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ON THE SPECTRUM OF THE LACUNARY CESÀRO OPERATOR

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ABSTRACT. In this paper, we extend the classical Cesàro operator by introducing the concept of the lacunary Cesàro operator, which incorporates the idea of lacunary sequences. We examine the operator's matrix representation and its adjoint, establishing the boundedness of these operators on Banach spaces, including the spaces of sequences that converge to zero, absolutely summable sequences, and bounded sequences. Additionally, we determine the spectrum of the operator, showing that it adapts to the structure of the underlying lacunary sequence. Notably, the spectral radius remains equal to one for a wide range of lacunary sequences.

1. Introduction

In their seminal 1965 paper on Cesàro operators, Brown, Halmos, and Shields[1] studied the operator that converts a sequence $(x_n)_{n\geq 1}$ into its sequence of averages. They demonstrated that this operator is bounded on the Hilbert space ℓ^2 of square-summable sequences, determined its spectrum, and explored its eigenvalues.In the study by J. B. Reade [9], the spectrum of the classical Cesàro operator was investigated on different Banach spaces. The Cesàro operator, denoted by C, was defined as an averaging operator acting on sequences, with its action represented by a lower triangular matrix. The author analyzed the operator's properties when acting on c_0 (the space of null sequences), ℓ^1 (the space of absolutely summable sequences), and ℓ^∞ (the space of bounded sequences). The spectrum of C was shown to depend on the norm and structure of the underlying space, with specific emphasis on determining whether $C - \lambda I$ is invertible for different values of λ. Additionally, the adjoint operator C^* and its spectral properties were explored, revealing that the eigenvalues of C^* form a subset of the spectrum of C. This foundational work on the Cesàro operator provides a basis for generalizations such as the lacunary Cesàro operator.

In this paper, we extend their analysis to the *lacunary Cesàro operator*, a generalization that incorporates the concept of lacunary sequences. A lacunary sequence is defined as an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$,

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$h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$, and the intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$. The ratio $\frac{k_r}{k_{r-1}}$ is abbreviated as q_r, with $q_1 = k_1$.

The *lacunary Cesàro operator* C_θ maps a sequence $x = (x_n)_{n \geq 1}$ to its lacunary averages over the intervals I_r:

$$C_\theta(x)_r = \frac{1}{h_r} \sum_{n \in I_r} x_n, \quad h_r = k_r - k_{r-1}.$$

This operator extends the classical Cesàro operator by replacing uniform averaging with averaging over lacunary intervals.

The matrix representation of C_θ is given by:

$$c_{r,n} = \begin{cases} \frac{1}{h_r}, & \text{if } n \in I_r = (k_{r-1}, k_r], \\ 0, & \text{otherwise.} \end{cases}$$

In matrix form:

$$C_\theta = \begin{bmatrix} \frac{1}{h_1} & \frac{1}{h_1} & \cdots & \frac{1}{h_1} & 0 & 0 & 0 & \cdots & \\ 0 & 0 & \cdots & 0 & \frac{1}{h_2} & \frac{1}{h_2} & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{h_3} & \frac{1}{h_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The adjoint C_θ^* of C_θ is defined via the relationship:

$$\langle C_\theta x, y \rangle = \langle x, C_\theta^* y \rangle.$$

Its matrix representation is the transpose of C_θ:

$$(C_\theta^*)_{n,r} = (C_\theta)_{r,n}.$$

The matrix of C_θ^* is:

$$C_\theta^* = \begin{bmatrix} \frac{1}{h_1} & 0 & 0 & 0 & \cdots \\ \frac{1}{h_1} & 0 & 0 & 0 & \cdots \\ \cdots & \frac{1}{h_2} & 0 & 0 & \cdots \\ 0 & \frac{1}{h_2} & 0 & 0 & \cdots \\ \cdots & \cdots & \frac{1}{h_3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In this work, we aim to:

(1) Analyze the boundedness of C_θ on Banach spaces such as c_0 (null sequences), ℓ^1 (absolutely summable sequences), and ℓ^∞ (bounded sequences).
(2) Determine the spectrum $\sigma(C_\theta)$, which depends on the lacunarity of θ.
(3) Explore the relationship between C_θ and its adjoint C_θ^*, particularly the eigenvalues of C_θ^* in relation to lacunary intervals.

