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ADDITIVITY OF MULTIPLICATIVE JORDAN HIGHER SEMI-DERIVATIONS ON RINGS AND STANDARD OPERATOR ALGEBRAS

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ABSTRACT. In this paper we study the additivity of multiplicative Jordan higher semi-derivations on rings and standard operator algebras.

Keywords. Additivity, multiplicative Jordan higher semi-derivations, prime rings, standard operator algebras
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1. INTRODUCTION

Recently, many mathematicians have devoted themselves to studying the additivity of maps. The first result about the additivity of multiplicative isomorphisms on rings was given by Martindale III [12]. Daif [6] introduced the definition of a multiplicative derivation of a ring and studied the additivity of such map. In the context of derivations, Lu [11] studied the additivity of multiplicative Jordan derivations on rings and Jing and Lu [10] generalized the results of Lu to a larger class of rings.

The concept of derivation was extended to higher derivation by Schmidt and Hasse [14]. Thus, in analogy to the concepts of multiplicative derivation and multiplicative Jordan derivation, on rings, defined in the literature [6, 10, 11], Ashraf and Parveen [2] introduced the notions of multiplicative higher derivation (originally called higher derivable map) and multiplicative Jordan higher derivation (originally called Jordan higher derivable map), of a ring, and they studied the additivity of the second class of these maps.

Bergen [3] introduced the notion of a semi-derivation of a ring, based on the concept of derivation, and Siddeeque and Khan [15] studied the additivity of multiplicative semi-derivation of a ring (a concept based on the notions of a semi-derivation and multiplicative derivation). Filippis et al. [8] introduced the notion of a Jordan semi-derivation of a ring, based on the concept of Jordan derivation, and Ferreira and Marietto [7] studied the additivity of multiplicative Jordan semi-derivation of a ring (a concept based on the notions of a Jordan semi-derivation and multiplicative Jordan derivation).

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Motivated by the facts mentioned above, in this paper we introduce the notions of a Jordan higher semi-derivation of a ring and multiplicative Jordan higher semi-derivation of a ring, based on the notions of a Jordan semi-derivation, Jordan (σ, τ)-higher derivations (introduced by Alkenani et al. [1]) and multiplicative Jordan higher derivation, and present a study on the additivity of the last class. We end the paper by presenting an application of obtained results on the standard operator algebras.

2. Multiplicative Jordan higher semi-derivations on rings

Let $\mathscr{R}$ be a ring and $g : \mathscr{R} \to \mathscr{R}$ an arbitrary map. We denote by $g^0 = id_{\mathscr{R}}$ (the identity map on $\mathscr{R}$) and $g^{n+1} = g^n \circ g$, for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\circ$ is the map composition operation.

A family $D = \{d_n\}_{n\in\mathbb{N}}$ of maps $d_n : \mathscr{R} \to \mathscr{R}$ is said to be a *multiplicative higher semi-derivation of $\mathscr{R}$ (associated with g)* (resp., *multiplicative Jordan higher semi-derivation of $\mathscr{R}$ (associated with g)*) if $d_0 = id_{\mathscr{R}}$,

$$d_n(ab) = \sum_{p+q=n} d_p(g^{n-p}(a))d_q(b) = \sum_{p+q=n} d_p(a)d_q(g^{n-q}(b)),$$

for all elements $a, b \in \mathscr{R}$ and for each $n \in \mathbb{N}$ (resp.,

$$\begin{aligned} d_n(ab + ba) &= \sum_{p+q=n} d_p(g^{n-p}(a))d_q(b) + d_p(g^{n-p}(b))d_q(a) \\ &= \sum_{p+q=n} d_p(a)d_q(g^{n-q}(b)) + d_p(b)d_q(g^{n-q}(a)), \end{aligned}$$

for all elements $a, b \in \mathscr{R}$ and for each $n \in \mathbb{N}$), and $d_n(g(a)) = g(d_n(a))$, for all element $a \in \mathscr{R}$ and for each integer $n \in \mathbb{N}$.

A multiplicative higher semi-derivation of $\mathscr{R}$ (associated with g) (resp., multiplicative Jordan higher semi-derivation of $\mathscr{R}$ (associated with g)) $D = \{d_n\}_{n\in\mathbb{N}}$ is said to be *additive* if d_n is an additive map, for each integer $n \in \mathbb{N}$. An additive multiplicative higher semi-derivation of $\mathscr{R}$ (associated with g) (resp., additive multiplicative Jordan higher semi-derivation of $\mathscr{R}$ (associated with g)) is called a *higher semi-derivation of $\mathscr{R}$ (associated with g)* (resp., *Jordan higher semi-derivation of $\mathscr{R}$ (associated with g)*).

The main result of this section reads as follows.

Theorem 2.1. *Let $\mathscr{R}$ be a ring containing a non-trivial idempotent e_1, $\mathscr{R} = \bigoplus_{i,j=1,2} \mathscr{R}_{ij}$ the Peirce decomposition of $\mathscr{R}$, relative to e_1, $g : \mathscr{R} \to \mathscr{R}$ an arbitrary endomorphism of $\mathscr{R}$ and t an element of $\mathscr{R}$. For each integer $n \in \mathbb{N}$, assume that the following property holds:*

$t = 0$ is the only element of $\mathscr{R}$ satisfying the identities

$$tr_{ij} + g^n(r_{ij})t = 0 \text{ and } tg^n(r_{ij}) + r_{ij}t = 0, \tag{♣}$$

for all $r_{ij} \in \mathscr{R}_{ij}$ $(i, j = 1, 2)$ resulting from the use of at least one of the five following cases, where each of them is composed of two subcases:

(1) *(a)* $(i, j) = (1, 1)$ *and (b)* $(i, j) = (1, 2)$,
(2) *(a)* $(i, j) = (1, 1)$ *and (b)* $(i, j) = (2, 1)$,
(3) *(a)* $(i, j) = (2, 2)$ *and (b)* $(i, j) = (1, 2)$,
(4) *(a)* $(i, j) = (2, 2)$ *and (b)* $(i, j) = (2, 1)$,
(5) *(a)* $(i, j) = (1, 2)$ *and (b)* $(i, j) = (2, 1)$, *provided the conditions:*

$$t = te_1 + g^n(e_1)t \text{ and } t = tg^n(e_1) + e_1t, \tag{2.1}$$

are satisfied.

