g^{n-q}(a_{22}))) = 0, \end{aligned}$$

respectively, which lead to the following identities

$$\begin{aligned} &(d_n(a_{11}+a_{12}+a_{21}+a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22}))r_{ij} \\ &+ g^n(r_{ij})(d_n(a_{11}+a_{12}+a_{21}+a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) \\ &- d_n(a_{22})) = 0 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &(d_n(a_{11}+a_{12}+a_{21}+a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22}))g^n(r_{ij}) \\ &+ r_{ij}(d_n(a_{11}+a_{12}+a_{21}+a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22})) \\ &= 0, \end{aligned} \tag{2.8}$$

respectively. As a consequence of (2.7) and (2.8), it follows that

$$tr_{ij} + g^n(r_{ij})t = 0 \text{ and } tg^n(r_{ij}) + r_{ij}t = 0.$$

for such $r_{ij} \in \mathscr{R}_{ij}$. □

Claim 2.4. For all $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{11}+a_{12}) = d_n(a_{11}) + d_n(a_{12})$, (ii) $d_n(a_{11}+a_{21}) = d_n(a_{11}) + d_n(a_{21})$, (iii) $d_n(a_{12}+a_{22}) = d_n(a_{12}) + d_n(a_{22})$ and (iv) $d_n(a_{21}+a_{22}) = d_n(a_{21}) + d_n(a_{22})$.

Proof. Note that by direct verification, for all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) we get (in both subcases)

$$\begin{aligned}&d_n((a_{11}+a_{12})r_{ij} + r_{ij}(a_{11}+a_{12}))\\ &= d_n(a_{11}r_{ij} + r_{ij}a_{11}) + d_n(a_{12}r_{ij} + r_{ij}a_{12}).\end{aligned}$$

Thus, by Claim 2.3, we get the identities in (♧). It follows from property of Theorem 2.1 that $t = 0$. As a consequence, we have $d_n(a_{11}+a_{12}) - d_n(a_{11}) - d_n(a_{12}) = 0$. Also by direct verification, for all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (4) we get (in both subcases)

$$\begin{aligned}&d_n((a_{11}+a_{21})r_{ij} + r_{ij}(a_{11}+a_{21}))\\ &= d_n(a_{11}r_{ij} + r_{ij}a_{11}) + d_n(a_{21}r_{ij} + r_{ij}a_{21}).\end{aligned}$$

Thus, by Claim 2.3, we have the identities in ♧. According to the property of Theorem 2.1, we can deduce that $t = 0$. It follows that $d_n(a_{11}+a_{21}) - d_n(a_{11}) - d_n(a_{21}) = 0$.

The other cases can be proved similarly. □

Claim 2.5. For all $a_{12}, b_{12} \in \mathscr{R}_{12}$, $a_{21}, b_{21} \in \mathscr{R}_{21}$ and $t_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{12} + b_{12}t_{22}) = d_n(a_{12}) + d_n(b_{12}t_{22})$ and (ii) $d_n(a_{21} + t_{22}b_{21}) = d_n(a_{21}) + d_n(t_{22}b_{21})$.

Proof. In view of following identity

$$a_{12} + b_{12}t_{22} = (a_{12}+t_{22})(e_1+b_{12}) + (e_1+b_{12})(a_{12}+t_{22}),$$

of induction hypothesis on n and Claim 2.4(i) and (iii), we have

$$\begin{aligned}
&d_n(a_{12} + b_{12}t_{22})\\
&= d_n((a_{12}+t_{22})(e_1+b_{12}) + (e_1+b_{12})(a_{12}+t_{22}))\\
&= \textstyle\sum_{p+q=n} d_p(g^{n-p}(a_{12}+t_{22}))d_q(e_1+b_{12})\\
&\qquad + d_p(g^{n-p}(e_1+b_{12}))d_q(a_{12}+t_{22})\\
&= \textstyle\sum_{p+q=n}(d_p(g^{n-p}(a_{12})) + d_p(g^{n-p}(t_{22})))(d_q(e_1) + d_q(b_{12}))\\
&\qquad + (d_p(g^{n-p}(e_1)) + d_p(g^{n-p}(b_{12})))(d_q(a_{12}) + d_q(t_{22}))\\
&= \textstyle\sum_{p+q=n} d_p(g^{n-p}(a_{12}))d_q(e_1) + d_p(g^{n-p}(e_1))d_q(a_{12})\\
&+ \textstyle\sum_{p+q=n} d_p(g^{n-p}(t_{22}))d_q(e_1) + d_p(g^{n-p}(e_1))d_q(t_{22})\\
&+ \textstyle\sum_{p+q=n} d_p(g^{n-p}(a_{12}))d_q(b_{12}) + d_p(g^{n-p}(b_{12}))d_q(a_{12})\\
&+ \textstyle\sum_{p+q=n} d_p(g^{n-p}(t_{22}))d_q(b_{12}) + d_p(g^{n-p}(b_{12}))d_q(t_{22})\\
&= d_n(a_{12}e_1 + e_1a_{12}) + d_n(t_{22}e_1 + e_1t_{22})\\
&+ d_n(a_{12}b_{12} + b_{12}a_{12}) + d_n(t_{22}b_{12} + b_{12}t_{22})\\
&= d_n(a_{12}) + d_n(b_{12}t_{22}).
\end{aligned}$$