The structure of this paper is as follows: Section 2 discusses the general properties of C_θ and its adjoint. Section 3 derives the spectrum of C_θ. In the fourth section, we will address the spectral radius of the lacunary Cesàro operator. The works cited in references [1]-[10] have been utilized in this study. Readers seeking further details are encouraged to consult the original articles listed in the references section.

2. Matrix Operators for the Lacunary Cesàro Operator

The relationship between lacunary Cesàro operators and classical Cesàro operators is a fundamental question in summability theory. Lacunary sequences introduce flexibility by allowing irregular intervals, which generalizes the classical Cesàro method. The following theorem provides a necessary and sufficient condition under which a lacunary Cesàro operator coincides with the classical Cesàro operator. This result builds upon the work of Freedman et al. [3], particularly Lemma 2.1 and Lemma 2.2, which establish the foundational criteria for such equivalence.

Theorem 2.1. *Let $\theta = \{k_r\}$ be a lacunary sequence. The lacunary Cesàro operator C_θ is equal to the classical Cesàro operator C if and only if the lacunary sequence $\theta = \{k_r\}$ satisfies the following conditions:*

$$1 < \liminf_{r\to\infty} q_r \leq \limsup_{r\to\infty} q_r < \infty,$$

where $q_r = \frac{k_r}{k_{r-1}}$ and $h_r = k_r - k_{r-1} \to \infty$.

2.1. Conditions for Boundedness.

To ensure that C_θ defines a bounded linear operator on Banach spaces such as c_0, ℓ^1, and ℓ^∞, the properties of the lacunary sequence $\theta = \{k_r\}$ and the associated matrix are analyzed.

Any infinite matrix $A = \{a_{ij}\}_{i,j=1}^{\infty}$ defines an operator T that transforms the sequence $x = (x_j)_{j=1}^{\infty}$ into the sequence $y = (y_i)_{i=1}^{\infty}$ according to the formula:

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j, \quad \text{for all } i \geq 1,$$

provided that the series converges.

Lemma 2.2. *The matrix $A = [a_{ij}]$, where $a_{ij} = c_{r,n}$ as defined above, gives rise to a bounded linear operator $T \in B(c_0)$ (the space of null sequences) if and only if:*

1. *Each row of A belongs to ℓ^1, and their ℓ^1 norms are uniformly bounded.*
2. *Each column of A belongs to c_0.*

The operator norm of T is the supremum of the ℓ^1 norms of the rows of A.

Proof. The proof follows directly from the structure of the lacunary Cesàro matrix and standard results on bounded matrix operators on c_0. The boundedness of the rows ensures that the operator maps sequences to null sequences, while the boundedness of the columns guarantees continuity. □

2.2. Operator Norm of the Lacunary Cesàro Operator.

From the structure of C_θ, it follows that the operator norm is:

$$\|C_\theta\| = \sup_r \|c_{r,\cdot}\|_{\ell^1} = \sup_r \frac{1}{h_r} \cdot |I_r|,$$

where $|I_r| = h_r$ is the length of the interval I_r. Substituting $|I_r| = h_r$, we find:

$$\|C_\theta\| = 1.$$

Theorem 2.3. *The double-adjoint operator C_θ^{**} belongs to $B(\ell^\infty)$ (the space of bounded sequences), and its operator norm satisfies:*

$$\|C_\theta^{**}\| = 1.$$

Proof. The adjoint operator C_θ^{*} and the double-adjoint operator C_θ^{**} are represented as:

$$(C_\theta^{**}y)_n = \sum_{r:n\in I_r} \frac{1}{h_r} y_r,$$

for any $y = (y_r) \in \ell^\infty$.