Then every multiplicative Jordan higher semi-derivation of $\mathcal{R}$ (associated with g) is additive.

In addition, if $\mathcal{R}$ is a prime ring of characteristic different from 2 and $g : \mathcal{R} \to \mathcal{R}$ is a ring isomorphism of $\mathcal{R}$, then every multiplicative Jordan higher semi-derivation of $\mathcal{R}$ (associated with g) is a higher semi-derivation of $\mathcal{R}$ (associated with g).

We will prove the Theorem 2.1 using the Second Principle of Mathematical Induction.

Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a multiplicative Jordan higher semi-derivation of $\mathcal{R}$ (associated with g). First of all, we note that $\{g^n\}_{n\in\mathbb{N}}$ is a family of endomorphism of $\mathcal{R}$ (this basic fact will be used throughout this paper without explicit mention) and that d_0 and d_1 are also additive maps, by definition and [7, Theorem 1], respectively. Hence, suppose that for each positive integer n, each map d_k $(0 \leq k < n)$ belonging to D is additive. Based on the techniques used by Ferreira and Marietto [7], we shall organize the proof of Theorem 2.1 in a series of claims.

Claim 2.2. $d_n(0) = 0$.

Proof. By induction hypothesis on n, we have $d_k(0) = 0$, for each integer $0 \leq k < n$. Since $d_p(g^{n-p}(0))d_q(0)+d_p(g^{n-p}(0))d_q(0) = 0$ (resp., $d_p(0)d_q(g^{n-q}(0))+d_p(0)d_q(g^{n-q}(0)) = 0$), for each integers $p, q \in \mathbb{N}$ satisfying $p + q = n$, then

$$d_n(0) = d_n(00 + 00) = \sum_{p+q=n} d_p(g^{n-p}(0))d_q(0) + d_p(g^{n-p}(0))d_q(0)$$
$$\text{(resp., } = \sum_{p+q=n} d_p(0)d_q(g^{n-q}(0)) + d_p(0)d_q(g^{n-q}(0)) = 0.$$

□

The following well known result will be used throughout this paper: Let e_1 be a non-trivial idempotent of $\mathcal{R}$ and formally set $e_2 = 1_{\mathcal{R}} - e_1$ ($\mathcal{R}$ need not have an identity element $1_{\mathcal{R}}$). Then $\mathcal{R}$ has a Peirce decomposition $\mathcal{R} = \mathcal{R}_{11} \oplus \mathcal{R}_{12} \oplus \mathcal{R}_{21} \oplus \mathcal{R}_{22}$, relative to e_1, where $\mathcal{R}_{ij} = e_i\mathcal{R}e_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: $\mathcal{R}_{ij}\mathcal{R}_{kl} \subseteq \delta_{jk}\mathcal{R}_{il}$, where δ_{jk} is the *Kronecker delta function*. More details about the Peirce decomposition and its properties, can be found in references [9] and [13].

Claim 2.3. For all $a_{11} \in \mathcal{R}_{11}$, $a_{12} \in \mathcal{R}_{12}$, $a_{21} \in \mathcal{R}_{21}$ and $a_{22} \in \mathcal{R}_{22}$, write $t = d_n(a_{11} + a_{12} + a_{21} + a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22})$. If

$$\begin{aligned} &d_n((a_{11} + a_{12} + a_{21} + a_{22})r_{ij} + r_{ij}(a_{11} + a_{12} + a_{21} + a_{22})) \\ &= d_n(a_{11}r_{ij} + r_{ij}a_{11}) + d_n(a_{12}r_{ij} + r_{ij}a_{12}) + d_n(a_{21}r_{ij} + r_{ij}a_{21}) \\ &+ d_n(a_{22}r_{ij} + r_{ij}a_{22}), \end{aligned} \tag{2.2}$$

for all $r_{ij} \in \mathcal{R}_{ij}$ resulting from the use of at least one of the five cases, then

$$tr_{ij} + g^n(r_{ij})t = 0 \text{ and } tg^n(r_{ij}) + r_{ij}t = 0,$$

for all $r_{ij} \in \mathcal{R}_{ij}$ satisfying this same case.

Proof. Let $r_{ij} \in \mathcal{R}_{ij}$ satisfying at least one of the five cases of Theorem 2.1. Note that by the definition of d_n, we have

$$d_n((a_{11} + a_{12} + a_{21} + a_{22})r_{ij} + r_{ij}(a_{11} + a_{12} + a_{21} + a_{22}))$$

$$= \sum_{p+q=n} d_p(g^{n-p}(a_{11}+a_{12}+a_{21}+a_{22}))d_q(r_{ij}) + d_p(g^{n-p}(r_{ij}))d_q(a_{11}+a_{12}+a_{21}+a_{22}) \tag{2.3}$$

$$= \sum_{p+q=n} d_p(a_{11}+a_{12}+a_{21}+a_{22})d_q(g^{n-q}(r_{ij})) + d_p(r_{ij})d_q(g^{n-q}(a_{11}+a_{12}+a_{21}+a_{22})) \tag{2.4}$$

and that

$$\begin{aligned} &d_n(a_{11}r_{ij}+r_{ij}a_{11})+d_n(a_{12}r_{ij}+r_{ij}a_{12})+d_n(a_{21}r_{ij}+r_{ij}a_{21}) \\ &+d_n(a_{22}r_{ij}+r_{ij}a_{22}) \\ &=\sum_{p+q=n} d_p(g^{n-p}(a_{11}))d_q(r_{ij})+d_p(g^{n-p}(r_{ij}))d_q(a_{11}) \\ &+\sum_{p+q=n} d_p(g^{n-p}(a_{12}))d_q(r_{ij})+d_p(g^{n-p}(r_{ij}))d_q(a_{12}) \\ &+\sum_{p+q=n} d_p(g^{n-p}(a_{21}))d_q(r_{ij})+d_p(g^{n-p}(r_{ij}))d_q(a_{21}) \\ &+\sum_{p+q=n} d_p(g^{n-p}(a_{22}))d_q(r_{ij})+d_p(g^{n-p}(r_{ij}))d_q(a_{22}) \qquad (2.5) \\ &=\sum_{p+q=n} d_p(a_{11})d_q(g^{n-q}(r_{ij}))+d_p(r_{ij})d_q(g^{n-q}(a_{11})) \\ &+\sum_{p+q=n} d_p(a_{12})d_q(g^{n-q}(r_{ij}))+d_p(r_{ij})d_q(g^{n-q}(a_{12})) \\ &+\sum_{p+q=n} d_p(a_{21})d_q(g^{n-q}(r_{ij}))+d_p(r_{ij})d_q(g^{n-q}(a_{21})) \\ &+\sum_{p+q=n} d_p(a_{22})d_q(g^{n-q}(r_{ij}))+d_p(r_{ij})d_q(g^{n-q}(a_{22})). \qquad (2.6) \end{aligned}$$

