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RIGHT ORTHOGONAL CLASS OF PURE PROJECTIVE MODULES OVER PURE HEREDITARY RINGS

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Abstract. We consider the class of all pure projective modules. Present article we investigate the right orthogonal class of all pure projective modules and these modules are defined via the vanishing cohomology of pure projective modules. We discuss the existence of preenvelope and the coresolution of these modules. Further, we show that the right orthogonal class of all pure projective modules is coresolving (injectively resolving) over a pure-hereditary ring and we analyze the dimensions of these modules. Finally, we proved the desirable properties of the dimension when the ring is semisimple artinian.

1. Introduction and preliminaries

In this work, R and R-Mod stand for an associative ring with identity and, respectively, the category of all left R-modules. All R-modules are left R-modules unless otherwise indicated, and $\mathcal{W}$ is the class of all pure projective R-modules.

Enochs first presented the ideas of module (pre)envelopes and (pre)covers in [3]. Since then, a great deal of research has been done on the existence and characteristics of (pre)envelopes and (pre)covers in relation to specific submodule categories. One of the primary study issues in relative homological algebra is the theory of (pre)covers and (pre)envelopes, which is fundamental to the domains of homological algebra and algebraic representation theory.

Let $\mathscr{G}$ represent a class of left R-modules. A homomorphism $g \in \operatorname{Hom}_R(G, M)$ with $G \in \mathscr{G}$ is a $\mathscr{G}$-precover [3] of M, if the group homomorphism

$$\operatorname{Hom}_R(G', g)\colon \operatorname{Hom}_R(G', G) \to \operatorname{Hom}_R(G', M)$$

is surjective for each $G' \in \mathscr{G}$. A $\mathscr{G}$-precover $g \in \operatorname{Hom}_R(G, M)$ of M is called a $\mathscr{G}$-cover of M if g is right minimal, that is, if $gf = g$ implies that g is an automorphism for each $f \in \operatorname{End}_R(G)$. $\mathscr{G} \subseteq R$- Mod is a precovering class (covering class) provided that each module has a $\mathscr{G}$-precover ($\mathscr{G}$-cover). Additionally, we have the dual definition of $\mathscr{G}$ preenvelope ($\mathscr{G}$ envelope).

The conceptions of pure subgroups were first investigated by Prüfer in [10]. Pure subgroups were generalized in numerous approaches in module theory. Pure

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projective modules come behind closely from the ideas of Prüfer's paper [10]. A pure projectiv emodule described as the direct summands of direct sums of finitely presented modules according to Warfield's criterion [18]. When a module is projective with regard to pure exact sequences, it is said to be pure projective [11].

Stenström utilising Ext and FP-injective dimensions of rings and modules in [15] and develop the notion of FP-injective modules. The smallest nonnegative integer t in such a way that an R-module G has FP-injective coresolution of length t is known as the FP-*injective dimension* of G. Mao and Ding establish the idea of $\mathcal{W}$-injective modules, where $\mathcal{W}$ represents the class of R-modules in [8]. In the next section, the definition for the $\mathcal{W}$-injective module (Definition 2.1) is given. The class of all pure projective modules is actually $\mathcal{W}$. It is clear that an R-module is pure projective if it is finitely presented, however this is not necessary for a pure projective module is finitely presented. Therefore, all $\mathcal{W}$-injective modules are FP-injective modules, but the reverse is not always true.

The notion of pure-hereditary rings were defined by Geng and Ding in [5] and the nontrivial generalization of hereditary rings includes all pure-hereditary rings. Given that pure projective modules are generalisations of projective modules and that hereditary rings exist whenever every ideal is projective, the question "when every ideal of a ring is pure-projective" makes sense. This condition is satisfied when a ring is pure-hereditary. All hereditary rings are pure-hereditary but the opposite is not always true since not all pure projective modules must be projective. In relative homological algebra, the classification of coresolutions and the dimensions of modules are significant and fascinating topics. Simson [14] introduced the idea of pure projective resolution and its module dimensions. The main motivation is the coresolutions (that is, right resolutions) and dimensions of the right orthogonal class of pure projective modules over a pure-hereditary ring will be examined in this article. Further, we give the characterisation of a semisimple artinian ring.

Recall that the class $\mathscr{G}$ of R-modules is stated to be *injectively resolving (coresolving)* [7] if the class $\mathcal{I}_0$ of all injective modules such that $\mathcal{I}_0 \subseteq \mathscr{G}$ and if given an exact sequence of left R-modules $0 \to G_1 \to G_2 \to G_3 \to 0$, with $G_1 \in \mathscr{G}$ the conditions $G_2 \in \mathscr{G}$ and $G_3 \in \mathscr{G}$ are equivalent.

An exact sequence of R-modules construct a projective resolution of an R-module G is

$$\cdots \xrightarrow{f_{n+1}} P_n \to \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} G \to 0$$

where each P_i represents a projective R-module. Further, nth *syzygy* of G is $\ker(f_n)$, which is represented by $\Omega_n(G)$. The following exact sequence of R-modules with each I^i is an injective R-module $i \geq 0$,

$$0 \to G \xrightarrow{d^0} I^0 \xrightarrow{d^1} \cdots \xrightarrow{d^n} I^n \to \cdots , \tag{1.1}$$

is called an injective resolution of G. The nth *cosyzygy* of G is $\mathrm{im}(d^{n-1})$ and it is denoted by $\Omega^{-n}(G)$. A $\mathcal{W}^{\perp}$-coresolution of M is an exact sequence $0 \to M \xrightarrow{f^0}$

$G^0 \xrightarrow{f^1} G^1 \xrightarrow{f^2} \cdots \to G^{n-1} \xrightarrow{f^n} G^n \to \cdots$ with $G^i \in \mathcal{W}^\perp$ for each $i \geq 0$. The $\operatorname{im} f^{n-1}$ is called nth $\mathcal{W}^\perp$-*cosyzygy* of M, denoted by $\Omega^{-n}_{\mathcal{W}^\perp}(M)$.

We construct the following from a class $\mathscr{G}$ of left R-modules.

$$\begin{aligned} \mathscr{G}^\perp &= \left\{G \in R\text{-}\operatorname{Mod} \mid \operatorname{Ext}^1_R(C,G)=0, \ \forall\, C \in \mathscr{C}\right\} \\ {}^\perp\mathscr{G} &= \left\{G \in R\text{-}\operatorname{Mod} \mid \operatorname{Ext}^1_R(G,C)=0, \ \forall\, C \in \mathscr{C}\right\}. \end{aligned}$$

The current paper is organized in the following description: In Section 2, we investigate the existence of $\mathcal{W}$-injective coresolution and its dimension. First we show that the existence of $\mathcal{W}$-injective preenvelope and hence we discuss over an arbitrary ring the existence of $\mathcal{W}$-injective coresolution. Further, we give the definition of pure-hereditary ring and we show that over this ring $\operatorname{id}(A) \leq 1$ for all $\mathcal{W}$-injective modules A. Clearly, the class of all $\mathcal{W}$-injective modules is coresolving over a pure-hereditary ring.