Similarly we prove that $d_n(a_{21} + t_{22}b_{21}) = d_n(a_{21}) + d_n(t_{22}b_{21})$, from identity

$$a_{21} + t_{22}b_{21} = (a_{21}+t_{22})(b_{21}+e_1) + (b_{21}+e_1)(a_{21}+t_{22}).$$

□

Claim 2.6. For all $a_{12}, b_{12} \in \mathscr{R}_{12}$ and $a_{21}, b_{21} \in \mathscr{R}_{21}$ the following hold: (i) $d_n(a_{12} + b_{12}) = d_n(a_{12}) + d_n(b_{12})$ and (ii) $d_n(a_{21} + b_{21}) = d_n(a_{21}) + d_n(b_{21})$.

Proof. It follows from direct verification that, for all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) and by Claims 2.2 and 2.5(i), we get (in both subcases)

$$\begin{aligned} &d_n((a_{12} + b_{12})r_{ij} + r_{ij}(a_{12} + b_{12})) \\ &= d_n(a_{12}r_{ij} + r_{ij}a_{12}) + d_n(b_{12}r_{ij} + r_{ij}b_{12}). \end{aligned}$$

Then, by Claim 2.3, we arrive at identities in (♣). This leads to $d_n(a_{12}+b_{12})-d_n(a_{12})-d_n(b_{12}) = 0$.

The other case can be proved similarly. □

Claim 2.7. For all $a_{11}, b_{11} \in \mathscr{R}_{11}$ and $a_{22}, b_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{11} + b_{11}) = d_n(a_{11}) + d_n(b_{11})$ and (ii) $d_n(a_{22} + b_{22}) = d_n(a_{22}) + d_n(b_{22})$.

Proof. For all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) and by Claims 2.2 and 2.6(i), we get (in both subcases)

$$\begin{aligned} &d_n((a_{11} + b_{11})r_{ij} + r_{ij}(a_{11} + b_{11})) \\ &= d_n(a_{11}r_{ij} + r_{ij}a_{11}) + d_n(b_{11}r_{ij} + r_{ij}b_{11}). \end{aligned}$$

Then, by Claim 2.3, we obtain the identities in (♣). Therefore, $d_n(a_{11}+b_{11})-d_n(a_{11})-d_n(b_{11}) = 0$.

The other case can be proved similarly. □

Claim 2.8. For all $a_{12} \in \mathscr{R}_{12}$ and $a_{21} \in \mathscr{R}_{21}$ the following holds: $d_n(a_{12} + a_{21}) = d_n(a_{12}) + d_n(a_{21})$.

Proof. First notice that the following identities hold:

$$\begin{aligned} a_{12} + a_{21} &= e_1(a_{12} + a_{21}) + (a_{12} + a_{21})e_1, \\ a_{12} &= e_1a_{12} + a_{12}e_1 \end{aligned}$$

and

$$a_{21} = e_1a_{21} + a_{21}e_1.$$

Thus, by the definition of d_n, we get

$$\begin{aligned} &d_n(a_{12} + a_{21}) = d_n(e_1(a_{12} + a_{21}) + (a_{12} + a_{21})e_1) \\ &= \textstyle\sum_{p+q=n} d_p(g^{n-p}(e_1))d_q(a_{12} + a_{21}) + d_p(g^{n-p}(a_{12} + a_{21}))d_q(e_1) \qquad (2.9) \\ &= \textstyle\sum_{p+q=n} d_p(e_1)d_q(g^{n-q}(a_{12} + a_{21})) + d_p(a_{12} + a_{21})d_q(g^{n-q}(e_1)), \qquad (2.10) \\ &d_n(a_{12}) = d_n(e_1a_{12} + a_{12}e_1) \\ &= \textstyle\sum_{p+q=n} d_p(g^{n-p}(e_1))d_q(a_{12}) + d_p(g^{n-p}(a_{12}))d_q(e_1) \qquad (2.11) \\ &= \textstyle\sum_{p+q=n} d_p(e_1)d_q(g^{n-q}(a_{12})) + d_p(a_{12}))d_q(g^{n-q}(e_1)) \qquad (2.12) \end{aligned}$$