Taking into account the hypothesis of the claim and subtracting (2.5) from (2.3) and (2.6) from (2.4), we get

$$\begin{aligned} &\sum_{p+q=n}(d_p(g^{n-p}(a_{11}+a_{12}+a_{21}+a_{22}))-d_p(g^{n-p}(a_{11}))-d_p(g^{n-p}(a_{12})) \\ &-d_p(g^{n-p}(a_{21}))-d_p(g^{n-p}(a_{22})))d_q(r_{ij})+d_p(g^{n-p}(r_{ij}))(d_q(a_{11}+a_{12} \\ &+a_{21}+a_{22})-d_q(a_{11})-d_q(a_{12})-d_q(a_{21})-d_q(a_{22}))=0 \end{aligned}$$

and

$$\begin{aligned} &\sum_{p+q=n}(d_p(a_{11}+a_{12}+a_{21}+a_{22})-d_p(a_{11})-d_p(a_{12})-d_p(a_{21}) \\ &-d_p(a_{22}))d_q(g^{n-q}(r_{ij}))+d_p(r_{ij})(d_q(g^{n-q}(a_{11}+a_{12}+a_{21}+a_{22})) \\ &-d_q(g^{n-q}(a_{11}))-d_q(g^{n-q}(a_{12}))-d_q(g^{n-q}(a_{21}))-d_q(g^{n-q}(a_{22})))=0, \end{aligned}$$

respectively, which lead to the following identities

$$\begin{aligned} &(d_n(a_{11}+a_{12}+a_{21}+a_{22})-d_n(a_{11})-d_n(a_{12})-d_n(a_{21})-d_n(a_{22}))r_{ij} \\ &+g^n(r_{ij})(d_n(a_{11}+a_{12}+a_{21}+a_{22})-d_n(a_{11})-d_n(a_{12})-d_n(a_{21}) \\ &-d_n(a_{22}))=0 \qquad (2.7) \end{aligned}$$

and

$$\begin{aligned} &(d_n(a_{11}+a_{12}+a_{21}+a_{22})-d_n(a_{11})-d_n(a_{12})-d_n(a_{21})-d_n(a_{22}))g^n(r_{ij}) \\ &+r_{ij}(d_n(a_{11}+a_{12}+a_{21}+a_{22})-d_n(a_{11})-d_n(a_{12})-d_n(a_{21})-d_n(a_{22})) \\ &=0, \qquad (2.8) \end{aligned}$$

respectively. As a consequence of (2.7) and (2.8), it follows that

$$tr_{ij}+g^n(r_{ij})t=0 \text{ and } tg^n(r_{ij})+r_{ij}t=0.$$

for such $r_{ij} \in \mathscr{R}_{ij}$. □

Claim 2.4. For all $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{11} + a_{12}) = d_n(a_{11}) + d_n(a_{12})$, (ii) $d_n(a_{11} + a_{21}) = d_n(a_{11}) + d_n(a_{21})$, (iii) $d_n(a_{12} + a_{22}) = d_n(a_{12}) + d_n(a_{22})$ and (iv) $d_n(a_{21} + a_{22}) = d_n(a_{21}) + d_n(a_{22})$.

Proof. Note that by direct verification, for all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) we get (in both subcases)

$$\begin{aligned} &d_n((a_{11} + a_{12})r_{ij} + r_{ij}(a_{11} + a_{12})) \\ &= d_n(a_{11}r_{ij} + r_{ij}a_{11}) + d_n(a_{12}r_{ij} + r_{ij}a_{12}). \end{aligned}$$

Thus, by Claim 2.3, we get the identities in (♧). It follows from property of Theorem 2.1 that $t = 0$. As a consequence, we have $d_n(a_{11} + a_{12}) - d_n(a_{11}) - d_n(a_{12}) = 0$. Also by direct verification, for all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (4) we get (in both subcases)

$$\begin{aligned} &d_n((a_{11} + a_{21})r_{ij} + r_{ij}(a_{11} + a_{21})) \\ &= d_n(a_{11}r_{ij} + r_{ij}a_{11}) + d_n(a_{21}r_{ij} + r_{ij}a_{21}). \end{aligned}$$

Thus, by Claim 2.3, we have the identities in ♧. According to the property of Theorem 2.1, we can deduce that $t = 0$. It follows that $d_n(a_{11} + a_{21}) - d_n(a_{11}) - d_n(a_{21}) = 0$.

The other cases can be proved similarly. □

Claim 2.5. For all $a_{12}, b_{12} \in \mathscr{R}_{12}$, $a_{21}, b_{21} \in \mathscr{R}_{21}$ and $t_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{12} + b_{12}t_{22}) = d_n(a_{12}) + d_n(b_{12}t_{22})$ and (ii) $d_n(a_{21} + t_{22}b_{21}) = d_n(a_{21}) + d_n(t_{22}b_{21})$.

