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NONLINEAR ELLIPTIC PROBLEM INVOLVING NATURAL GROWTH TERM, L^1-DATA AND VARIABLE EXPONENT

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Abstract. Using approximation techniques and the theory of maximal monotone operators in Banach spaces, we prove the existence of at least one solution of a wide class of multivalued nonlinear elliptic problems involving natural growth terms and general $p(.)$-Leray-Lions operator.

1. Introduction

Nonlinear PDEs occupy one of the important places in modeling of natural phenomena in several fields notably in physics, biology and engineering sciences. In general, the equations resulting from this modeling are solved in classical Sobolev spaces with constant exponent. But, to model phenomena involving non-homogeneous materials, classical Sobolev spaces are insufficient as they do not capture all the properties of these materials (for example the change of state). So, we must use a more flexible approach, this led to solve equations in Sobolev spaces with variable exponent, commonly denoted by $W^{1,p(.)}(\Omega)$ (to be defined later).

Nowadays, the study of problems in Sobolev spaces with $p(.)$ growth is currently gaining considerable attention in mathematical research since it has been discovered that they are essential in various applications, including physics, mechanical process, electro-rheological fluids, stationary thermo-rheological viscous flows of non-Newtonian fluids (see [4, 17, 22, 29, 30] for more details). They are also used in image processing ([14]).

In this paper we deal with the following nonlinear multivalued problem

$$(E,f)\begin{cases} \beta(u)-\mathrm{div}a(x,u,\nabla u)+h(x,u,\nabla u)\ni f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Here, we assume that Ω is an open, bounded domain in $\mathbb{R}^N$ ($N\geq 3$), with a smooth boundary $\partial\Omega$ and $f\in L^1(\Omega)$. The operator $A(u)=-\mathrm{div}a(x,u,\nabla u)$ is a $p(.)$-Leray-Lions operator mapping from $W_0^{1,p(.)}(\Omega)$ to its dual space $W^{-1,p'(.)}(\Omega)$. The function $\beta:\mathbb{R}\to 2^{\mathbb{R}}$ is a maximal monotone mapping with $0\in\beta(0)$. The

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nonlinear term $h(x, u, \nabla u)$, which has natural growth of order $p(.)$ in ∇u, satisfies a sign-condition.

It is worth noting that if the graph β has a single-valued range, the problem (E, f) has already been studied within the context of constant exponents [8, 9, 11, 12]. These problems were subsequently adapted to the context of variable exponents [5, 33]. In [1] the authors established the existence of a renormalized solution for a specific case of problem (E, f) in the setting of weighted exponent spaces (see also [16]). When $\beta \equiv 0$ and $g \equiv 0$, Boccardo, Gallouët and Murat [10] demonstrated the existence of at least one solution to the problem (E, f) (we also refer readers to [8]). In [3], the existence of a solution to the problem (E, f) is proven within the framework of constant exponent spaces. Another significant contribution to the L^1-theory for the $p-$Laplacian or $p(.)-$Laplacian, involving differential inclusion equations, has focused on the following problem.

$$(P) \begin{cases} \gamma(u) - \operatorname{div}(a(x, \nabla u) + F(u)) \ni \Upsilon & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

A lot of results of existence and uniqueness have been found concerning the problem (P) (see [2, 7, 23]). As far as problems like (E, f) are concerned, we refer the reader to [25, 32], where the authors analyzed the existence and uniqueness of solutions of related problem under Neumann and Robin boundaries conditions. In a recent paper [3], Akdim and Ouboufettal examined the existence of solutions to (E, f) within the framework of Sobolev space with constant exponent. The key focus of our work lies in addressing general nonlinear operators $-\operatorname{div}(a(x, u, \nabla u))$, coupled with a graph, within the framework of Sobolev spaces with variable exponent.

In this paper, following the ideas of the authors in [3], we generalize their main result to the framework of Sobolev spaces with variable exponent.
Here is an overview of the paper's contents. In Section 2, we recall some preliminaries on variable exponent spaces. Section 3 presents our key assumptions along with some fundamental results. In Section 4, we investigate the existence of solutions when $f \in L^\infty(\Omega)$. Lastly, in Section 5, we establish the existence of weak, entropy, and renormalized solutions when $f \in L^1(\Omega)$.