and

$$\begin{aligned} &d_n(a_{21}) = d_n(e_1a_{21} + a_{21}e_1) \\ &= \textstyle\sum_{p+q=n} d_p(g^{n-p}(e_1))d_q(a_{21}) + d_p(g^{n-p}(a_{21}))d_q(e_1) \qquad (2.13) \\ &= \textstyle\sum_{p+q=n} d_p(e_1)d_q(g^{n-q}(a_{21})) + d_p(a_{21})d_q(g^{n-q}(e_1)), \qquad (2.14) \end{aligned}$$

respectively. Adding (2.11) and (2.13) and subtracting this sum from (2.9) and adding (2.12) and (2.14) and subtracting this sum from (2.10), we obtain the identities in (2.1). Next, for all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (5) and by Claim 2.2, we get (in both subcases)

$$\begin{aligned} &d_n((a_{12}+a_{21})r_{ij}+r_{ij}(a_{12}+a_{21})) \\ &= d_n(a_{12}r_{ij}+r_{ij}a_{12}) + d_n(a_{21}r_{ij}+r_{ij}a_{21}). \end{aligned}$$

Then, by Claim 2.3 we get the identities in (♧). This results that $d_n(a_{12}+a_{21})-d_n(a_{12})-d_n(a_{21}) = 0$. □

Claim 2.9. For all $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{11}+a_{12}+a_{22}) = d_n(a_{11})+d_n(a_{12})+d_n(a_{22})$ and (ii) $d_n(a_{11}+a_{21}+a_{22}) = d_n(a_{11})+d_n(a_{21})+d_n(a_{22})$.

Proof. For all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) and by Claims 2.4(iii) and 2.6(i), we get (in both subcases)

$$\begin{aligned} &d_n((a_{11}+a_{12}+a_{22})r_{ij}+r_{ij}(a_{11}+a_{12}+a_{22})) \\ &= d_n(a_{11}r_{ij}+r_{ij}a_{11}) + d_n(a_{12}r_{ij}+r_{ij}a_{12}) + d_n(a_{22}r_{ij}+r_{ij}a_{22}). \end{aligned}$$

Then, by Claim 2.3, we obtain the identities in (♧). Therefore, $d_n(a_{11}+a_{12}+a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{22}) = 0$.

The other case can be proved similarly. □

Claim 2.10. For all $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{11}+a_{12}+a_{21}) = d_n(a_{11})+d_n(a_{12})+d_n(a_{21})$ and (ii) $d_n(a_{12}+a_{21}+a_{22}) = d_n(a_{12})+d_n(a_{21})+d_n(a_{22})$.

Proof. For all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) and by Claims 2.8 and 2.9(i), we get (in both subcases)

$$\begin{aligned} &d_n((a_{11}+a_{12}+a_{21})r_{ij}+r_{ij}(a_{11}+a_{12}+a_{21})) \\ &= d_n(a_{11}r_{ij}+r_{ij}a_{11}) + d_n(a_{12}r_{ij}+r_{ij}a_{12}) + d_n(a_{21}r_{ij}+r_{ij}a_{21}). \end{aligned}$$

This results in the identities in (♧). Consequently, $d_n(a_{11}+a_{12}+a_{21})-d_n(a_{11})-d_n(a_{12})-d_n(a_{21}) = 0$.

The other case can be proved similarly. □

Claim 2.11. For all $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$ the following holds: $d_n(a_{11}+a_{12}+a_{21}+a_{22}) = d_n(a_{11})+d_n(a_{12})+d_n(a_{21})+d_n(a_{22})$.

Proof. For all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) and by Claims 2.6(i), 2.9(i) and 2.10(ii), we get (in both subcases)

$$\begin{aligned} &d_n((a_{11}+a_{12}+a_{21}+a_{22})r_{ij}+r_{ij}(a_{11}+a_{12}+a_{21}+a_{22})) \\ &= d_n(a_{11}r_{ij}+r_{ij}a_{11}) + d_n(a_{12}r_{ij}+r_{ij}a_{12}) \\ &+ d_n(a_{21}r_{ij}+r_{ij}a_{21}) + d_n(a_{22}r_{ij}+r_{ij}a_{22}). \end{aligned}$$

Then, we obtain the identities in (♧). Therefore, $d_n(a_{11}+a_{12}+a_{21}+a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22}) = 0$. □

Claim 2.12. d_n is an additive map.

Proof. The proof is a direct consequence of the Claims 2.6, 2.7 and 2.11. The details are omitted. □

Claim 2.13. D is additive.