In Section 3, first we define $\mathcal{W}$-injective coresolution dimension and then we examine the coresolution dimensions $\operatorname{cores.dim}_{\mathcal{W}^\perp}(-)$ for the class $\mathcal{W}^\perp$ of all $\mathcal{W}$-injective modules. We proved that $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M) \leq m$ if and only if $\Omega^{-n}(M) \in \mathcal{W}^\perp$ for $n \geq m$ if and only if $\Omega^{-n}_{\mathcal{W}^\perp}(M) \in \mathcal{W}^\perp$ for $n \geq m$ over a pure-hereditary ring. In addition, we shown that $\mathcal{W}$-injective coresolution over a pure-hereditary ring possesses the standard homological dimension properties.

In Section 4, we discuss the characterizations of $\mathcal{W}$-injective coresolution dimensions of modules over a pure-hereditary ring. We first demonstrated that $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M) \leq n$ if and only if $\operatorname{Ext}^m_R(G,M)=0$ for all pure projective R-modules G and $m > n$. Further, we define the finitistic $\mathcal{W}$-injective coresolution dimension, which is denoted by $\operatorname{Fcores.dim}_{\mathcal{W}^\perp}(R)$. It is shown that

$$\begin{aligned} \operatorname{Fcores.dim}_{\mathcal{W}^\perp}(R) &= \sup\{\operatorname{cores.dim}_{\mathcal{W}^\perp}(M) \mid M \text{ is any } R\text{-module } \} \\ &= \sup\{\operatorname{pd}(F) \mid F \text{ is a pure projective} R\text{-module } \}. \end{aligned}$$

Finally, we prove that $\operatorname{Fcores.dim}_{\mathcal{W}^\perp}(R) \leq n$ if and only if $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M) \leq n$ for all pure projective R-modules M if and only if $\operatorname{pd}_R(M) \leq n$ for all pure projective R-modules M if and only if $\operatorname{pd}_R(M) \leq n$ for all R-modules M that are both $\mathcal{W}$-injective and pure projective and $\operatorname{Fcores.dim}_{\mathcal{W}^\perp}(R) < \infty$.

In Section 5, we provide the equivalent conditions of the unique mapping property of $\mathcal{W}$-injective envelope. It is proven that R is semisimple artinian ring if and only if there is a $\mathcal{W}$-injective envelope with the unique mapping property for each pure projective R-module if and only if $\operatorname{Ext}^1_R(U,U')=0$ for all pure projective R-modules U and U'; if and only if every pure projective R-module is injective.

2. Existence of $\mathcal{W}$-injective coresolution

The existence of the $\mathcal{W}$-injective preenvelope and coresolution of modules will be investigated in this section. Also, we study the injectively resolving (coresolving) of the class of $\mathcal{W}$-injective modules.

The following is where we start.

Definition 2.1. [17] A left R-module G is called $\mathcal{W}$-injective if $\operatorname{Ext}^1_R(W,G)=0$ for all $W \in \mathcal{W}$. We denote $\mathcal{W}^\perp$ by the class of all $\mathcal{W}$-injective modules.

Proposition 2.2. *The class $\mathcal{W}^{\perp}$ of all $\mathcal{W}$-injective modules is closed under pure submodules.*

Proof. Let A be a pure submodule of a $\mathcal{W}$-injective module M. Then there is a pure exact sequence $0 \to A \to M \to M/A \to 0$ and a functor $\mathrm{Hom}_R(G, -)$ preserves this sequence is exact whenever $G \in \mathcal{W}$. This implies that the sequence $0 \to \mathrm{Hom}_R(G, A) \to \mathrm{Hom}_R(G, M) \to \mathrm{Hom}_R(G, M/A) \to \mathrm{Ext}^1_R(G, A) \to 0$ is also exact for all $G \in \mathcal{W}$. It follows that $Ext^1_R(G, A) = 0$ for all $G \in \mathcal{W}$. Thus A is $\mathcal{W}$-injective. □

Theorem 2.3. *Each R-module provides a $\mathcal{W}$-injective preenvelope.*

Proof. Let M be an R-module. According to [4, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ such that for any R-homomorphism $\psi\colon M \to W$ with W a $\mathcal{W}$-injective R-module, there exists a pure submodule A of W such that $|A| \leq \aleph_\alpha$ and $\psi(M) \subset A$. It is clear that A is $\mathcal{W}$-injective by Proposition 2.2 and because of the class of all $\mathcal{W}$-injective R-modules is closed under direct products. In the consequence, the theorem is proved by [4, Proposition 6.2.1]. □

The reminder of the construction of an injective coresolution of modules from (1.1). We give an analogous sequence using $\mathcal{W}$-injective modules instead of injective modules. According to Theorem 2.3 and [16, Lemma 1.9], every R-module over an arbitrary ring R has a special $\mathcal{W}$-injective preenvelope.

Definition 2.4. A $\mathcal{W}$-injective coresolution of M is an exact sequence of R-modules

$$0 \to M \to G^0 \to G^1 \to \cdots$$

with G^i an $\mathcal{W}$-injective R-module for each $i \geq 0$ and if it still holds exactness after applying the functor $\mathrm{Hom}(-, W)$, where W is $\mathcal{W}$-injective.

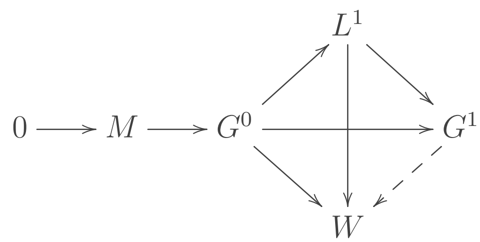
The question arises when all modules have $\mathcal{W}$-injective coresolution. The following description demonstrate this. This is the case over any ring, according to the theorem.