Proof. In view of following identity

$$a_{12} + b_{12}t_{22} = (a_{12} + t_{22})(e_1 + b_{12}) + (e_1 + b_{12})(a_{12} + t_{22}),$$

of induction hypothesis on n and Claim 2.4(i) and (iii), we have

$$\begin{aligned} &d_n(a_{12} + b_{12}t_{22}) \\ &= d_n((a_{12} + t_{22})(e_1 + b_{12}) + (e_1 + b_{12})(a_{12} + t_{22})) \\ &= \textstyle\sum_{p+q=n} d_p(g^{n-p}(a_{12} + t_{22}))d_q(e_1 + b_{12}) \\ &\qquad + d_p(g^{n-p}(e_1 + b_{12}))d_q(a_{12} + t_{22}) \\ &= \textstyle\sum_{p+q=n}(d_p(g^{n-p}(a_{12})) + d_p(g^{n-p}(t_{22})))(d_q(e_1) + d_q(b_{12})) \\ &\qquad + (d_p(g^{n-p}(e_1)) + d_p(g^{n-p}(b_{12})))(d_q(a_{12}) + d_q(t_{22})) \\ &= \textstyle\sum_{p+q=n} d_p(g^{n-p}(a_{12}))d_q(e_1) + d_p(g^{n-p}(e_1))d_q(a_{12}) \\ &+ \textstyle\sum_{p+q=n} d_p(g^{n-p}(t_{22}))d_q(e_1) + d_p(g^{n-p}(e_1))d_q(t_{22}) \\ &+ \textstyle\sum_{p+q=n} d_p(g^{n-p}(a_{12}))d_q(b_{12}) + d_p(g^{n-p}(b_{12}))d_q(a_{12}) \\ &+ \textstyle\sum_{p+q=n} d_p(g^{n-p}(t_{22}))d_q(b_{12}) + d_p(g^{n-p}(b_{12}))d_q(t_{22}) \\ &= d_n(a_{12}e_1 + e_1a_{12}) + d_n(t_{22}e_1 + e_1t_{22}) \\ &+ d_n(a_{12}b_{12} + b_{12}a_{12}) + d_n(t_{22}b_{12} + b_{12}t_{22}) \\ &= d_n(a_{12}) + d_n(b_{12}t_{22}). \end{aligned}$$

Similarly we prove that $d_n(a_{21} + t_{22}b_{21}) = d_n(a_{21}) + d_n(t_{22}b_{21})$, from identity

$$a_{21} + t_{22}b_{21} = (a_{21} + t_{22})(b_{21} + e_1) + (b_{21} + e_1)(a_{21} + t_{22}).$$

□

Claim 2.6. For all $a_{12}, b_{12} \in \mathscr{R}_{12}$ and $a_{21}, b_{21} \in \mathscr{R}_{21}$ the following hold: (i) $d_n(a_{12} + b_{12}) = d_n(a_{12}) + d_n(b_{12})$ and (ii) $d_n(a_{21} + b_{21}) = d_n(a_{21}) + d_n(b_{21})$.

Proof. It follows from direct verification that, for all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) and by Claims 2.2 and 2.5(i), we get (in both subcases)

$$\begin{aligned} &d_n((a_{12} + b_{12})r_{ij} + r_{ij}(a_{12} + b_{12})) \\ &= d_n(a_{12}r_{ij} + r_{ij}a_{12}) + d_n(b_{12}r_{ij} + r_{ij}b_{12}). \end{aligned}$$

Then, by Claim 2.3, we arrive at identities in (♧). This leads to $d_n(a_{12}+b_{12})-d_n(a_{12})-d_n(b_{12}) = 0$.

The other case can be proved similarly. □

Claim 2.7. For all $a_{11}, b_{11} \in \mathscr{R}_{11}$ and $a_{22}, b_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{11} + b_{11}) = d_n(a_{11}) + d_n(b_{11})$ and (ii) $d_n(a_{22} + b_{22}) = d_n(a_{22}) + d_n(b_{22})$.

Proof. For all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) and by Claims 2.2 and 2.6(i), we get (in both subcases)

$$\begin{aligned} &d_n((a_{11} + b_{11})r_{ij} + r_{ij}(a_{11} + b_{11})) \\ &= d_n(a_{11}r_{ij} + r_{ij}a_{11}) + d_n(b_{11}r_{ij} + r_{ij}b_{11}). \end{aligned}$$

Then, by Claim 2.3, we obtain the identities in (♧). Therefore, $d_n(a_{11}+b_{11})-d_n(a_{11})-d_n(b_{11}) = 0$.

The other case can be proved similarly. □

Claim 2.8. For all $a_{12} \in \mathscr{R}_{12}$ and $a_{21} \in \mathscr{R}_{21}$ the following holds: $d_n(a_{12} + a_{21}) = d_n(a_{12}) + d_n(a_{21})$.

Proof. First notice that the following identities hold:

$$\begin{aligned} a_{12} + a_{21} &= e_1(a_{12} + a_{21}) + (a_{12} + a_{21})e_1, \\ a_{12} &= e_1a_{12} + a_{12}e_1 \end{aligned}$$

and

$$a_{21} = e_1a_{21} + a_{21}e_1.$$

Thus, by the definition of d_n, we get

$$\begin{aligned} &d_n(a_{12} + a_{21}) = d_n(e_1(a_{12} + a_{21}) + (a_{12} + a_{21})e_1) \\ &= \textstyle\sum_{p+q=n} d_p(g^{n-p}(e_1))d_q(a_{12} + a_{21}) + d_p(g^{n-p}(a_{12} + a_{21}))d_q(e_1) && (2.9) \\ &= \textstyle\sum_{p+q=n} d_p(e_1)d_q(g^{n-q}(a_{12} + a_{21})) + d_p(a_{12} + a_{21})d_q(g^{n-q}(e_1)), && (2.10) \\ &d_n(a_{12}) = d_n(e_1a_{12} + a_{12}e_1) \\ &= \textstyle\sum_{p+q=n} d_p(g^{n-p}(e_1))d_q(a_{12}) + d_p(g^{n-p}(a_{12}))d_q(e_1) && (2.11) \\ &= \textstyle\sum_{p+q=n} d_p(e_1)d_q(g^{n-q}(a_{12})) + d_p(a_{12}))d_q(g^{n-q}(e_1)) && (2.12) \end{aligned}$$

and

$$\begin{aligned} &d_n(a_{21}) = d_n(e_1a_{21} + a_{21}e_1) \\ &= \textstyle\sum_{p+q=n} d_p(g^{n-p}(e_1))d_q(a_{21}) + d_p(g^{n-p}(a_{21}))d_q(e_1) && (2.13) \\ &= \textstyle\sum_{p+q=n} d_p(e_1)d_q(g^{n-q}(a_{21})) + d_p(a_{21})d_q(g^{n-q}(e_1)), && (2.14) \end{aligned}$$

respectively. Adding (2.11) and (2.13) and subtracting this sum from (2.9) and adding (2.12) and (2.14) and subtracting this sum from (2.10), we obtain the identities in (2.1). Next, for all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (5) and by Claim 2.2, we get (in both subcases)

$$\begin{aligned} &d_n((a_{12}+a_{21})r_{ij}+r_{ij}(a_{12}+a_{21})) \\ &= d_n(a_{12}r_{ij}+r_{ij}a_{12})+d_n(a_{21}r_{ij}+r_{ij}a_{21}). \end{aligned}$$