Proof. Applying the Second Principle of Mathematical Induction on D, in view of Claim 2.12, we concluded that D is additive. □

To complete the proof of the Theorem 2.1 we will assume in addition that $\mathscr{R}$ is a prime ring of characteristic different from 2 and $g : \mathscr{R} \to \mathscr{R}$ is a ring isomorphism of $\mathscr{R}$.

Claim 2.14. D is a higher semi-derivation of $\mathscr{R}$ (associated with g).

Proof. First, note that according to the definition presented in [1, pp. 23], the family D is simultaneously a Jordan $(g, id_{\mathscr{R}})$-higher derivation of $\mathscr{R}$ and a Jordan $(id_{\mathscr{R}}, g)$-higher derivation of $\mathscr{R}$. Thus, by [1, Theorem 2.11], D is simultaneously a $(g, id_{\mathscr{R}})$-higher derivation of $\mathscr{R}$ and a $(id_{\mathscr{R}}, g)$-higher derivation of $\mathscr{R}$, respectively. These last two results show that D is a higher semi-derivation of $\mathscr{R}$ (associated with g). □

3. Multiplicative Jordan higher semi-derivations on standard operator algebras

Let $\mathscr{X}$ be a Banach space. We denote by $\mathscr{B}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$, and $\mathscr{F}(\mathscr{X})$ the ideal of all bounded finite rank operators in $\mathscr{B}(\mathscr{X})$. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{X})$ is called *prime* if $a\mathscr{A}b = 0$ implies $a = 0$ or $b = 0$. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{X})$ is called a *standard operator algebra* if $\mathscr{A}$ contain $\mathscr{F}(\mathscr{X})$. It is clear $\mathscr{B}(\mathscr{X})$ is a standard operator algebra.

Let $\mathscr{X}$ be a Banach space with $\dim \mathscr{X} \geq 2$ and $\mathscr{A} \subset \mathscr{B}(\mathscr{X})$ a standard operator algebra. It is well known that $\mathscr{B}(\mathscr{X})$ and $\mathscr{A}$ are prime rings and $\mathscr{A}$ contains a non-trivial idempotent e_1. Also, $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$ under the strong operator topology and, by [4, Corollary 7.3.], there exist a linear invertible operator of $\mathscr{B}(\mathscr{X})$, $y : \mathscr{X} \to \mathscr{X}$, such that $g(a) = yay^{-1}$, for all element $a \in \mathscr{A}$. We will write $e_2 = 1_{\mathscr{B}(\mathscr{X})} - e_1$, where $1_{\mathscr{B}(\mathscr{X})}$ is a multiplicative identity of $\mathscr{B}(\mathscr{X})$.

In order to understand the motivations of hypotheses of the theorem of this section, we will take into consideration the results of Chang [5, Theorem 1], that proved that all maps which are associated with arbitrary semi-derivations of a prime ring are always homomorphisms, of Brešar and Šemrl [4, Corollary 7.3.], that characterized the class of isomorphisms between standard operator algebras and of Ferreira and Marietto [7].

In light of this, we can state the following.

Theorem 3.1. *Let $\mathscr{X}$ be a Banach space with $\dim \mathscr{X} \geq 2$, $\mathscr{A} \subset \mathscr{B}(\mathscr{X})$ a standard operator algebra and $g : \mathscr{A} \to \mathscr{A}$ an isomorphism of $\mathscr{A}$. Then the following statement holds: every multiplicative Jordan higher semi-derivation of $\mathscr{A}$ (associated with g) is a higher semi-derivation of $\mathscr{A}$ (associated with g).*

Proof. The proof is based on the techniques used by Ferreira and Marietto [7].

Let $\mathscr{B}(\mathscr{X}) = \oplus_{i,j=1,2}\mathscr{B}(\mathscr{X})_{ij}$ and $\mathscr{A} = \oplus_{i,j=1,2}\mathscr{A}_{ij}$ be the Peirce decompositions of $\mathscr{B}(\mathscr{X})$ and $\mathscr{A}$, relative to e_1, respectively, and an arbitrary element $t \in \mathscr{A}$. For each integer $n \in \mathbb{N}$, we consider two cases, according as whether $n = 0$ or $n \geq 1$. Case 1: $n = 0$. If the identities in (♧) result from the use of the case (1), then

$$tr_{11} + r_{11}t = 0 \text{ for all } r_{11} \in \mathscr{A}_{11}, \text{ and } tr_{12} + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{A}_{12}. \tag{3.1}$$