Theorem 2.5. *There is a $\mathcal{W}$-injective coresolution for every R-module M.*

Proof. Assume M is an R-module. According to Theorem 2.3, M has a $\mathcal{W}$-injective preenvelope

$$0 \to M \xrightarrow{f} G^0 \to L^1 \to 0,$$

where G^0 is the $\mathcal{W}$-injective and L^1 is the cokernal of f. Because of the presence of the $\mathcal{W}$-injective preenvelope of M, $\mathrm{Hom}_R(G^0, W) \to \mathrm{Hom}_R(M, W)$ is surjective for all $\mathcal{W}$-injective R-modules W. Now L^1 has $\mathcal{W}$-injective preenvelope G^1, $0 \to L^1 \to G^1 \to L^2 \to 0$. Thus, we are given a commutative diagram



Then the sequence $\mathrm{Hom}_R(G^1, W) \to \mathrm{Hom}_R(G^0, W) \to \mathrm{Hom}_R(M, W) \to 0$ is exact. As we continue in this manner, we obtain an exact sequence.

$$\cdots \to \mathrm{Hom}_R(G^1, W) \to \mathrm{Hom}_R(G^0, W) \to \mathrm{Hom}_R(M, W) \to 0.$$

Therefore, $0 \to M \to G^0 \to G^1 \to \cdots$ is a $\mathcal{W}$-injective coresolution of M. □

For the purposes of continue to developing this article, we require the following definition.

Definition 2.6. [9] A ring R is said to be pure-hereditary, whereas all of its ideals are pure projective.

Remark 2.7. (1) Each hereditary ring is pure-hereditary, however the opposite is not always true.
(2) Each Noetherian ring is pure-hereditary, while the reverse is not always true.

Example 2.8. (1) $\mathbb{Z}$ is a pure-hereditary ring.
(2) $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is a left semihereditary ring and a right (pure) hereditary. It is not left pure-hereditary.

Lemma 2.9. *An injective dimension of an $\mathcal{W}$-injective R-module is at most one over a pure-hereditary ring.*

Proof. Take into consideration an exact sequence $0 \to J \to R \to R/J \to 0$ with J is an ideal of R and consider an R-module U to be $\mathcal{W}$-injective. We consequently obtain an exact sequence $0 \to \mathrm{Ext}^1_R(J, U) \to \mathrm{Ext}^2_R(R/J, U) \to 0$. Therefore $\mathrm{Ext}^1_R(J, U) = 0$ since R is Pure-hereditary. Hence $\mathrm{id}(U) \leq 1$. □

Proposition 2.10. *The class $\mathcal{W}^{\perp}$ of all $\mathcal{W}$-injective modules is coresolving over a Pure-hereditary ring R.*

Proof. Consider $0 \to U_1 \xrightarrow{\phi} U_2 \xrightarrow{\psi} U_3 \to 0$ to be an exact sequence of left R-modules with $U_1, U_2 \in \mathcal{W}^{\perp}$. The sequence $\mathrm{Ext}^1_R(G, U_2) \to \mathrm{Ext}^1_R(G, U_3) \to \mathrm{Ext}^2_R(G, U_1)$ is exact for each R-module $G \in \mathcal{W}$. According to Lemma 2.9, $\mathrm{Ext}^1_R(G, U_3) = 0$. It follows that U_3 is $\mathcal{W}$-injective. □

Lemma 2.11. *Assume that M is a R-module and that R is a Pure-hereditary ring. If the two subsequent sequences*

$$0 \to M \to G^0 \to G^1 \to \cdots \to G^{n-1} \to G^n \to 0$$
$$\text{and } 0 \to M \to H^0 \to H^1 \to \cdots \to H^{n-1} \to H^n \to 0$$

are exact with G^i, $H^i \in \mathcal{W}^\perp$ *where* $0 \leq i \leq n-1$, *then* $G^n \in \mathcal{W}^\perp$ *if and only if* $H^n \in \mathcal{W}^\perp$.

Proof. Consideration is given to the exact sequence of the R-modules below

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-1} \to L^n \to 0$$

with $I^i \in \mathcal{I}$ where $0 \leq i \leq n-1$. From the following complexes

$$\begin{aligned} \mathbb{G}^\bullet &: \quad 0 \to G^0 \to G^1 \to \cdots \to G^{n-1} \to G^n \to 0 \\ \mathbb{H}^\bullet &: \quad 0 \to H^0 \to H^1 \to \cdots \to H^{n-1} \to H^n \to 0 \\ \mathbb{I}^\bullet &: \quad 0 \to I^0 \to I^1 \to \cdots \to I^{n-1} \to L^n \to 0, \end{aligned}$$

we can choose morphisms $\mathbb{G}^\bullet \to \mathbb{I}^\bullet$ and $\mathbb{H}^\bullet \to \mathbb{I}^\bullet$. Then the following two sequences:

$$\text{cone}(\mathbb{G}^\bullet \to \mathbb{I}^\bullet) \colon 0 \to G^0 \to G^1 \oplus I^0 \to \cdots \to G^{n-1} \oplus I^{n-2} \to G^n \oplus I^{n-1} \to L^n \to 0$$

and

$$\text{cone}(\mathbb{H}^\bullet \to \mathbb{I}^\bullet) \colon 0 \to H^0 \to H^1 \oplus I^0 \to \cdots \to H^{n-1} \oplus I^{n-2} \to H^n \oplus I^{n-1} \to L^n \to 0$$

are exact. Let

$$G = \text{im}(G^{n-2} \oplus I^{n-3} \to G^{n-1} \oplus I^{n-2})$$

and

$$H = \text{im}(H^{n-2} \oplus I^{n-3} \to H^{n-1} \oplus I^{n-2}).$$

By Proposition 2.10, $\mathcal{W}^\perp$ is coresolving. Thus G and H are in $\mathcal{W}^\perp$. We get that $G^n \oplus I^{n-1} \in \mathcal{W}^\perp$ if and only if $H^n \oplus I^{n-1} \in \mathcal{W}^\perp$ from the following short exact sequences, $0 \to G \to G^n \oplus I^{n-1} \to L^n \to 0$ and $0 \to H \to H^n \oplus I^{n-1} \to L^n \to 0$, We now take into consideration the exact sequences below:

$$0 \to I^{n-1} \to G^n \oplus I^{n-1} \to G^n \to 0 \text{ and } 0 \to I^{n-1} \to H^n \oplus I^{n-1} \to H^n \to 0.$$

Then $G^n \in \mathcal{W}^\perp$ if and only if $G^n \oplus I^{n-1} \in \mathcal{W}^\perp$ and $H^n \in \mathcal{W}^\perp$ if and only if $H^n \oplus I^{n-1} \in \mathcal{W}^\perp$. Hence $G^n \in \mathcal{W}^\perp$ if and only if $H^n \in \mathcal{W}^\perp$. □

3. $\mathcal{W}$-injective dimension

This section examines the homological dimensions for a coresolving class of a right orthogonal class of pure projective modules are investigated. Examining specifically a coresolution dimensions $\text{cores.}\dim_{\mathcal{W}^\perp}(-)$ for a coresolving class $\mathcal{W}^\perp$.