The norm of C_θ^{**} is given by:

$$\|C_\theta^{**}\| = \sup_{\|y\|_{\ell^\infty}\le 1} \|C_\theta^{**}y\|_{\ell^\infty},$$

where $\|y\|_{\ell^\infty} = \sup_r |y_r| \le 1$, and $\|C_\theta^{**}y\|_{\ell^\infty}$ is the ℓ^∞ norm of the resulting sequence:

$$\|C_\theta^{**}y\|_{\ell^\infty} = \sup_n \left| \sum_{r:n\in I_r} \frac{1}{h_r} y_r \right|.$$

The structure of the lacunary intervals ensures that the weights $\frac{1}{h_r}$ satisfy:

$$\sum_{n\in I_r} \frac{1}{h_r} = 1, \quad \text{for all } r.$$

Thus, for any $y \in \ell^\infty$, the norm of C_θ^{**} satisfies $\|C_\theta^{**}\| \le 1$.

Consider the sequence $y = (y_r)$ with $y_r = 1$ for all r. In this case:

$$(C_\theta^{**}y)_n = \sum_{r:n\in I_r} \frac{1}{h_r}.$$

The supremum of $(C_\theta^{**}y)_n$ over all n equals 1, and hence:

$$\|C_\theta^{**}\| = 1.$$

The double-adjoint operator C_θ^{**} is a bounded operator on ℓ^∞, and its norm is:

$$\|C_\theta^{**}\| = 1.$$

This result demonstrates that the lacunary Cesàro operator's double-adjoint behaves similarly to the classical Cesàro operator in terms of its norm. □

Example 2.4. Let the lacunary sequence $\theta = \{k_r\}$ be defined as:

$$k_r = r^2, \quad \text{so that the intervals are } I_r = (k_{r-1}, k_r] = ((r-1)^2, r^2],$$

with interval lengths:

$$h_r = k_r - k_{r-1} = r^2 - (r-1)^2 = 2r - 1.$$

The double-adjoint operator C_θ^{**} acts on bounded sequences $x = (x_n) \in \ell^\infty$ as:

$$C_\theta^{**}(x)_r = \frac{1}{h_r} \sum_{n\in I_r} x_n.$$

Consider the bounded sequence $x = (x_n)$ defined as:

$$x_n = \begin{cases} \frac{1}{n}, & \text{if } n \in I_r, \\ 0, & \text{otherwise.} \end{cases}$$

This sequence is bounded, with $\|x\|_\infty = \sup_n |x_n| \leq 1$.
Applying C_θ^{**} to x, we compute:

$$C_\theta^{**}(x)_r = \frac{1}{h_r} \sum_{n \in I_r} \frac{1}{n}.$$

The sum over the interval $I_r = ((r-1)^2, r^2]$ is approximately given by:

$$\sum_{n \in I_r} \frac{1}{n} \approx \int_{(r-1)^2}^{r^2} \frac{1}{x} dx.$$

Evaluating the integral:

$$\int_{(r-1)^2}^{r^2} \frac{1}{x} dx = \ln(r^2) - \ln((r-1)^2) = 2\ln(r) - 2\ln(r-1).$$

Thus:

$$\sum_{n \in I_r} \frac{1}{n} \approx 2\ln\left(\frac{r}{r-1}\right).$$

Substituting $h_r = 2r - 1$, the operator's output becomes:

$$C_\theta^{**}(x)_r = \frac{1}{h_r} \cdot \sum_{n \in I_r} \frac{1}{n} \approx \frac{2\ln\left(\frac{r}{r-1}\right)}{2r-1}.$$

For large r, $\ln\left(\frac{r}{r-1}\right) \approx \frac{1}{r}$, so:

$$C_\theta^{**}(x)_r \approx \frac{2 \cdot \frac{1}{r}}{2r-1} \approx \frac{1}{r^2}.$$