In particular, taking $r_{11} = e_1$, in the first identity, in (3.1), we get $2t_{11} + t_{12} + t_{21} = 0$ which implies that $t_{11} = 0$, $t_{12} = 0$ and $t_{21} = 0$. As a consequence, the second identity, in (3.1), becomes $r_{12}t_{22} = 0$, for all $r_{12} \in \mathscr{A}_{12}$, which implies that $t_{22} = 0$, in view of the primeness of $\mathscr{A}$. Thus, we must have $t = 0$. If the identities in (♧) result from the use of the case (3), then

$$tr_{22} + r_{22}t = 0 \text{ for all } r_{22} \in \mathscr{A}_{22}, \text{ and } tr_{12} + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{A}_{12}.$$

Since $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$, the identities above lead us to

$$tr_{22} + r_{22}t = 0 \text{ for all } r_{22} \in \mathscr{B}(\mathscr{X})_{22}, \text{ and } tr_{12} + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{B}(\mathscr{X})_{12}. \tag{3.2}$$

Thus, taking $r_{22} = e_2$, in the first identity, in (3.2), we get $t_{12} + t_{21} + 2t_{22} = 0$ which shows that $t_{12} = 0$, $t_{21} = 0$ and $t_{22} = 0$. As a consequence, the second identity, in (3.2), becomes $t_{11}r_{12} = 0$, for all $r_{12} \in \mathscr{B}(\mathscr{X})_{12}$, which results in $t_{11} = 0$, in view of the primeness of $\mathscr{B}(\mathscr{X})$. Therefore $t = 0$. The cases (2) and (4) can be proved similarly. Now, if the identities in (♧) result from the use of the case (5), then the conditions in (2.1) imply that

$$t = te_1 + e_1t$$

which immediately implies that $t_{11} = 0$ and $t_{22} = 0$. Also, from the remaining properties of case (5), we have

$$tr_{12} + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{A}_{12}, \text{ and } tr_{21} + r_{21}t = 0, \text{ for all } r_{21} \in \mathscr{A}_{21},$$

which leads to $t_{12} = 0$ and $t_{21} = 0$. Hence it follows that $t = 0$. Therefore the Case 1 is proved. Case 2: $n \geq 1$. Consider $(y^{-1})^n = ((y^{-1})^n)_{11} + ((y^{-1})^n)_{12} + ((y^{-1})^n)_{21} + ((y^{-1})^n)_{22}$ and $y^n = (y^n)_{11} + (y^n)_{12} + (y^n)_{21} + (y^n)_{22}$ be the Peirce decompositions of $(y^{-1})^n$ and y^n, relative to e_1, respectively. If the identities in (♧) result from the use of the case (1), then

$$tr_{11} + g^n(r_{11})t = 0 \text{ and } tg^n(r_{11}) + r_{11}t = 0, \text{ for all } r_{11} \in \mathscr{A}_{11}, \tag{3.3}$$

and

$$tr_{12} + g^n(r_{12})t = 0 \text{ and } tg^n(r_{12}) + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{A}_{12}. \tag{3.4}$$

Taking $r_{11} = e_1$ and multiplying the first identity, in (3.3), on the left by $g^n(e_1)$ and on the right by e_1 we obtain that $g^n(e_1)te_1 = 0$. Also, multiplying the second identity, in (3.3), on the left by e_1 and on the right by $g^n(e_1)$ we obtain that $e_1tg^n(e_1) = 0$. These two last results allow to deduce that

$$(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{11} + t_{21}) = 0 \text{ and } (t_{11} + t_{12})((y^n)_{11} + (y^n)_{21}) = 0, \tag{3.5}$$

respectively. Also, multiplying the first identity, in (3.4), on the left by $g^n(r_{12})$ we get $g^n(r_{12})tr_{12} = 0$ and multiplying the second identity, in (3.4), on the right by $g^n(r_{12})$ we get $r_{12}tg^n(r_{12}) = 0$. As a consequence of these two last identities we obtain

$$r_{12}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{11} + t_{21})r_{12} = 0$$

and

$$r_{12}(t_{21}+t_{22})((y^n)_{11}+(y^n)_{21})r_{12}=0,$$

for all $r_{12}\in\mathscr{A}_{12}$, respectively. Since $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$, then we have

$$r_{12}(((y^{-1})^n)_{21}+((y^{-1})^n)_{22})(t_{11}+t_{21})r_{12}=0$$

and

$$r_{12}(t_{21}+t_{22})((y^n)_{11}+(y^n)_{21})r_{12}=0,$$

for all $r_{12}\in\mathscr{B}(\mathscr{X})_{12}$, respectively. Using the [10, Lemma 1.6(P4)], we obtain

$$(((y^{-1})^n)_{21}+((y^{-1})^n)_{22})(t_{11}+t_{21})=0 \text{ and } (t_{21}+t_{22})((y^n)_{11}+(y^n)_{21})=0. \tag{3.6}$$