Theorem 2.5 states that we can handle the $\mathcal{W}$-injective coresolution dimension of an R-module. Further, $\mathcal{W}$-injective coresolution dimension of an R-module M is indicated by the symbol $\text{cores.}\dim_{\mathcal{W}^\perp}(M)$ and is the least nonnegative integer n with M has a $\mathcal{W}$-injective coresolution of length n. In other words, for any pure projective R-module G, $\text{Ext}_R^{n+1}(G, M) = 0$. Set $\text{cores.}\dim_{\mathcal{W}^\perp}(M) = \infty$ if such a n does not exists.

Example 3.1. Consider the sequence $0 \to U \to G^0 \to 0$ with G^0 is $\mathcal{W}$-injective. From this, $U \cong G^0$ is a $\mathcal{W}$-injective coresolution of U and its length is zero if U is $\mathcal{W}$-injective. Hence $\mathcal{W}$-injective R-module has a $\mathcal{W}$-injective coresolution dimension of zero. To put it another way, $\operatorname{cores.dim}_{\mathcal{W}^\perp}(U) = 0$ if U is $\mathcal{W}$-injective. The reverse implication is also true. If $0 \to U \to G^0 \to 0$ is a $\mathcal{W}$-injective coresolution of U of length zero, then $U \cong G^0$. Hence U is $\mathcal{W}$-injective.

From Lemma 2.11, we instantly discover the next.

Proposition 3.2. *Assume R is a Pure-hereditary ring and M is an R-module. Consequently, the subsequent conditions are equivalent:*

(1) $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M) \leq m$;
(2) $\Omega^{-n}(M) \in \mathcal{W}^\perp$ *for* $n \geq m$;
(3) $\Omega^{-n}_{\mathcal{W}^\perp}(M) \in \mathcal{W}^\perp$ *for* $n \geq m$.

The subclass of R-Mod whose elements have finite $\mathcal{W}^\perp$-coresolution dimensions is denoted by $\widetilde{\mathcal{W}^\perp}$. Then comes what concerns next

Lemma 3.3. *Assume R is a Pure-hereditary ring. Consequently, the following holds true:*

(1) *Suppose the sequence $0 \to M \to G \to N \to 0$ of R-modules is exact with $G \in \mathcal{W}^\perp$. Hence $M \in \widetilde{\mathcal{W}^\perp}$ if and only if $N \in \widetilde{\mathcal{W}^\perp}$. In this case, either all three objects are in $\mathcal{W}^\perp$ or* $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(N) + 1$.
(2) *Suppose the exact sequence of R-modules with $G \in \mathcal{W}^\perp$ is $0 \to G \to M \to N \to 0$. Hence $M \in \widetilde{\mathcal{W}^\perp}$ if and only if $N \in \widetilde{\mathcal{W}^\perp}$ and* $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(N)$.
(3) *Suppose the exact sequence of R-modules with $G \in \mathcal{W}^\perp$ is $0 \to M \to N \to G \to 0$. Hence $M \in \widetilde{\mathcal{W}^\perp}$ if and only if $N \in \widetilde{\mathcal{W}^\perp}$ and* $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(N)$, *except the case of $N \notin \mathcal{W}^\perp$ and $M \in \mathcal{W}^\perp$.*

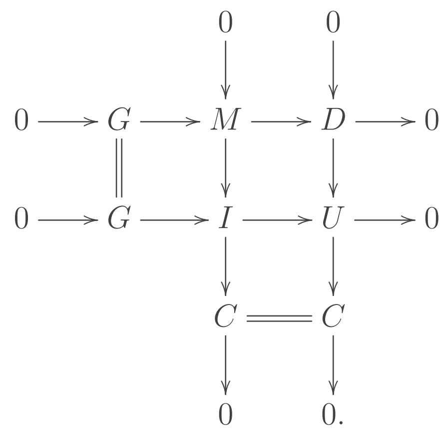
Proof. (1). Theorem 2.5 states that an $\mathcal{W}^\perp$-coresolution exists for an R-module N. Let the $\mathcal{W}^\perp$-coresolution of N be $0 \to N \to G^0 \to G^1 \to \cdots$. Consequently, $0 \to M \to G \to G^0 \to G^1 \to \cdots$ be an $\mathcal{W}^\perp$-coresolution of M. Therefore the following inequality

$$\operatorname{cores.dim}_{\mathcal{W}^\perp}(N) \leq \operatorname{cores.dim}_{\mathcal{W}^\perp}(M) \leq \operatorname{cores.dim}_{\mathcal{W}^\perp}(N) + 1$$

holds by Proposition 3.2. Thus $N \in \widetilde{\mathcal{W}^\perp}$ if and only if $M \in \widetilde{\mathcal{W}^\perp}$. Now we show that either all the three objects are in $\mathcal{M}$ or $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(N)+1$. The assertion is vacuously true when any one of the objects is zero. Suppose all the three objects are non zero. That is $\mathcal{W}^\perp$-coresolution dimension of all objects are non negative. There is nothing to verify when one of $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M)$ and $\operatorname{cores.dim}_{\mathcal{W}^\perp}(N)$ is infinite. We shall assume that $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M)$ and $\operatorname{cores.dim}_{\mathcal{W}^\perp}(N)$ are finite. If $M \in \mathcal{W}^\perp$, then all the three objects are in $\mathcal{W}^\perp$ since $N \in \mathcal{W}^\perp$. If $M \notin \mathcal{W}^\perp$, let t_1 be a $\operatorname{cores.dim}_{\mathcal{W}^\perp}(M)$ and t_2 be a $\operatorname{cores.dim}_{\mathcal{W}^\perp}(N)$. Consider $0 \to N \to G^0 \to G^1 \to \cdots \to G^n \to 0$ to be an $\mathcal{W}^\perp$-coresolution of N. It follows that $0 \to M \to G \to G^0 \to G^1 \to \cdots \to G^n \to 0$ is an $\mathcal{W}^\perp$-coresolution of M. Therefore $m \leq t_2+1$. If $t_1 < t_2+1$, then $\Omega^{-m}_{\mathcal{W}^\perp}(M) = \operatorname{im}(G^{t_1-2} \to G^{t_1-1}) \in \mathcal{W}^\perp$

by Proposition 3.2. This is in contradiction to $\operatorname{cores.dim}_{\mathcal{W}^{\perp}}(N) = t_2$. Hence $t_1 = t_2 + 1$.

(2). We consider the pushout diagram of $M \to I$ and $M \to D$ as well as we take into account an exact sequence $0 \to M \to I \to C \to 0$ with $I \in \mathcal{I}$:



In the middle row, $U \in \mathcal{W}^{\perp}$ since $\mathcal{W}^{\perp}$ is coresolving. From the right column and the middle column, D has finite $\mathcal{W}$-injective coresolution dimension if and only if C has finite $\mathcal{W}$-injective coresolution dimension if and only if M has finite $\mathcal{W}$-injective coresolution dimension.