2. Preliminaries

In the following, we review some definitions and fundamental properties of Lebesgue and Sobolev spaces with variable exponents.
Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, with smooth boundary $\partial\Omega$. We define the set

$$C_+(\overline{\Omega}) = \left\{ p(.) : \overline{\Omega} \longrightarrow (1, \infty) \text{ continuous such that } 1 < p^- \leq p^+ < \infty \right\},$$

where $p^- := \min_{x \in \overline{\Omega}} p(x)$ and $p^+ := \max_{x \in \overline{\Omega}} p(x)$.
Let $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as the

set of measurable $f:\Omega\to\mathbb{R}$ such that its $p(.)$-modular $\rho_{p(.)}:L^{p(.)}(\Omega)\longrightarrow\mathbb{R}$ verifies

$$\rho_{p(.)}(f):=\int_\Omega |f|^{p(x)}dx<\infty.$$

The space $L^{p(.)}(\Omega)$ is equipped with the Luxembourg norm

$$\|f\|_{p(.)}:=\inf\left\{\theta>0:\int_\Omega\left|\frac{f(x)}{\theta}\right|^{p(x)}dx\le 1\right\}.$$

For any f in $L^{p(.)}(\Omega)$, one has ([20, 21])

$$\min\left\{\|f\|_{p(.)}^{p^-};\ \|f\|_{p(.)}^{p^+}\right\}\le\rho_{p(.)}(f)\le\max\left\{\ \|f\|_{p(.)}^{p^-};\ \|f\|_{p(.)}^{p^+}\right\}. \tag{2.1}$$

Lemma 2.1 (**Hölder type inequality, see** [26]).

(i) *For any $f\in L^{p(.)}(\Omega)$ and $g\in L^{q(.)}(\Omega)$ such that $\frac{1}{p(x)}+\frac{1}{q(x)}=1$ a.e. in Ω, one has*

$$\left|\int_\Omega fgdx\right|\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)\|f\|_{p(.)}\|g\|_{q(.)}. \tag{2.2}$$

(ii) *For all $p_1,\ p_2\in C_+(\overline{\Omega})$ such that $p_1(x)\le p_2(x)$ a.e. $x\in\Omega$, one has that $L^{p_2(.)}(\Omega)\hookrightarrow L^{p_1(.)}(\Omega)$ and the embedding is continuous.*

Proposition 2.2. [26] *For any $u_n,u\in L^{p(x)}(\Omega)$, the following assertions hold true.*

(i) $\|f\|_{p(.)}<1$ *(resp, $=1$, >1) if and only if* $\rho_{p(.)}(f)<1$ *(resp, $=1$, >1);*
(ii) $\|f\|_{p(.)}>1$ *imply* $\|f\|_{p(.)}^{p-}\le\rho_{p(.)}(f)\le\|u\|_{p(.)}^{p+}$, *and* $\|f\|_{p(.)}<1$ *imply* $\|f\|_{p(.)}^{p+}\le\rho_{p(.)}(u)\le\|u\|_{p(.)}^{p-}$;
(iii) $\|f_n\|_{p(.)}\to 0$ *if and only if* $\rho_{p(.)}(f_n)\to 0$, *and* $\|f_n\|_{p(.)}\to\infty$ *if and only* $\rho_{p(.)}(f_n)\to\infty$.

We now define the Sobolev space with variable exponents as follows

$$W^{1,p(.)}(\Omega):=\left\{f\in L^{p(.)}(\Omega):\ |\nabla f|\in L^{p(.)}(\Omega)\right\},$$

with the norm

$$||f||_{1,p(.)}=\|f\|_{p(.)}+\|\nabla f\|_{p(.)}.$$

We denote by $W_0^{1,p(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$.
The Sobolev exponent is defined as $p^*(x)=\frac{Np(x)}{N-p(x)}$ if $p(x)<N$ and $p^*(x)=\infty$ if $p(x)\ge N$.