Adding the first identity, in (3.5), to the first one, in (3.6), we obtain $(y^{-1})^n(t_{11}+t_{21})=0$ which leads to $t_{11}+t_{21}=0$. This results that $t_{11}=0$ and $t_{21}=0$. Next, multiplying the second identity, in (3.3), on the right by $g^n(r_{22})$ and taking $r_{11}=e_1$ we get $e_1tg^n(r_{22})=0$. As a consequence of this we have

$$t_{12}((y^n)_{12}+(y^n)_{22})r_{22}=0,$$

for all $r_{22}\in\mathscr{A}_{22}$, which allows to conclude that

$$t_{12}((y^n)_{12}+(y^n)_{22})r_{22}=0,$$

for all $r_{22}\in\mathscr{B}(\mathscr{X})_{22}$, in view of the density of $\mathscr{A}$ in $\mathscr{B}(\mathscr{X})$. This leads to the conclusion that

$$t_{12}((y^n)_{12}+(y^n)_{22})=0. \tag{3.7}$$

Adding the second identity, in (3.5), to the identity one, in (3.7), we have $t_{12}y^n=0$ which results in $t_{12}=0$. Thus, taking $r_{11}=e_1$ in the first identity, in (3.3), we get $g^n(e_1)t_{22}=0$ and the first identity, in (3.4), becomes $g^n(r_{12})t_{22}=0$, for all $r_{12}\in\mathscr{A}_{12}$. As a consequence of these last two results, we obtain

$$(((y^{-1})^n)_{11}+((y^{-1})^n)_{12})t_{22}=0 \text{ and } r_{12}(((y^{-1})^n)_{21}+((y^{-1})^n)_{22})t_{22}=0,$$

for all $r_{12}\in\mathscr{A}_{12}$, respectively, which implies that

$$(((y^{-1})^n)_{11}+((y^{-1})^n)_{12})t_{22}=0 \text{ and } r_{12}(((y^{-1})^n)_{21}+((y^{-1})^n)_{22})t_{22}=0,$$

for all $r_{12}\in\mathscr{B}(\mathscr{X})_{12}$. From this we deduce that

$$(((y^{-1})^n)_{11}+((y^{-1})^n)_{12})t_{22}=0 \text{ and } (((y^{-1})^n)_{21}+((y^{-1})^n)_{22})t_{22}=0,$$

in view of primeness of $\mathscr{B}(\mathscr{X})$. Adding the last two identities we obtain $(y^{-1})^nt_{22}=0$ which leads to $t_{22}=0$. Consequently, we have $t=0$. Now, if the identities in (♧) result from the use of the case (3), then

$$tr_{22}+g^n(r_{22})t=0 \text{ and } tg^n(r_{22})+r_{22}t=0, \text{ for all } r_{22}\in\mathscr{A}_{22}, \tag{3.8}$$

and

$$tr_{12}+g^n(r_{12})t=0 \text{ and } tg^n(r_{12})+r_{12}t=0, \text{ for all } r_{12}\in\mathscr{A}_{12}. \tag{3.9}$$

Note that the identities, in (3.8), are equivalent to

$$(y^{-1})^ntr_{22}+r_{22}(y^{-1})^nt=0 \text{ and } ty^nr_{22}+r_{22}ty^n=0, \text{ for all } r_{22}\in\mathscr{A}_{22},$$

respectively, that lead to identities

$$(y^{-1})^n t r_{22} + r_{22}(y^{-1})^n t = 0 \text{ and } t y^n r_{22} + r_{22} t y^n = 0, \text{ for all } r_{22} \in \mathscr{B}(\mathscr{X})_{22}. \quad (3.10)$$

Thus, multiplying both identities, in (3.10), from the left and from the right by e_2 we get

$$(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{12} + t_{22})r_{22} + r_{22}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{12} + t_{22}) = 0$$

and

$$(t_{21} + t_{22})((y^n)_{12} + (y^n)_{22})r_{22} + r_{22}(t_{21} + t_{22})((y^n)_{12} + (y^n)_{22}) = 0,$$

for all $r_{22} \in \mathscr{B}(\mathscr{X})_{22}$. This results that

$$(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{12} + t_{22}) = 0 \text{ and } (t_{21} + t_{22})((y^n)_{12} + (y^n)_{22}) = 0. \quad (3.11)$$

Next, multiplying the first identity, in (3.10), on the left by e_1 and the second identity, in (3.10), on the right by e_1, we get

$$(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{12} + t_{22})r_{22} = 0 \text{ and } r_{22}(t_{21} + t_{22})((y^n)_{11} + (y^n)_{21}) = 0,$$