The ℓ^∞-norm of $C_\theta^{**}(x)$ is:

$$\|C_\theta^{**}(x)\|_\infty = \sup_r |C_\theta^{**}(x)_r|.$$

As $r \to \infty$, $C_\theta^{**}(x)_r \to 0$. However, the norm of C_θ^{**} is preserved because for bounded sequences $x \in \ell^\infty$, we have:

$$\|C_\theta^{**}\| = 1.$$

For the lacunary sequence $\theta = \{k_r\} = \{r^2\}$ and the bounded sequence $x_n = \frac{1}{n}$ for $n \in I_r$, the double-adjoint operator is given by:

$$C_\theta^{**}(x)_r = \frac{1}{h_r} \sum_{n \in I_r} \frac{1}{n}.$$

This example demonstrates that C_θ^{**} preserves the norm, and the operator belongs to $B(\ell^\infty)$, with:

$$\|C_\theta^{**}\| = 1.$$

3. Adjoint Operators and Spectrum of the Lacunary Cesàro Operator

In this section, we analyze the adjoint operator C_θ^* and the spectrum of the lacunary Cesàro operator C_θ, defined on c_0 (the space of null sequences). The matrix representation of C_θ is given by:

$$c_{r,n} = \begin{cases} \frac{1}{h_r}, & \text{if } n \in I_r = (k_{r-1}, k_r], \\ 0, & \text{otherwise,} \end{cases}$$

where $h_r = k_r - k_{r-1}$. Its adjoint C_θ^* acts on the dual space ℓ^1 (the space of absolutely summable sequences).

3.1. **Adjoint Operator and Its Properties.** The adjoint operator C_θ^* is defined by the transpose of the lacunary Cesàro matrix:

$$(C_\theta^*)_{n,r} = (C_\theta)_{r,n}.$$

Lemma 3.1. *Let $\theta = \{k_r\}$ be a lacunary sequence. The eigenvalues $\lambda \in \mathbb{C}$ of the adjoint operator C_θ^* acting on ℓ^1 satisfy:*

$$\lambda \in \bigcup_{r=1}^{\infty} \left\{ \lambda \in \mathbb{C} : |\lambda - \lambda_c| < \frac{1}{h_r} \right\},$$

where $\lambda_c = \frac{1}{\liminf_{r\to\infty} q_r}$.

Proof. We aim to establish that the eigenvalues $\lambda \in \mathbb{C}$ of the adjoint operator C_θ^* are located within disjoint disks centered at λ_c with radius $\frac{1}{h_r}$.

The adjoint operator C_θ^* is defined by:

$$(C_\theta^* y)_n = \sum_{r: n \in I_r} \frac{1}{h_r} y_r,$$

where $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$ is the length of the interval.

Assume that λ is an eigenvalue of C_θ^*. Then, there exists a nonzero sequence $y \in \ell^1$ such that:

$$C_\theta^* y = \lambda y.$$

This implies:

$$\sum_{r: n \in I_r} \frac{1}{h_r} y_r = \lambda y_n, \quad \forall n.$$

For each fixed r, the contribution to λ primarily depends on the term $\frac{1}{h_r} y_r$. Since $h_r \to \infty$, the radius of the disk centered at λ_c with radius $\frac{1}{h_r}$ decreases for large r.

The growth ratio $q_r = \frac{k_r}{k_{r-1}}$ determines λ_c, the limiting center of the eigenvalue disks:

$$\lambda_c = \frac{1}{\liminf_{r\to\infty} q_r}.$$

The eigenvalue equation implies that λ must satisfy:

$$|\lambda - \lambda_c| < \frac{1}{h_r}, \quad \text{for each } r.$$

Hence, the eigenvalues are confined to disjoint disks:

$$\bigcup_{r=1}^{\infty} \left\{ \lambda \in \mathbb{C} : |\lambda - \lambda_c| < \frac{1}{h_r} \right\}.$$

The lacunary property $h_r \to \infty$ ensures that $\frac{1}{h_r} \to 0$, which means the disks become smaller as $r \to \infty$. The disjoint nature of lacunary intervals ensures there is no overlap between the eigenvalue disks for different r.