(3) Similar to the proof of (2). □

Proposition 3.4. *Suppose the sequence $0 \to J_1 \to J_2 \to J_3 \to 0$ of R-modules is exact over a pure-hereditary ring R. Then the third is in the $\widetilde{\mathcal{W}^{\perp}}$ if any two of J_1, J_2 and J_3 are in $\widetilde{\mathcal{W}^{\perp}}$.*

Proof. We let the following

$$n = \min\{\operatorname{cores.dim}_{\mathcal{W}^{\perp}}(J_1), \operatorname{cores.dim}_{\mathcal{W}^{\perp}}(J_2), \operatorname{cores.dim}_{\mathcal{W}^{\perp}}(J_3)\}.$$

Clearly, $n < \infty$. Next, we look at the commutative diagram based on the Horseshoe Lemma.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_1 & \longrightarrow & J_2 & \longrightarrow & J_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^0 & \longrightarrow & G^0 \oplus H^0 & \longrightarrow & H^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^{n-1} & \longrightarrow & G^{n-1} \oplus H^{n-1} & \longrightarrow & H^{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1^n & \longrightarrow & L_2^n & \longrightarrow & L_3^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with G^i and H^i are $\mathcal{W}$-injective modules where $0 \leq i \leq n-1$. Consider the last row $0 \to L_1^n \to L_2^n \to L_3^n \to 0$. By the assumption and Proposition 3.2, at least one of L_1^n, L_2^n and L_3^n is in $\mathcal{W}^{\perp}$. Since any two of J_1, J_2 and J_3 are in $\widetilde{\mathcal{W}^{\perp}}$, two of L_1^n, L_2^n and L_3^n are in $\mathcal{W}^{\perp}$ by Lemma 2.11. Therefore all the three objects L_1^n, L_2^n and L_3^n are in $\mathcal{W}^{\perp}$ by Lemma 3.3. But $L_i^n \in \widetilde{\mathcal{W}^{\perp}}$ if and only if $J_i \in \widetilde{\mathcal{W}^{\perp}}$ by Lemma 2.11 for each $i \in \{1, 2, 3\}$. Hence it completes the proof. □

Proposition 3.5. *Let J_1, J_2 and J_3 be R-modules in $\widetilde{\mathcal{W}^{\perp}}$ over a pure-hereditary ring R. Consequently the subsequent conditions are true.*

(1) *Suppose the sequence $0 \to J_1 \to J_2 \to J_3 \to 0$ of R-modules is exact. Then*
$$\text{cores.}\dim_{\mathcal{W}^{\perp}}(J_2) \leq \max\{\text{cores.}\dim_{\mathcal{W}^{\perp}}(J_1), \text{cores.}\dim_{\mathcal{W}^{\perp}}(J_3)\}$$
with strict inequality possible only if
$$\text{cores.}\dim_{\mathcal{W}^{\perp}}(J_1) = \text{cores.}\dim_{\mathcal{W}^{\perp}}(J_3) + 1.$$

(2) *Suppose the sequence $0 \to J_1 \to J_2 \to J_3 \to 0$ of R-modules is exact. Consequently, we get*
$$\text{cores.}\dim_{\mathcal{W}^{\perp}}(J_1) \leq \max\{\text{cores.}\dim_{\mathcal{W}^{\perp}}(J_3), \text{cores.}\dim_{\mathcal{W}^{\perp}}(J_2)\} + 1$$

(3) *Suppose the exact sequences $0 \to J_1 \to J_3 \to J_2 \to 0$ and $0 \to J_2 \to J_3 \to J_1 \to 0$ of R-modules with satisfies $J_3 = J_1 \oplus J_2$. Then*
$$\text{cores.}\dim_{\mathcal{W}^{\perp}}(J_3) = \max\{\text{cores.}\dim_{\mathcal{W}^{\perp}}(J_1), \text{cores.}\dim_{\mathcal{W}^{\perp}}(J_2)\}.$$

Proof. We first demonstrate condition (1), and condition (2) is then similarly demonstrated to condition (1).

(1). With the use of the analogue proof for Proposition 3.4 and the Horseshoe Lemma,

$$\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_2) \le \max\{\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1), \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_3)\}.$$

It remains to show that the above equation with strict inequality is possible only if $\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_3) + 1$. Suppose

$$n = \min\{\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1), \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_3)\}$$

$$N = \max\{\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1), \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_3)\}$$

and $\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_2) = t$. If $t \le n$, we get an exact sequence according to the Horseshoe Lemma

$$0 \to \Omega^{-t}(J_1) \to \Omega^{-t}(J_2) \to \Omega^{-t}(J_3) \to 0.$$

Thus $\Omega^{-t}(J_3) \notin \mathcal{W}^\perp$ and $\Omega^{-t}(J_2) \in \mathcal{W}^\perp$ when $t < n$. By Lemma 3.3,

$$\operatorname{cores.dim}_{\mathcal{W}^\perp}(\Omega^{-n}(J_1)) = \operatorname{cores.dim}_{\mathcal{W}^\perp} \Omega^{-n}(J_3) + 1$$

and hence $\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_3)+1$. If $t = n$, then $\Omega^{-n}(J_2) \in \mathcal{W}^\perp$. Hence by Lemma 3.3, either $\operatorname{cores.dim}_{\mathcal{W}^\perp}(\Omega^{-n}(J_1)) = \operatorname{cores.dim}_{\mathcal{W}^\perp} \Omega^{-n}(J_3)+1$ or both $\Omega^{-n}(J_1)$ and $\Omega^{-n}(J_3)$ are in $\mathcal{W}^\perp$. Thus $\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_3)+1$. If $n < t$, then again become an exact sequence of the form $0 \to \Omega^{-t}(J_1) \to \Omega^{-t}(J_2) \to \Omega^{-t}(J_3) \to 0$ such that $\Omega^{-n}(J_2) \notin \mathcal{W}^\perp$ and either $\Omega^{-n}(J_1)$ or $\Omega^{-n}(J_3) \in \mathcal{W}^\perp$. If $\Omega^{-n}(J_1) \in \mathcal{W}^\perp$, then $\Omega^{-n}(J_3) \notin \mathcal{W}^\perp$. By Lemma 3.3(2),

$$\operatorname{cores.dim}_{\mathcal{W}^\perp}(\Omega^{-n}(J_2)) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(\Omega^{-n}(J_3))$$

and hence $\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_2) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_3) = N$. If $\Omega^{-n}(J_3) \in \mathcal{W}^\perp$, then $\Omega^{-n}(J_1) \notin \mathcal{W}^\perp$. By Lemma 3.3 (3), $\operatorname{cores.dim}_{\mathcal{W}^\perp}(\Omega^{-n}(J_1)) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(\Omega^{-n}(J_2))$ and hence

$$\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_2) = N.$$

This concludes the proof.