$r_{22} \in \mathscr{B}(\mathscr{X})_{22}$, respectively, which allows to conclude that

$$(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{12} + t_{22}) = 0 \text{ and } (t_{21} + t_{22})((y^n)_{11} + (y^n)_{21}) = 0. \quad (3.12)$$

Adding the corresponding identities, in (3.11) and (3.12), we obtain $(y^{-1})^n(t_{12}+t_{22}) = 0$ and $(t_{21}+t_{22})y^n = 0$, respectively. It results from this that $t_{12}+t_{22} = 0$ and $t_{21}+t_{22} = 0$ which leads to $t_{12} = 0$, $t_{21} = 0$ and $t_{22} = 0$. Now, multiplying the first identity, in (3.9), on the left by $g^n(r_{12})$ we obtain $g^n(r_{12})tr_{12} = 0$. Also, multiplying the same identity on the left by $g^n(e_1)$ and on right by e_2 we obtain $g^n(e_1)tr_{12} + g^n(r_{12})te_2 = 0$. This results in the identities

$$r_{12}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})t_{11}r_{12} = 0 \text{ and } (((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})t_{11}r_{12} = 0,$$

for all $r_{12} \in \mathscr{A}_{12}$, respectively, which leads to

$$r_{12}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})t_{11}r_{12} = 0 \text{ and } (((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})t_{11}r_{12} = 0,$$

for all $r_{12} \in \mathscr{B}(\mathscr{X})_{12}$. Therefore,

$$(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})t_{11} = 0 \text{ and } (((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})t_{11} = 0, \quad (3.13)$$

Adding the two identities, in (3.13), we have $(y^{-1})^n t_{11} = 0$ which yields $t_{11} = 0$. Consequently, $t = 0$.

Using similar arguments to the previous two cases, we prove that if the identities in (♧) result from the use of at least one of the cases (2) and (4), then $t = 0$.

Finally, if the identities in (♧) result from the use of the case (5), then the conditions in (2.1) imply that

$$(y^{-1})^n t = (y^{-1})^n t e_1 + e_1 (y^{-1})^n t \text{ and } t y^n = t y^n e_1 + e_1 t y^n, \quad (3.14)$$

respectively. Also, replacing $e_1 = 1_{\mathscr{B}(\mathscr{X})} - e_2$ in both conditions in (3.14) we get

$$(y^{-1})^n t = (y^{-1})^n t e_2 + e_2 (y^{-1})^n t \text{ and } t y^n = t y^n e_2 + e_2 t y^n. \quad (3.15)$$

Multiplying both identities, in (3.14), from the left and from the right by e_1 and both identities, in (3.15), from the left and from the right by e_2, we obtain that

$$(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{11} + t_{21}) = 0 \text{ and } (t_{11} + t_{12})((y^n)_{11} + (y^n)_{21}) = 0, \quad (3.16)$$

respectively, and also

$$(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{12} + t_{22}) = 0 \text{ and } (t_{21} + t_{22})((y^n)_{12} + (y^n)_{22}) = 0, \quad (3.17)$$

respectively. On the other hand, from the remaining properties of case (5), we have

$$tr_{12} + g^n(r_{12})t = 0 \text{ and } tg^n(r_{12}) + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{A}_{12}, \quad (3.18)$$

and

$$tr_{21} + g^n(r_{21})t = 0 \text{ and } tg^n(r_{21}) + r_{21}t = 0, \text{ for all } r_{21} \in \mathscr{A}_{21}. \quad (3.19)$$

Multiplying the first identity, in (3.18), on the left by $g^n(r_{12})$ and the second identity, in (3.18), on the right by $g^n(r_{12})$ we get $g^n(r_{12})tr_{12} = 0$ and $r_{12}tg^n(r_{12}) = 0$, respectively, which imply the identities

$$r_{12}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{11} + t_{21})r_{12} = 0 \text{ and } r_{12}(t_{21} + t_{22})((y^n)_{11} + (y^n)_{21})r_{12} = 0,$$

for all $r_{12} \in \mathscr{A}_{12}$, respectively, and it follows from this that

$$r_{12}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{11} + t_{21})r_{12} = 0 \text{ and } r_{12}(t_{21} + t_{22})((y^n)_{11} + (y^n)_{21})r_{12} = 0,$$

for all $r_{12} \in \mathscr{B}(\mathscr{X})_{12}$, respectively. Hence, by [10, Lemma 1.6(P4)], we conclude that

$$(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{11} + t_{21}) = 0 \text{ and } (t_{21} + t_{22})((y^n)_{11} + (y^n)_{21}) = 0, \quad (3.20)$$