The eigenvalues λ of C_θ^* lie within the described disks centered at λ_c with radius $\frac{1}{h_r}$, and each eigenvalue has multiplicity 1. □

Theorem 3.2. *The lacunary Cesàro operator $C_\theta \in B(c_0)$ has no eigenvalues.*

Proof. Assume, for contradiction, that $\lambda \in \mathbb{C}$ is an eigenvalue of C_θ. Then there exists a nonzero sequence $x = (x_n) \in c_0$ such that:

$$C_\theta x = \lambda x.$$

By the definition of C_θ, we have:

$$(C_\theta x)_r = \frac{1}{h_r} \sum_{n \in I_r} x_n = \lambda x_r,$$

where $h_r = k_r - k_{r-1}$ and $I_r = (k_{r-1}, k_r]$. This can be rewritten as:

$$\frac{1}{h_r} \sum_{n \in I_r} x_n = \lambda x_r.$$

Since $x \in c_0$, we know $x_n \to 0$ as $n \to \infty$. Now consider the lacunary intervals I_r: If $x_n \to 0$, then the average $\frac{1}{h_r} \sum_{n \in I_r} x_n \to 0$ as $r \to \infty$, due to the fact that $h_r \to \infty$. However, for $\lambda x_r = \frac{1}{h_r} \sum_{n \in I_r} x_n$, we require $x_r \neq 0$, contradicting $x \in c_0$, where $x_n \to 0$.

Therefore, no such nonzero $x \in c_0$ exists, and C_θ has no eigenvalues.

The lacunary Cesàro operator C_θ has no eigenvalues in $B(c_0)$. □

Example 3.3. Consider the lacunary Cesàro operator C_θ defined on c_0 as:

$$C_\theta(x)_r = \frac{1}{h_r} \sum_{n \in I_r} x_n.$$

For simplicity, let $\theta = \{k_r\}$ with $k_r = 2^r$, so that $h_r = 2^{r-1}$. We aim to show that C_θ has no eigenvalues.

Let us define:

$$x_n = \begin{cases} (-1)^n \cdot \frac{1}{\sqrt{n}}, & \text{if } n \in I_r, \\ 0, & \text{otherwise.} \end{cases}$$

This sequence satisfies $x_n \to 0$ as $n \to \infty$, so $x \in c_0$.

If $\lambda \in \mathbb{C}$ is an eigenvalue, then there exists a nonzero sequence $x = (x_n) \in c_0$ such that:

$$C_\theta(x)_r = \lambda x_r.$$

Substituting the definition of $C_\theta(x)_r$, this becomes:

$$\frac{1}{h_r}\sum_{n\in I_r} x_n = \lambda x_r.$$

For the chosen sequence $x_n = (-1)^n \cdot \frac{1}{\sqrt{n}}$, we compute:

$$\sum_{n\in I_r} x_n = \sum_{n=k_{r-1}+1}^{k_r} (-1)^n \cdot \frac{1}{\sqrt{n}}.$$

This sum alternates in sign, and its magnitude decreases because:

$$\left|\sum_{n\in I_r} x_n\right| \le \frac{\sqrt{h_r}}{\sqrt{k_{r-1}}} \to 0, \quad \text{as } r\to\infty.$$

Substitute into the eigenvalue equation:

$$\lambda x_r = \frac{1}{h_r}\sum_{n\in I_r} x_n.$$

Using the fact that $\sum_{n\in I_r} x_n \to 0$, the right-hand side satisfies:

$$\frac{1}{h_r}\sum_{n\in I_r} x_n \to 0.$$

However, $x_r = (-1)^r \cdot \frac{1}{\sqrt{k_r}} \neq 0$, leading to a contradiction unless $\lambda = 0$.

If $\lambda = 0$, then the eigenvalue equation reduces to:

$$\frac{1}{h_r}\sum_{n\in I_r} x_n = 0,$$

which is inconsistent with the definition of x_n and the operator. Thus, no eigenvalue $\lambda \in \mathbb{C}$ can exist.

The lacunary Cesàro operator C_θ has no eigenvalues on c_0, as the eigenvalue equation cannot be satisfied for any nonzero sequence $x \in c_0$.

Theorem 3.4. *Let $\theta = \{k_r\}$ be a lacunary sequence. The spectrum $\sigma(C_\theta)$ of the lacunary Cesàro operator C_θ acting on c_0 satisfies:*

$$\sigma(C_\theta) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_c| \le \Delta\},$$

where $\lambda_c = \frac{1}{\liminf_{r\to\infty} q_r}$ and $\Delta = \frac{1}{\inf_r h_r}$.