Then, by Claim 2.3 we get the identities in (♧). This results that $d_n(a_{12}+a_{21})-d_n(a_{12})-d_n(a_{21})=0$. □

Claim 2.9. For all $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{11}+a_{12}+a_{22}) = d_n(a_{11})+d_n(a_{12})+d_n(a_{22})$ and (ii) $d_n(a_{11}+a_{21}+a_{22}) = d_n(a_{11})+d_n(a_{21})+d_n(a_{22})$.

Proof. For all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) and by Claims 2.4(iii) and 2.6(i), we get (in both subcases)

$$\begin{aligned} &d_n((a_{11}+a_{12}+a_{22})r_{ij}+r_{ij}(a_{11}+a_{12}+a_{22})) \\ &= d_n(a_{11}r_{ij}+r_{ij}a_{11})+d_n(a_{12}r_{ij}+r_{ij}a_{12})+d_n(a_{22}r_{ij}+r_{ij}a_{22}). \end{aligned}$$

Then, by Claim 2.3, we obtain the identities in (♧). Therefore, $d_n(a_{11}+a_{12}+a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{22}) = 0$.

The other case can be proved similarly. □

Claim 2.10. For all $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_n(a_{11}+a_{12}+a_{21}) = d_n(a_{11})+d_n(a_{12})+d_n(a_{21})$ and (ii) $d_n(a_{12}+a_{21}+a_{22}) = d_n(a_{12})+d_n(a_{21})+d_n(a_{22})$.

Proof. For all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) and by Claims 2.8 and 2.9(i), we get (in both subcases)

$$\begin{aligned} &d_n((a_{11}+a_{12}+a_{21})r_{ij}+r_{ij}(a_{11}+a_{12}+a_{21})) \\ &= d_n(a_{11}r_{ij}+r_{ij}a_{11})+d_n(a_{12}r_{ij}+r_{ij}a_{12})+d_n(a_{21}r_{ij}+r_{ij}a_{21}). \end{aligned}$$

This results in the identities in (♧). Consequently, $d_n(a_{11}+a_{12}+a_{21})-d_n(a_{11})-d_n(a_{12})-d_n(a_{21})=0$.

The other case can be proved similarly. □

Claim 2.11. For all $a_{11} \in \mathscr{R}_{11}$, $a_{12} \in \mathscr{R}_{12}$, $a_{21} \in \mathscr{R}_{21}$ and $a_{22} \in \mathscr{R}_{22}$ the following holds: $d_n(a_{11}+a_{12}+a_{21}+a_{22}) = d_n(a_{11})+d_n(a_{12})+d_n(a_{21})+d_n(a_{22})$.

Proof. For all $r_{ij} \in \mathscr{R}_{ij}$ satisfying the case (3) and by Claims 2.6(i), 2.9(i) and 2.10(ii), we get (in both subcases)

$$\begin{aligned} &d_n((a_{11}+a_{12}+a_{21}+a_{22})r_{ij}+r_{ij}(a_{11}+a_{12}+a_{21}+a_{22})) \\ &= d_n(a_{11}r_{ij}+r_{ij}a_{11})+d_n(a_{12}r_{ij}+r_{ij}a_{12}) \\ &+ d_n(a_{21}r_{ij}+r_{ij}a_{21})+d_n(a_{22}r_{ij}+r_{ij}a_{22}). \end{aligned}$$

Then, we obtain the identities in (♧). Therefore, $d_n(a_{11}+a_{12}+a_{21}+a_{22}) - d_n(a_{11}) - d_n(a_{12}) - d_n(a_{21}) - d_n(a_{22}) = 0$. □

Claim 2.12. d_n is an additive map.

Proof. The proof is a direct consequence of the Claims 2.6, 2.7 and 2.11. The details are omitted. □

Claim 2.13. D is additive.

Proof. Applying the Second Principle of Mathematical Induction on D, in view of Claim 2.12, we concluded that D is additive. □

To complete the proof of the Theorem 2.1 we will assume in addition that $\mathscr{R}$ is a prime ring of characteristic different from 2 and $g : \mathscr{R} \to \mathscr{R}$ is a ring isomorphism of $\mathscr{R}$.

Claim 2.14. D is a higher semi-derivation of $\mathscr{R}$ (associated with g).

Proof. First, note that according to the definition presented in [1, pp. 23], the family D is simultaneously a Jordan $(g, id_{\mathscr{R}})$-higher derivation of $\mathscr{R}$ and a Jordan $(id_{\mathscr{R}}, g)$-higher derivation of $\mathscr{R}$. Thus, by [1, Theorem 2.11], D is simultaneously a $(g, id_{\mathscr{R}})$-higher derivation of $\mathscr{R}$ and a $(id_{\mathscr{R}}, g)$-higher derivation of $\mathscr{R}$, respectively. These last two results show that D is a higher semi-derivation of $\mathscr{R}$ (associated with g). □

3. Multiplicative Jordan higher semi-derivations on standard operator algebras

Let $\mathscr{X}$ be a Banach space. We denote by $\mathscr{B}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$, and $\mathscr{F}(\mathscr{X})$ the ideal of all bounded finite rank operators in $\mathscr{B}(\mathscr{X})$. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{X})$ is called *prime* if $a\mathscr{A}b = 0$ implies $a = 0$ or $b = 0$. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{X})$ is called a *standard operator algebra* if $\mathscr{A}$ contain $\mathscr{F}(\mathscr{X})$. It is clear $\mathscr{B}(\mathscr{X})$ is a standard operator algebra.