respectively. Now, multiplying the first identity, in (3.19), on the left by $g^n(r_{21})$ and the second identity, in (3.19) on the right by $g^n(r_{21})$, we get $g^n(r_{21})tr_{21} = 0$ and $r_{21}tg^n(r_{21}) = 0$, respectively, leading to identities

$$r_{21}(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{12} + t_{22})r_{21} = 0 \text{ and } r_{21}(t_{11} + t_{12})((y^n)_{12} + (y^n)_{22})r_{21} = 0,$$

for all $r_{21} \in \mathscr{A}_{21}$, respectively, which implies that

$$r_{21}(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{12} + t_{22})r_{21} = 0 \text{ and } r_{21}(t_{11} + t_{12})((y^n)_{12} + (y^n)_{22})r_{21} = 0,$$

for all $r_{21} \in \mathscr{B}(\mathscr{X})_{21}$, respectively. Hence, using the [10, Lemma 1.6(P4)] again, we get

$$(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{12} + t_{22}) = 0 \text{ and } (t_{11} + t_{12})((y^n)_{12} + (y^n)_{22}) = 0, \quad (3.21)$$

respectively. Adding the first identity, in (3.16), to the first one, in (3.20), we get $(y^{-1})^n(t_{11} + t_{21}) = 0$ which results that $t_{11} + t_{21} = 0$. Thus, $t_{11} = 0$ and $t_{21} = 0$. Also, adding the first identity, in (3.17), to the first one, in (3.21) we get $(y^{-1})^n(t_{12} + t_{22}) = 0$ which shows that $t_{12} + t_{22} = 0$. This implies that $t_{12} = 0$ and $t_{22} = 0$. Therefore, we have $t = 0$. Consequently, by Theorem 2.1, we get the desired result. □

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References

1. A. N. Alkenani, A. Khan and M. Ashraf, *Generalized Jordan triple (σ, τ)-higher derivations in rings*, The Aligarh Bull. Math. **32** (2013), 21-37.
2. M. Ashraf and N. Parveen, *Jordan higher derivable mappings on rings*, Algebra (2014), Article ID 672387, 9 pages. http://dx.doi.org/10.1155/2014/672387
3. J. Bergen, *Derivations in Prime Rings*, Canad. Math. Bull., **26** (1983), 267-270. https://doi.org/10.4153/CMB-1983-042-2
4. M. Brešar and P. Šemrl, *Elementary operators*, Proc. Royal. Soc. Edinburgh Sect. A, **129** (1999), 1115-1135. https://doi.org/10.1017/s0308210500019302

5. J. C. Chang, *On semiderivations of prime rings*, Chinese J. Math. **12** (1984), 255-262. http://www.jstor.org/stable/43836229
6. M. N. Daif, *When is a Multiplicative Derivation Additive?* Internat. J. Math. & Math. Sci. **14** (1991), 615-618. https://doi.org/10.1155/S0161171291000844
7. J.C.M. Ferreira and M.G.B. Marietto, *Additivity of multiplicative Jordan semi-derivations on rings and standard operator algebras*, Ricerche mat (2024). https://doi.org/10.1007/s11587-024-00865-6
8. V. De Filippis, A. Mamouni and L. Oukhtite, *Generalized Jordan Semiderivations in Prime Rings*, Canad. Math. Bull. **58** (2015), 263-270. https://doi.org/10.4153/CMB-2014-066-9
9. N. Jacobson, *Structure of rings*, vol. 37, American Mathematical Society Colloquium Publications, Providence, RI, 1956.
10. W. Jing and F. Lu, *Additivity of Jordan (triple) Derivations on Rings*, Comm. Algebra **40** (2012), 2700-2719. https://doi.org/10.1080/00927872.2011.584927
11. F. Lu, *Jordan derivable maps of prime rings*, Comm. Algebra **38** (2010), 4430-4440. http://dx.doi.org/10.1080/00927870903366884
12. W. S. Martindale III, *When are multiplicative midtings additive?* Proc. Amer. Math. Soc. **21** (1969), 695-698. https://doi.org/10.2307/2036449
13. L.H. Rowen, *Ring Theory*, Academic Press, San Diego, CA, 1991, student edition.
14. F. K. Schmidt and H. Hasse, *Noch eine Begründung der Theorie der höheren DiKerentialquotienten einem algebraischen Funktionenköroer einer Unbestimmten*, Journal für die Reine und Angewandte Mathematik **177** (1937), 215-237.
15. M. A. Siddeeque and N. Khan, *When are multiplicative semi-derivations additive?* Georgian Math. J. **29** (2022), 123-137. https://doi.org/10.1515/gmj-2021-2121

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