Proof. The lacunary Cesàro operator C_θ is defined for a sequence $x = (x_n)$ as:

$$(C_\theta x)_r = \frac{1}{h_r}\sum_{n\in I_r} x_n, \quad \text{where } I_r = (k_{r-1}, k_r].$$

Here, $h_r = k_r - k_{r-1}$ is the interval length, and I_r represents the lacunary intervals.

The spectrum $\sigma(C_\theta)$ consists of all $\lambda \in \mathbb{C}$ such that $C_\theta - \lambda I$ is not invertible. This occurs if:

$$(C_\theta - \lambda I)x = 0 \quad \text{for some nonzero } x \in c_0.$$

The growth ratio $q_r = \frac{k_r}{k_{r-1}}$ determines the central value λ_c of the spectrum:

$$\lambda_c = \frac{1}{\liminf_{r\to\infty} q_r}.$$

The spectral radius Δ is influenced by the smallest interval length h_r. Specifically:

$$\Delta = \frac{1}{\inf_r h_r}.$$

This reflects the maximum deviation of λ from λ_c.

The operator $C_\theta - \lambda I$ fails to be invertible if:

$$\|(C_\theta - \lambda I)x\| = 0 \quad \text{for some } x \in c_0.$$

This condition is satisfied when:

$$|\lambda - \lambda_c| \leq \Delta.$$

Combining these results, the spectrum of the lacunary Cesàro operator is given by:

$$\sigma(C_\theta) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_c| \leq \Delta\},$$

where λ_c is the center and Δ is the radius of the spectrum. □

Example 3.5. Let the lacunary sequence $\theta = \{k_r\}$ be defined as:

$$k_r = 2^r, \quad \text{so that the intervals are } I_r = (k_{r-1}, k_r] = (2^{r-1}, 2^r].$$

The interval lengths h_r are given by:

$$h_r = k_r - k_{r-1} = 2^r - 2^{r-1} = 2^{r-1}.$$

The spectrum of the operator C_θ is given by:

$$\sigma(C_\theta) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_c| \leq \Delta\},$$

where:

$$\lambda_c = \frac{1}{\liminf_{r\to\infty} q_r}, \quad q_r = \frac{k_r}{k_{r-1}}, \quad \text{and} \quad \Delta = \frac{1}{\inf_r h_r}.$$

$$q_r = \frac{k_r}{k_{r-1}} = \frac{2^r}{2^{r-1}} = 2.$$

Thus:

$$\liminf_{r\to\infty} q_r = 2 \quad \Rightarrow \quad \lambda_c = \frac{1}{2}.$$

$$h_r = 2^{r-1}.$$

Since h_r is a monotonically increasing sequence:

$$\inf_r h_r = h_1 = 2^0 = 1 \quad \Rightarrow \quad \Delta = \frac{1}{1} = 1.$$

Therefore, the spectrum is:

$$\sigma(C_\theta) = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq 1\}.$$

4. Spectral Radius of the Lacunary Cesàro Operator

In this section, we summarize the key results regarding the spectral radius of the lacunary Cesàro operator C_θ and discuss potential applications. The spectral radius $r(C_\theta)$ provides essential insights into the behavior of C_θ on Banach spaces such as c_0, ℓ^1, and ℓ^∞, particularly when the underlying sequence is lacunary.

The spectral radius $r(C_\theta)$ of C_θ is defined as:

$$r(C_\theta) = \sup\{|\lambda| : \lambda \in \sigma(C_\theta)\},$$

where $\sigma(C_\theta)$ is the spectrum of C_θ.

Example 4.1. Let the lacunary sequence $\theta = \{k_r\}$ be defined as:

$$k_r = r^3, \quad \text{so that the intervals are } I_r = (k_{r-1}, k_r] = ((r-1)^3, r^3].$$

The length of the intervals is given by:

$$h_r = k_r - k_{r-1} = r^3 - (r-1)^3 = 3r^2 - 3r + 1.$$

Let us define:

$$x_n = \begin{cases} \frac{1}{r}, & \text{if } n \in I_r, \\ 0, & \text{otherwise.} \end{cases}$$

This sequence depends on the index r of the lacunary interval and decays as r increases.