(3). Following by (1),

$$\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1 \oplus J_2) \le \max\{\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1), \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_2)\}$$

with strict inequality is possible only if

$$\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_2) \pm 1.$$

Now we only to show that

$$\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1 \oplus J_2) = \max\{\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1), \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_2)\}$$

if $\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_2) \pm 1$. Suppose

$$\operatorname{cores.dim}_{\mathcal{W}^\perp}(J_1) = \operatorname{cores.dim}_{\mathcal{W}^\perp}(J_2) + 1 = n + 1.$$

Then there are two exact sequences $0 \to J_1 \xrightarrow{t_0} I_0 \xrightarrow{t_1} \cdots \xrightarrow{t_n} I_n \xrightarrow{t_{n+1}} G_{n+1} \to 0$ and $0 \to J_2 \xrightarrow{r_0} E_0 \xrightarrow{r_1} \cdots \xrightarrow{r_{n-1}} E_{n-1} \xrightarrow{r_n} H_n \to 0$ with all I_j and E_k being injective for every $j \in \{0, 1, \cdots, n\}$ and $k \in \{0, 1, \cdots n-1\}$, G_{n+1} and H_n being in $\mathcal{W}^\perp$. Hence

$$0 \to J_1 \oplus J_2 \xrightarrow{t_0 \oplus r_0} I_0 \oplus E_0 \xrightarrow{t_1 \oplus r_1} \cdots \to I_n \oplus E_n \xrightarrow{t_n \oplus r_n} I_n \oplus H_n \xrightarrow{0 \oplus t^{n+1}} G_{n+1} \to 0$$

is an $\mathcal{W}^{\perp}$-coresolution of $J_1 \oplus J_2$. If $\operatorname{cores.dim}_{\mathcal{W}^{\perp}}(J_1 \oplus J_2) = m < n+1$, then by Proposition 3.2, $\operatorname{im}(I_{n-1} \oplus E_{n-1} \overset{t_{n-1}\oplus r_{n-1}}{\longrightarrow} I_n \oplus E_n) \in \mathcal{W}^{\perp}$ for $m \leq n$. Thus $\Omega^{-m}(J_1) \oplus \Omega^{-m}(J_2) \in \mathcal{W}^{\perp}$. Consequently, $\Omega^{-m}(J_2) \in \mathcal{W}^{\perp}$ because under direct summands of the class $\mathcal{W}^{\perp}$ is closed. This follows that $\operatorname{cores.dim}_{\mathcal{W}^{\perp}}(J_2) \leq m < n+1$. This is a contradiction to our assumption. Thus $\operatorname{cores.dim}_{\mathcal{W}^{\perp}}(J_1 \oplus J_2) = \max\{\operatorname{cores.dim}_{\mathcal{W}^{\perp}}(J_1), \operatorname{cores.dim}_{\mathcal{W}^{\perp}}(J_2)\}$.

□

4. $\mathcal{W}$-injective dimensions with derived functors

We describe the characterizations of the $\mathcal{W}$-injective coresolution dimension.

Proposition 4.1. *The following are equivalent if $M \in \widetilde{\mathcal{W}^{\perp}}$ and an integer $n \geq 0$ over a Pure-hereditary ring R.*

(1) $\operatorname{cores.dim}_{\mathcal{W}^{\perp}}(M) \leq n$;
(2) $\operatorname{Ext}_R^m(G, M) = 0$ *for all pure projective R-modules G and $m > n$;*
(3) $\Omega^{-m}(M) \in \mathcal{W}^{\perp}$ *for all $m \geq n$;*
(4) $\Omega^{-m}_{\mathcal{W}^{\perp}}(M) \in \mathcal{W}^{\perp}$ *for all $m \geq n$;*
(5) *If there is an exact sequence $0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0$ with $G^i \in \mathcal{W}^{\perp}$ where $0 \leq i \leq n$, then G^n is $\mathcal{W}$-injective;*
(6) $\Omega_n(N) \in {}^{\perp \geq}M$ *for all pure projective R-modules N;*
(7) $\operatorname{cores.dim}_{\mathcal{W}^{\perp}}(N) \leq n$ *for all $N \in ({}^{\perp\geq}M)^{\perp\geq}$;*
(8) *There exists an exact sequence $0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0$, where each G^i is $\mathcal{W}$-injective.*

Proof. (1) ⇒ (2). Consider M to be an R-module. Then, we have the exact sequence of R-modules

$$0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0$$

with $G^i \in \mathcal{W}^{\perp}$ where $0 \leq i \leq n$. This implies that, for any pure projective R-module G and for $m > n$,

$$\operatorname{Ext}_R^m(G, M) \cong \operatorname{Ext}_R^{m-n}(G, G^n)$$

is given by Lemma 2.9. Then $\operatorname{Ext}_R^{n+1}(G, M) = 0$ because G^n is $\mathcal{W}$-injective and if we take $m = n+1$. Further, if $m > n+1$, then $\operatorname{Ext}_R^m(G, M) \cong \operatorname{Ext}_R^{m-n}(G, G^n) = 0$ for $m > n$ by Lemma 2.9.

(2) ⇒ (1) is trivial.

The following are hold from an isomorphism $\operatorname{Ext}_R^m(G, E_n) \cong \operatorname{Ext}_R^{m-n}(G, M)$,

$$(2) \Leftrightarrow (5) \text{ and } (2) \Leftrightarrow (8).$$

By Proposition 3.2, (1) ⇔ (3) and also (3) ⇔ (4) are hold.

(1) ⇒ (6). Assume that R-module N is pure projective. Then we have an exact sequence with P_i a projective R-module and $0 \leq i \leq n-1$

$$0 \to \Omega_n(N) \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0.$$

Then, for any $j \geq 1$, $\operatorname{Ext}_R^j(\Omega_n(N), M) \cong \operatorname{Ext}_R^{n+j}(N, M) = 0$. Consequently, $\Omega_n(N) \in {}^{\perp\geq}M$.

(6) $\Rightarrow$ (7). Assume that G is a pure projective R-module. Therefore, according to the hypothesis, $\Omega_n(G) \in {}^{\perp\geq}M$. This implies that $\Omega_n(G) \in {}^{\perp\geq}N$ for any $N \in {}^{\perp\geq}(M^{\perp\geq})$ because ${}^{\perp\geq}M = {}^{\perp\geq}(({}^{\perp\geq}M)^{\perp\geq}) \subseteq {}^{\perp\geq}N$. Thus, for $j \geq 1$,

$$\mathrm{Ext}_R^{n+j}(G, N) \cong \mathrm{Ext}_R^{j}(\Omega_n(G), N) = 0.$$

By using the (2) $\Leftrightarrow$ (4) condition, $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(N) \leq n$.