Let $\mathscr{X}$ be a Banach space with $\dim \mathscr{X} \geq 2$ and $\mathscr{A} \subset \mathscr{B}(\mathscr{X})$ a standard operator algebra. It is well known that $\mathscr{B}(\mathscr{X})$ and $\mathscr{A}$ are prime rings and $\mathscr{A}$ contains a non-trivial idempotent e_1. Also, $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$ under the strong operator topology and, by [4, Corollary 7.3.], there exist a linear invertible operator of $\mathscr{B}(\mathscr{X})$, $y : \mathscr{X} \to \mathscr{X}$, such that $g(a) = yay^{-1}$, for all element $a \in \mathscr{A}$. We will write $e_2 = 1_{\mathscr{B}(\mathscr{X})} - e_1$, where $1_{\mathscr{B}(\mathscr{X})}$ is a multiplicative identity of $\mathscr{B}(\mathscr{X})$.

In order to understand the motivations of hypotheses of the theorem of this section, we will take into consideration the results of Chang [5, Theorem 1], that proved that all maps which are associated with arbitrary semi-derivations of a prime ring are always homomorphisms, of Brešar and Šemrl [4, Corollary 7.3.], that characterized the class of isomorphisms between standard operator algebras and of Ferreira and Marietto [7].

In light of this, we can state the following.

Theorem 3.1. *Let $\mathscr{X}$ be a Banach space with $\dim \mathscr{X} \geq 2$, $\mathscr{A} \subset \mathscr{B}(\mathscr{X})$ a standard operator algebra and $g : \mathscr{A} \to \mathscr{A}$ an isomorphism of $\mathscr{A}$. Then the following statement holds: every multiplicative Jordan higher semi-derivation of $\mathscr{A}$ (associated with g) is a higher semi-derivation of $\mathscr{A}$ (associated with g).*

Proof. The proof is based on the techniques used by Ferreira and Marietto [7].

Let $\mathscr{B}(\mathscr{X}) = \oplus_{i,j=1,2}\mathscr{B}(\mathscr{X})_{ij}$ and $\mathscr{A} = \oplus_{i,j=1,2}\mathscr{A}_{ij}$ be the Peirce decompositions of $\mathscr{B}(\mathscr{X})$ and $\mathscr{A}$, relative to e_1, respectively, and an arbitrary element $t \in \mathscr{A}$. For each

integer $n \in \mathbb{N}$, we consider two cases, according as whether $n = 0$ or $n \geq 1$. Case 1: $n = 0$. If the identities in (♧) result from the use of the case (1), then

$$tr_{11} + r_{11}t = 0 \text{ for all } r_{11} \in \mathscr{A}_{11}, \text{ and } tr_{12} + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{A}_{12}. \tag{3.1}$$

In particular, taking $r_{11} = e_1$, in the first identity, in (3.1), we get $2t_{11} + t_{12} + t_{21} = 0$ which implies that $t_{11} = 0$, $t_{12} = 0$ and $t_{21} = 0$. As a consequence, the second identity, in (3.1), becomes $r_{12}t_{22} = 0$, for all $r_{12} \in \mathscr{A}_{12}$, which implies that $t_{22} = 0$, in view of the primeness of $\mathscr{A}$. Thus, we must have $t = 0$. If the identities in (♧) result from the use of the case (3), then

$$tr_{22} + r_{22}t = 0 \text{ for all } r_{22} \in \mathscr{A}_{22}, \text{ and } tr_{12} + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{A}_{12}.$$

Since $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$, the identities above lead us to

$$tr_{22} + r_{22}t = 0 \text{ for all } r_{22} \in \mathscr{B}(\mathscr{X})_{22}, \text{ and } tr_{12} + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{B}(\mathscr{X})_{12}. \tag{3.2}$$

Thus, taking $r_{22} = e_2$, in the first identity, in (3.2), we get $t_{12} + t_{21} + 2t_{22} = 0$ which shows that $t_{12} = 0$, $t_{21} = 0$ and $t_{22} = 0$. As a consequence, the second identity, in (3.2), becomes $t_{11}r_{12} = 0$, for all $r_{12} \in \mathscr{B}(\mathscr{X})_{12}$, which results in $t_{11} = 0$, in view of the primeness of $\mathscr{B}(\mathscr{X})$. Therefore $t = 0$. The cases (2) and (4) can be proved similarly. Now, if the identities in (♧) result from the use of the case (5), then the conditions in (2.1) imply that

$$t = te_1 + e_1t$$

which immediately implies that $t_{11} = 0$ and $t_{22} = 0$. Also, from the remaining properties of case (5), we have

$$tr_{12} + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{A}_{12}, \text{ and } tr_{21} + r_{21}t = 0, \text{ for all } r_{21} \in \mathscr{A}_{21},$$

which leads to $t_{12} = 0$ and $t_{21} = 0$. Hence it follows that $t = 0$. Therefore the Case 1 is proved. Case 2: $n \geq 1$. Consider $(y^{-1})^n = ((y^{-1})^n)_{11} + ((y^{-1})^n)_{12} + ((y^{-1})^n)_{21} + ((y^{-1})^n)_{22}$ and $y^n = (y^n)_{11} + (y^n)_{12} + (y^n)_{21} + (y^n)_{22}$ be the Peirce decompositions of $(y^{-1})^n$ and y^n, relative to e_1, respectively. If the identities in (♧) result from the use of the case (1), then

$$tr_{11} + g^n(r_{11})t = 0 \text{ and } tg^n(r_{11}) + r_{11}t = 0, \text{ for all } r_{11} \in \mathscr{A}_{11}, \tag{3.3}$$

and

$$tr_{12} + g^n(r_{12})t = 0 \text{ and } tg^n(r_{12}) + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{A}_{12}. \tag{3.4}$$