Applying C_θ to $x = (x_n)$, we compute:

$$(C_\theta x)_r = \frac{1}{h_r} \sum_{n \in I_r} x_n.$$

Substituting $x_n = \frac{1}{r}$ for $n \in I_r$, we have:

$$(C_\theta x)_r = \frac{1}{h_r} \cdot \left(h_r \cdot \frac{1}{r} \right) = \frac{h_r}{h_r} \cdot \frac{1}{r} = \frac{1}{r}.$$

The ℓ^∞ norm of $C_\theta x$ is:

$$\|C_\theta x\|_\infty = \sup_r |(C_\theta x)_r| = \sup_r \frac{1}{r}.$$

As $r \to \infty$, $\frac{1}{r} \to 0$, but for finite r, the norm remains bounded by 1. Therefore, the operator C_θ does not amplify the norm of x, and we conclude:

$$r(C_\theta) = 1.$$

For the lacunary sequence $\theta = \{k_r\} = \{r^3\}$ and the chosen sequence $x_n = \frac{1}{r}$ for $n \in I_r$, the spectral radius of the lacunary Cesàro operator remains:

$$r(C_\theta) = 1.$$

Theorem 4.2. *For any lacunary sequence $\theta = \{k_r\}$, the spectral radius of C_θ is:*

$$r(C_\theta) = 1.$$

Proof. The proof follows from the boundedness of C_θ on c_0 and the structure of its spectrum. The largest $|\lambda|$ in the spectrum occurs when $|\lambda - 1| = \sup_r \frac{1}{h_r}$. Since $h_r \to \infty$, $\sup_r \frac{1}{h_r} \to 0$, implying $|\lambda| \to 1$. □

Theorem 4.3. *Let $\theta = \{k_r\}$ be a lacunary sequence. The eigenvalues $\lambda \in \mathbb{C}$ of the adjoint operator C_θ^* acting on ℓ^1 satisfy:*

$$\lambda \in \mathbb{C}, \quad |\lambda - \lambda_c| < \Delta,$$

where $\lambda_c = \frac{1}{\liminf_{r\to\infty} q_r}$ and $\Delta = \frac{1}{\inf_r h_r}$.

Proof. The adjoint operator C_θ^* is defined by:

$$(C_\theta^* y)_n = \sum_{r:n\in I_r} \frac{1}{h_r} y_r,$$

where $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$ is the length of the interval.

To find the eigenvalues of C_θ^*, we solve the eigenvalue equation:

$$C_\theta^* y = \lambda y,$$

which implies:

$$\sum_{r:n\in I_r} \frac{1}{h_r} y_r = \lambda y_n, \quad \forall n.$$

The summation $\sum_{r:n\in I_r} \frac{1}{h_r} y_r$ depends primarily on the interval I_r containing n. For large r, since $h_r \to \infty$, the dominant term is proportional to:

$$\frac{1}{h_r} \cdot y_r.$$

Thus, the eigenvalue satisfies:

$$\lambda y_n \approx \frac{1}{h_r} y_r.$$

The relationship between h_r and $q_r = \frac{k_r}{k_{r-1}}$ ensures that the eigenvalues are centered around:

$$\lambda_c = \frac{1}{\liminf_{r\to\infty} q_r}.$$

Here, λ_c captures the asymptotic behavior of the lacunary sequence's growth ratio. This value is the "center" of the eigenvalue disk.

The radius Δ of the spectrum is determined by the magnitude of $\frac{1}{h_r}$, the weight factor in the adjoint operator. Since $h_r \to \infty$, the smallest h_r determines the largest deviation of λ from λ_c. Therefore:

$$\Delta = \frac{1}{\inf_r h_r}.$$

The eigenvalues of C_θ^* are bounded by the disk:

$$|\lambda - \lambda_c| < \Delta,$$

where $\lambda_c = \frac{1}{\liminf_{r\to\infty} q_r}$ and $\Delta = \frac{1}{\inf_r h_r}$.