(7) $\Rightarrow$ (1) is trivial. □

The definition of the *finitistic $\mathcal{W}$-injective coresolution dimension* is

$$\sup\{\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(U) \colon U \in \widetilde{\mathcal{W}^\perp}\}.$$

It is identified by $\mathrm{Fcores.\,dim}_{\mathcal{W}^\perp}(R)$.

Proposition 4.2. *Assume R is a Pure-hereditary ring and take into consideration an exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$. Then* cores. dim *of the third must also be a finite in $\mathcal{W}^\perp$ if any two of* $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_1)$, $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_2)$ *and* $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_3)$ *are finite. Moreover,*

1. $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_2) \leq \sup\{\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_1), \mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_3)\}$;
2. $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_1) \leq \sup\{\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_2), \mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_3) + 1\}$;
3. $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_3) \leq \sup\{\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_2), \mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M_1) - 1\}$.

Proof. By Proposition 3.4 and Proposition 4.1. □

Theorem 4.3. *Assume R is a Pure-hereditary ring. In such case, the following are then equivalent:*

1. $\mathrm{Fcores.\,dim}_{\mathcal{W}^\perp}(R)$;
2. $\sup\{\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M) \colon M \in \widetilde{\mathcal{W}^\perp}\}$;
3. $\sup\{\mathrm{pd}_R(U) \colon U$ *is a pure projective R-module*$\}$;
4. $\sup\{\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M) \colon M$ *is an R-module*$\}$.

Proof. The argument of the proof of (1) = (2) and (4) $\leq$ (2) are clear.

(2) $\leq$ (3). Suppose $\sup\{\mathrm{pd}(U) \colon U$ is a pure projective R-module$\} = t < \infty$. Consider an R-module M and assume that U is a pure projective R-module. We have, $\mathrm{Ext}_R^{t+1}(U, M) = 0$ since $\mathrm{pd}(U) \leq t$. Hence $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M) \leq t$.

(3) $\leq$ (1). Consider U to be a pure projective R-module and then consider its projective resolution: $\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to U \to 0$. We need to prove that $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(U) \leq \mathrm{Fcores.\,dim}_{\mathcal{W}^\perp}(R)$. Consider that $\mathrm{Fcores.\,dim}_{\mathcal{W}^\perp}(R) = n < \infty$ and V to be an R-module. Following that, (1) provides the sequence which is exact,

$$0 \to V \to G^0 \to G^1 \to \cdots \to G^{n-1} \to G^n \to 0$$

with $G^i \in \mathcal{W}^\perp$. We get a double complex

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & & & 0 & & \\
 & & & & \downarrow & & & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathrm{Hom}(P_0, V) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}(P_n, V) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(U, G^0) & \longrightarrow & \mathrm{Hom}(P_0, G^0) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}(P_n, G^0) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(U, G^{n-1}) & \longrightarrow & \mathrm{Hom}(P_0, G^{n-1}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}(P_n, G^{n-1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(U, G^n) & \longrightarrow & \mathrm{Hom}(P_0, G^n) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}(P_n, G^n) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & & & 0. & &
\end{array}
$$

Since from the above diagram, U is pure projective and all G^i's are $\mathcal{W}$-injective. Consequently, excluding the top row, every row is exact and from the projectivity of P_i's each column is exact other than the left column.

In the aforementioned double complex, we are taking into consideration two complexes,

$$0 \to \mathrm{Hom}(M, G^0) \to \mathrm{Hom}(M, G^1) \to \cdots \to \mathrm{Hom}(M, G^n) \to 0$$

and

$$0 \to \mathrm{Hom}(P_0, V) \to \mathrm{Hom}(P_1, V) \to \cdots \to \mathrm{Hom}(P_n, V) \to \cdots .$$

Then, we have isomorphic homology groups since by the argument of spectral sequence. For all $j \geq 1$, $\mathrm{Ext}_R^{n+j}(M, V) = 0$. Consequently, $\mathrm{pd}(M) \leq n$.

$(2) \leq (4)$. Consider M to be an R-module and suppose $\sup\{\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(H) \colon H$ is a pure projective R-module$\} = m < \infty$. However, a pure projective preenvelope is admissible for any R-module. It follows that we consider the following short exact sequence

$$0 \to M \to H \to H/M \to 0,$$

where H is pure projective. By the second condition of Proposition 4.2,

$$\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M) \leq \mathrm{cores.\,dim}_{\mathcal{W}^\perp}(F) \leq m.$$

□

Corollary 4.4. *The following equivalent conditions are true for an integer $n \geq 0$ over a pure-hereditary ring R:*

(1) $\mathrm{Fcores.\,dim}_{\mathcal{W}^\perp}(R) \leq n$;
(2) $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M) \leq n$ *for all pure projective R-modules M*;

(3) $\mathrm{pd}_R(M) \leq n$ *for all pure projective* R*-modules* M;
(4) $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M) \leq n$ *for all projective* R*-modules* M, *and* $\mathrm{Fcores.\,dim}_{\mathcal{W}^\perp}(R) < \infty$;
(5) $\mathrm{pd}_R(M) \leq n$ *for all* R*-modules* M *that are both pure projective and* $\mathcal{W}$*-injective and* $\mathrm{Fcores.\,dim}_{\mathcal{W}^\perp}(R) < \infty$;
(6) $\mathrm{Ext}_R^{n+1}(M, N) = 0$ *for all pure projective* R*-modules* M *and* N*;*
(7) $\mathrm{Ext}_R^{n+i}(M, N) = 0$ *for all pure projective* R*-modules* M, N *and* $i \geq 1$.

Proof. We merely need to demonstrate that (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3).

(4) $\Rightarrow$ (2). Consider M to be a pure projective R-module. Theorem 4.3 (4) states that $\mathrm{pd}_R(M) = m$ for some integer $m \geq 0$ since $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(R) < \infty$. Hence M has a projective resolution

$$0 \to P_m \to P_{m-1} \to \ldots \to P_1 \to P_0 \to M \to 0.$$

Note that $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(P_i) \leq n$ for each P_i by (5). Hence by Proposition 4.1, $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M) \leq n$.