Taking $r_{11} = e_1$ and multiplying the first identity, in (3.3), on the left by $g^n(e_1)$ and on the right by e_1 we obtain that $g^n(e_1)te_1 = 0$. Also, multiplying the second identity, in (3.3), on the left by e_1 and on the right by $g^n(e_1)$ we obtain that $e_1tg^n(e_1) = 0$. These two last results allow to deduce that

$$(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{11} + t_{21}) = 0 \text{ and } (t_{11} + t_{12})((y^n)_{11} + (y^n)_{21}) = 0, \tag{3.5}$$

respectively. Also, multiplying the first identity, in (3.4), on the left by $g^n(r_{12})$ we get $g^n(r_{12})tr_{12} = 0$ and multiplying the second identity, in (3.4), on the right by $g^n(r_{12})$ we get $r_{12}tg^n(r_{12}) = 0$. As a consequence of these two last identities we obtain

$$r_{12}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{11} + t_{21})r_{12} = 0$$

and

$$r_{12}(t_{21}+t_{22})((y^n)_{11}+(y^n)_{21})r_{12}=0,$$

for all $r_{12}\in\mathscr{A}_{12}$, respectively. Since $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$, then we have

$$r_{12}(((y^{-1})^n)_{21}+((y^{-1})^n)_{22})(t_{11}+t_{21})r_{12}=0$$

and

$$r_{12}(t_{21}+t_{22})((y^n)_{11}+(y^n)_{21})r_{12}=0,$$

for all $r_{12}\in\mathscr{B}(\mathscr{X})_{12}$, respectively. Using the [10, Lemma 1.6(P4)], we obtain

$$(((y^{-1})^n)_{21}+((y^{-1})^n)_{22})(t_{11}+t_{21})=0 \text{ and } (t_{21}+t_{22})((y^n)_{11}+(y^n)_{21})=0. \tag{3.6}$$

Adding the first identity, in (3.5), to the first one, in (3.6), we obtain $(y^{-1})^n(t_{11}+t_{21})=0$ which leads to $t_{11}+t_{21}=0$. This results that $t_{11}=0$ and $t_{21}=0$. Next, multiplying the second identity, in (3.3), on the right by $g^n(r_{22})$ and taking $r_{11}=e_1$ we get $e_1tg^n(r_{22})=0$. As a consequence of this we have

$$t_{12}((y^n)_{12}+(y^n)_{22})r_{22}=0,$$

for all $r_{22}\in\mathscr{A}_{22}$, which allows to conclude that

$$t_{12}((y^n)_{12}+(y^n)_{22})r_{22}=0,$$

for all $r_{22}\in\mathscr{B}(\mathscr{X})_{22}$, in view of the density of $\mathscr{A}$ in $\mathscr{B}(\mathscr{X})$. This leads to the conclusion that

$$t_{12}((y^n)_{12}+(y^n)_{22})=0. \tag{3.7}$$

Adding the second identity, in (3.5), to the identity one, in (3.7), we have $t_{12}y^n=0$ which results in $t_{12}=0$. Thus, taking $r_{11}=e_1$ in the first identity, in (3.3), we get $g^n(e_1)t_{22}=0$ and the first identity, in (3.4), becomes $g^n(r_{12})t_{22}=0$, for all $r_{12}\in\mathscr{A}_{12}$. As a consequence of these last two results, we obtain

$$(((y^{-1})^n)_{11}+((y^{-1})^n)_{12})t_{22}=0 \text{ and } r_{12}(((y^{-1})^n)_{21}+((y^{-1})^n)_{22})t_{22}=0,$$

for all $r_{12}\in\mathscr{A}_{12}$, respectively, which implies that

$$(((y^{-1})^n)_{11}+((y^{-1})^n)_{12})t_{22}=0 \text{ and } r_{12}(((y^{-1})^n)_{21}+((y^{-1})^n)_{22})t_{22}=0,$$

for all $r_{12}\in\mathscr{B}(\mathscr{X})_{12}$. From this we deduce that

$$(((y^{-1})^n)_{11}+((y^{-1})^n)_{12})t_{22}=0 \text{ and } (((y^{-1})^n)_{21}+((y^{-1})^n)_{22})t_{22}=0,$$

in view of primeness of $\mathscr{B}(\mathscr{X})$. Adding the last two identities we obtain $(y^{-1})^nt_{22}=0$ which leads to $t_{22}=0$. Consequently, we have $t=0$. Now, if the identities in (♣) result from the use of the case (3), then

$$tr_{22}+g^n(r_{22})t=0 \text{ and } tg^n(r_{22})+r_{22}t=0, \text{ for all } r_{22}\in\mathscr{A}_{22}, \tag{3.8}$$

and

$$tr_{12}+g^n(r_{12})t=0 \text{ and } tg^n(r_{12})+r_{12}t=0, \text{ for all } r_{12}\in\mathscr{A}_{12}. \tag{3.9}$$

Note that the identities, in (3.8), are equivalent to

$$(y^{-1})^ntr_{22}+r_{22}(y^{-1})^nt=0 \text{ and } ty^nr_{22}+r_{22}ty^n=0, \text{ for all } r_{22}\in\mathscr{A}_{22},$$

respectively, that lead to identities

$$(y^{-1})^n t r_{22} + r_{22}(y^{-1})^n t = 0 \text{ and } t y^n r_{22} + r_{22} t y^n = 0, \text{ for all } r_{22} \in \mathscr{B}(\mathscr{X})_{22}. \quad (3.10)$$

Thus, multiplying both identities, in (3.10), from the left and from the right by e_2 we get

$$(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{12} + t_{22})r_{22} + r_{22}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{12} + t_{22}) = 0$$

and

$$(t_{21} + t_{22})((y^n)_{12} + (y^n)_{22})r_{22} + r_{22}(t_{21} + t_{22})((y^n)_{12} + (y^n)_{22}) = 0,$$

for all $r_{22} \in \mathscr{B}(\mathscr{X})_{22}$. This results that

$$(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{12} + t_{22}) = 0 \text{ and } (t_{21} + t_{22})((y^n)_{12} + (y^n)_{22}) = 0. \quad (3.11)$$

Next, multiplying the first identity, in (3.10), on the left by e_1 and the second identity, in (3.10), on the right by e_1, we get

$$(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{12} + t_{22})r_{22} = 0 \text{ and } r_{22}(t_{21} + t_{22})((y^n)_{11} + (y^n)_{21}) = 0,$$

$r_{22} \in \mathscr{B}(\mathscr{X})_{22}$, respectively, which allows to conclude that

$$(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{12} + t_{22}) = 0 \text{ and } (t_{21} + t_{22})((y^n)_{11} + (y^n)_{21}) = 0. \quad (3.12)$$