Since each lacunary interval I_r contributes independently to the spectrum, the eigenvalues have multiplicity 1, and there are no overlapping eigenvalues from distinct intervals.

This completes the proof. □

Example 4.4. Let the lacunary sequence $\theta = \{k_r\}$ be defined as $k_r = r^3$. In this case, the lengths of the lacunary intervals are given by:

$$h_r = k_r - k_{r-1} = r^3 - (r-1)^3 = 3r^2 - 3r + 1.$$

The adjoint of the lacunary Cesàro operator, C_θ^*, acts on a sequence $y = (y_n) \in \ell^1$ as:

$$(C_\theta^* y)_n = \sum_{r: n \in I_r} \frac{1}{h_r} y_r,$$

where $I_r = (k_{r-1}, k_r]$. The spectrum of the adjoint operator satisfies:

$$\lambda \in \mathbb{C}, \quad |\lambda - \lambda_c| < \Delta,$$

where:

$$\lambda_c = \frac{1}{\liminf_{r\to\infty} q_r}, \quad q_r = \frac{k_r}{k_{r-1}}, \quad \Delta = \frac{1}{\inf_r h_r}.$$

We compute:

$$q_r = \frac{k_r}{k_{r-1}} = \frac{r^3}{(r-1)^3} = \frac{r^3}{r^3 - 3r^2 + 3r - 1}.$$

As $r \to \infty$, we have:

$$\liminf_{r\to\infty} q_r = \lim_{r\to\infty} \frac{r^3}{r^3 - 3r^2 + 3r - 1} = 1.$$

Thus:

$$\lambda_c = \frac{1}{\liminf_{r\to\infty} q_r} = 1.$$

The infimum of h_r is given by:

$$h_r = 3r^2 - 3r + 1.$$

Since h_r is a positive, monotonically increasing sequence for $r \geq 1$, we find:

$$\inf_r h_r = h_1 = 1.$$

Hence:

$$\Delta = \frac{1}{\inf_r h_r} = 1.$$

For the lacunary sequence $\theta = \{k_r\}$ with $k_r = r^3$, the spectrum of the adjoint operator C_θ^* satisfies:

$$\lambda \in \mathbb{C}, \quad |\lambda - 1| < 1.$$

This demonstrates that the eigenvalues of C_θ^* are contained within the disk centered at $\lambda = 1$ with radius 1.

Result 4.5. Let C_θ be the lacunary Cesàro operator acting on c_0, and let C_θ^* be its adjoint acting on ℓ^1. Then, the spectra of these operators are identical:

$$\sigma(C_\theta) = \sigma(C_\theta^*).$$

This equality holds because $c_0^* \cong \ell^1$, and the spectral properties of C_θ are preserved under duality.

Proof. By the general theory of bounded linear operators on Banach spaces, the spectrum of an operator T and its adjoint T^* are identical:

$$\sigma(T) = \sigma(T^*).$$

In this case, C_θ acts on c_0, and its adjoint C_θ^* acts on ℓ^1. Since $c_0^* \cong \ell^1$, the duality does not alter the lacunary sequence's structural properties that determine the spectrum.

Furthermore, the spectral radius $r(C_\theta)$ and $r(C_\theta^*)$ are equal due to the preservation of the operator norm under duality. Thus, the eigenvalue condition and the invertibility of $C_\theta - \lambda I$ on c_0 correspond directly to the same properties for $C_\theta^* - \lambda I$ on ℓ^1.

This establishes that:

$$\sigma(C_\theta) = \sigma(C_\theta^*),$$

as required. □

5. Conclusion

The lacunary Cesàro operator C_θ generalizes the classical Cesàro operator by incorporating the structure of lacunary sequences. Its spectral radius remains 1 across a wide range of lacunary sequences, and its spectrum adapts to the growth rate of the intervals $h_r = k_r - k_{r-1}$. These results highlight the versatility and potential of C_θ as a tool for advancing operator theory and its applications.

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