(5) $\Rightarrow$ (3). Assume that an R-module M is pure projective. We deduce that $\mathrm{cores.\,dim}_{\mathcal{W}^\perp}(M) = l$ for some non-negative integer l from Theorem 4.3 (2) and the fact that $\mathrm{Fcores.\,dim}_{\mathcal{W}^\perp}(R) < \infty$. Consequently, according to [17, Theorem 6.2] every R-module has an $\mathcal{W}$-injective preenvelope. An exact sequence of R-modules with each E_i is both pure projective and $\mathcal{W}$-injective is then obtained:

$$0 \to M \to E_0 \to E_1 \to \ldots \to E_l \to 0.$$

It provided that $\mathrm{pd}_R(M) \leq n$ since $\mathrm{pd}_R(E_i) \leq n$ for each $i \in \{0, 1, \cdots, l\}$.

□

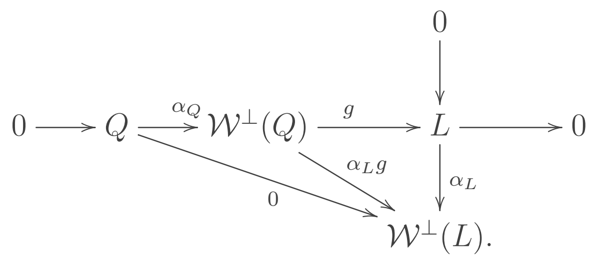
5. Application

Notice that an injective envelope $\alpha_G : G \to E(G)$ of G has the unique mapping property in [2] if there exists a unique homomorphism $f : E(G) \to G'$ such that $f\alpha_G = g$ for any homomorphism $g : G \to G'$ with G' an injective module. Similarly, we can define the unique mapping property of $\mathcal{W}$-injective envelope. The equivalent conditions for the unique mapping property of the $\mathcal{W}$-injective envelope are now provided.

Proposition 5.1. *The following specifications apply to any ring* R *and are equivalent.*

(i) R *is semisimple artinian.*
(ii) $\mathrm{Ext}_R^1(U, U') = 0$ *for all pure projective* R*-modules* U *and* U';
(iii) *Every pure projective* R*-module has an* $\mathcal{W}$*-injective envelope with the unique mapping property;*
(iv) *Every pure projective* R*-module is* $\mathcal{W}$*-injective;*
(v) *Every pure projective* R*-module is injective.*
(vi) *Every projective* R*-module is* $\mathcal{W}$*-injective;*
(vii) *Every pure projective* R*-module has an injective envelope with the unique mapping property;*

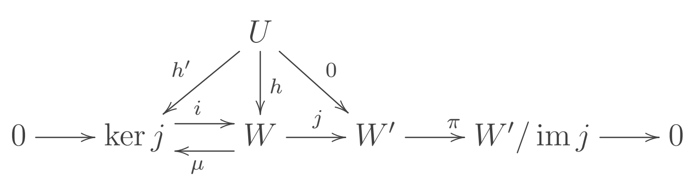
Proof. *(iii)* $\Rightarrow$ *(iv)*. Assume that Q is a pure projective R-module. Then there is a diagram



is commutative with exact row. Note that $\alpha_L g \alpha_Q = 0 = 0\alpha_Q$. By (iii), $\alpha_L g = 0$ and hence $L = \operatorname{im} g \subseteq \ker\ \alpha_L = 0$ since α_L is monic. Consequently, Q is $\mathcal{W}$-injective.

It is clear to the proof of $(iv) \Leftrightarrow (ii)$.

$(ii) \Rightarrow (iii)$. Consider a pure projective R-module U. According to [17, Theorem 6.2], U has a $\mathcal{W}$-injective envelope $h\colon U \to W$, where W is $\mathcal{W}$-injective. To show that the sufficient condition for this, for any homomorphism $j\colon W \to W'$ with $W' \in \mathcal{W}^{\perp}$ in such a way that $jh = 0$, thus $j = 0$. Clearly, there exists $h'\colon U \to \ker j$ with $ih' = h$ because $\operatorname{im} h \subseteq \operatorname{im} j$, where $i\colon \ker j \to W$ is the inclusion map. For every pure projective R-module H, $\operatorname{Ext}^1_R(H, L) = 0$ by (ii). Consequently, according to Corollary 4.4, $\operatorname{pd}_R(H) = 0$. Therefore, for every pure projective R-module H would also be true the condition $\operatorname{Ext}^1_R(H, \ker j) = 0$ and $\ker j$ is hence $\mathcal{W}$-injective. Thus there exists $\mu\colon W \to \ker j$ such that $h' = \mu h$. A commutative diagram is then presented below with an exact row.



Thus $(i\mu)h = i(\mu h) = ih' = h$, and hence $i\mu$ is an isomorphism. Hence $j = 0$ since i is an onto homomorphism.

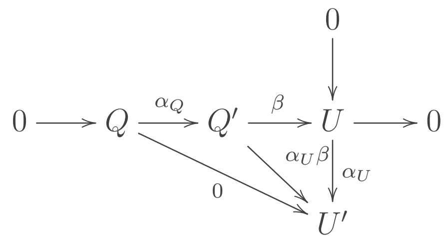
$(ii) \Rightarrow (vi)$. By Corollary 4.4, $\operatorname{Fcores.dim}_{\mathcal{W}^{\perp}}(R) = 0$. Therefor, for any projective R-modules Q, $\operatorname{cores.dim}_{\mathcal{W}^{\perp}}(Q) = 0$. Thus, any projective R-module becomes into $\mathcal{W}$-injective.

$(vi) \Rightarrow (ii)$. Consider a pure projective R-module U. Thus, $\operatorname{cores.dim}_{\mathcal{W}^{\perp}}(U) = 0$ by following (vi). Therefore, all pure projective R-modules U and U' have $\operatorname{Ext}^1_R(U, U') = 0$ according to Corollary 4.4.

$(v) \Rightarrow (ii)$ is clear.

$(ii) \Rightarrow (vii)$ is comparable with the proof of $(3) \Rightarrow (1)$.

$(vii) \Rightarrow (v)$. Assume that an R-module Q is pure projective. Consequently, the diagram is provided



commutative with the exact row, where Q' and U' are the injective envelope of Q and U, respectively. Note that $\alpha_U \beta \alpha_Q = 0 = 0\alpha_Q$. By (vii), $\alpha_U \beta = 0$ and hence $U = im(\beta) \subseteq \ker \alpha_U = 0$ since α_U is monic. Thus M is injective since α_M is an isomorphism.

$(i) \Rightarrow (v)$ is trivial.

$(v) \Rightarrow (i)$. An R-module L perfectly matches into an exact sequence of the type $0 \to G \to G \to L \to 0$ with L is pure projective because a direct limit of finitely presented modules exist for an R-module according to [13, Lemma 2]. Hence the sequence splits. This follows that L is also injective since by hypothesis. Hence R is semisimple artinian. □

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