Adding the corresponding identities, in (3.11) and (3.12), we obtain $(y^{-1})^n(t_{12}+t_{22}) = 0$ and $(t_{21} + t_{22})y^n = 0$, respectively. It results from this that $t_{12} + t_{22} = 0$ and $t_{21} + t_{22} = 0$ which leads to $t_{12} = 0$, $t_{21} = 0$ and $t_{22} = 0$. Now, multiplying the first identity, in (3.9), on the left by $g^n(r_{12})$ we obtain $g^n(r_{12})tr_{12} = 0$. Also, multiplying the same identity on the left by $g^n(e_1)$ and on right by e_2 we obtain $g^n(e_1)tr_{12} + g^n(r_{12})te_2 = 0$. This results in the identities

$$r_{12}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})t_{11}r_{12} = 0 \text{ and } (((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})t_{11}r_{12} = 0,$$

for all $r_{12} \in \mathscr{A}_{12}$, respectively, which leads to

$$r_{12}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})t_{11}r_{12} = 0 \text{ and } (((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})t_{11}r_{12} = 0,$$

for all $r_{12} \in \mathscr{B}(\mathscr{X})_{12}$. Therefore,

$$(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})t_{11} = 0 \text{ and } (((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})t_{11} = 0, \quad (3.13)$$

Adding the two identities, in (3.13), we have $(y^{-1})^n t_{11} = 0$ which yields $t_{11} = 0$. Consequently, $t = 0$.

Using similar arguments to the previous two cases, we prove that if the identities in (♧) result from the use of at least one of the cases (2) and (4), then $t = 0$.

Finally, if the identities in (♧) result from the use of the case (5), then the conditions in (2.1) imply that

$$(y^{-1})^n t = (y^{-1})^n t e_1 + e_1(y^{-1})^n t \text{ and } t y^n = t y^n e_1 + e_1 t y^n, \quad (3.14)$$

respectively. Also, replacing $e_1 = 1_{\mathscr{B}(\mathscr{X})} - e_2$ in both conditions in (3.14) we get

$$(y^{-1})^n t = (y^{-1})^n t e_2 + e_2(y^{-1})^n t \text{ and } t y^n = t y^n e_2 + e_2 t y^n. \quad (3.15)$$

Multiplying both identities, in (3.14), from the left and from the right by e_1 and both identities, in (3.15), from the left and from the right by e_2, we obtain that

$$(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{11} + t_{21}) = 0 \text{ and } (t_{11} + t_{12})((y^n)_{11} + (y^n)_{21}) = 0, \quad (3.16)$$

respectively, and also

$$(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{12} + t_{22}) = 0 \text{ and } (t_{21} + t_{22})((y^n)_{12} + (y^n)_{22}) = 0, \quad (3.17)$$

respectively. On the other hand, from the remaining properties of case (5), we have

$$tr_{12} + g^n(r_{12})t = 0 \text{ and } tg^n(r_{12}) + r_{12}t = 0, \text{ for all } r_{12} \in \mathscr{A}_{12}, \quad (3.18)$$

and

$$tr_{21} + g^n(r_{21})t = 0 \text{ and } tg^n(r_{21}) + r_{21}t = 0, \text{ for all } r_{21} \in \mathscr{A}_{21}. \quad (3.19)$$

Multiplying the first identity, in (3.18), on the left by $g^n(r_{12})$ and the second identity, in (3.18), on the right by $g^n(r_{12})$ we get $g^n(r_{12})tr_{12} = 0$ and $r_{12}tg^n(r_{12}) = 0$, respectively, which imply the identities

$$r_{12}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{11} + t_{21})r_{12} = 0 \text{ and } r_{12}(t_{21} + t_{22})((y^n)_{11} + (y^n)_{21})r_{12} = 0,$$

for all $r_{12} \in \mathscr{A}_{12}$, respectively, and it follows from this that

$$r_{12}(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{11} + t_{21})r_{12} = 0 \text{ and } r_{12}(t_{21} + t_{22})((y^n)_{11} + (y^n)_{21})r_{12} = 0,$$

for all $r_{12} \in \mathscr{B}(\mathscr{X})_{12}$, respectively. Hence, by [10, Lemma 1.6(P4)], we conclude that

$$(((y^{-1})^n)_{21} + ((y^{-1})^n)_{22})(t_{11} + t_{21}) = 0 \text{ and } (t_{21} + t_{22})((y^n)_{11} + (y^n)_{21}) = 0, \quad (3.20)$$

respectively. Now, multiplying the first identity, in (3.19), on the left by $g^n(r_{21})$ and the second identity, in (3.19) on the right by $g^n(r_{21})$, we get $g^n(r_{21})tr_{21} = 0$ and $r_{21}tg^n(r_{21}) = 0$, respectively, leading to identities

$$r_{21}(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{12} + t_{22})r_{21} = 0 \text{ and } r_{21}(t_{11} + t_{12})((y^n)_{12} + (y^n)_{22})r_{21} = 0,$$

for all $r_{21} \in \mathscr{A}_{21}$, respectively, which implies that

$$r_{21}(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{12} + t_{22})r_{21} = 0 \text{ and } r_{21}(t_{11} + t_{12})((y^n)_{12} + (y^n)_{22})r_{21} = 0,$$

for all $r_{21} \in \mathscr{B}(\mathscr{X})_{21}$, respectively. Hence, using the [10, Lemma 1.6(P4)] again, we get

$$(((y^{-1})^n)_{11} + ((y^{-1})^n)_{12})(t_{12} + t_{22}) = 0 \text{ and } (t_{11} + t_{12})((y^n)_{12} + (y^n)_{22}) = 0, \quad (3.21)$$

respectively. Adding the first identity, in (3.16), to the first one, in (3.20), we get $(y^{-1})^n(t_{11} + t_{21}) = 0$ which results that $t_{11} + t_{21} = 0$. Thus, $t_{11} = 0$ and $t_{21} = 0$. Also, adding the first identity, in (3.17), to the first one, in (3.21) we get $(y^{-1})^n(t_{12} + t_{22}) = 0$ which shows that $t_{12} + t_{22} = 0$. This implies that $t_{12} = 0$ and $t_{22} = 0$. Therefore, we have $t = 0$. Consequently, by Theorem 2.1, we get the desired result. □

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