Lemma 2.3. *Let p be in $C_+(\overline{\Omega})$, and $f,f_n\in L^{p(.)}(\Omega)$ such that $\|f_n\|_{p(.)}\le C$. If $f_n(.)\to f(.)$ a.e. in Ω, then $f_n\rightharpoonup f$ in $L^{p(.)}(\Omega)$.*

Theorem 2.4. [21, 24]

(i) *Let p be in $C_+(\overline{\Omega})$. Then, the sets $L^{p(.)}(\Omega)$ and $W_0^{1,p(.)}(\Omega)$ are separable and reflexive Banach spaces.*
(ii) *If $q\in C_+(\overline{\Omega})$ and $q(x)<p^*(x)$ for every $x\in\Omega$, then the embedding $W_0^{1,p(.)}(\Omega)\hookrightarrow\hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.*

(iii) *There exists a constant $M > 0$, such that*

$$\|u\|_{p(.)} \leq M\|\nabla f\|_{p(.)},\ \forall f \in W_0^{1,p(.)}(\Omega).$$

(iv) *There exists a constant $M' > 0$, such that*

$$\|f\|_{p^*(.)} \leq M'\|\nabla f\|_{p(.)},\ \forall f \in W_0^{1,p(.)}(\Omega).$$

Remark 2.5. By Poincaré's inequality (iii) of Theorem 2.4, we deduce that $\|\nabla f\|_{p(.)}$ and $\|f\|_{1,p(.)}$ are equivalent norms in $W_0^{1,p(.)}(\Omega)$.

Definition 2.6. [18] Let $W^{-1,p'(.)}(\Omega)$ be the dual of the set $W_0^{1,p(.)}(\Omega)$ and $G \in W^{-1,p'(.)}(\Omega)$. Then, there exists $g_0, g_1, ..., g_N \in L^{p'(.)}(\Omega)$ such that $G = g_0 + \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}$.
Moreover, for every $v \in W_0^{1,p(.)}(\Omega)$,

$$\langle G, v\rangle = \int_\Omega g_0 v dx - \sum_{i=1}^{N} \int_\Omega g_i \frac{\partial v}{\partial x_i} dx,$$

and the dual space is equipped with the following norm

$$\|G\|_{-1,p'(.)} \simeq \sum_{i=0}^{N} \|g_i\|_{p'(.)}.$$

We will use throughout the paper, for any $\gamma > 0$, the truncation function defined by $T_\gamma(s) = \max\{-\gamma, \min\{\gamma, s\}\}$. It is obvious that $\lim_{\gamma\to\infty} T_\gamma(s) = s$ and $|T_\gamma(s)| = \min\{|s|; \gamma\}$.
$\mathcal{T}_0^{1,p(.)}(\Omega) := \{u : \Omega \to \mathbb{R}$ measurable function such that $T_\gamma(s) \in W_0^{1,p(.)}(\Omega)\}$.
We stress that a function $u \in \mathcal{T}_0^{1,p(.)}(\Omega)$ does not necessarily belong to $W_0^{1,1}(\Omega)$. However, one can define, as follows, the weak gradient of u still denoted by ∇u (see [7, 31]).

Proposition 2.7. *Let $u \in \mathcal{T}_0^{1,p(.)}(\Omega)$. Then, there exists a unique measurable function $\vartheta : \Omega \longrightarrow \mathbb{R}^N$ such that $\nabla T_\gamma(u) = \vartheta\chi_{\{|u|<\gamma\}}$, for a.e. $x \in \Omega$ and for all $\gamma > 0$. The function ϑ is called the weak gradient of u and is still denoted by ∇u. Moreover, if $u \in W_0^{1,p(.)}(\Omega)$, then $\vartheta \in (L^{p(.)}(\Omega))^N$ and $\vartheta = \nabla u$ coincides with the standard distributional gradient of u.*

The following lemma provides an useful result of convergence in the sense of graph (see [32]).

Lemma 2.8. *Let $(B_m)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $B_n \to B$ in the sense of graphs (for $(r, s) \in B$, there exists $(r_n, s_n) \in B_n$ such that $r_n \to r$ and $s_n \to s$). Let $(z_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$ be two sequences of $L^1(\Omega)$. We assume that $\forall n \geq 1, w_n \in B_n(\zeta)$, $(w_n)_{n\geq 1}$ are bounded in $L^1(\Omega)$ and $\zeta_n \to \zeta$ in $L^1(\Omega)$. Then,*

$$\zeta \in dom(B).$$

We now define some functions that will be used frequently throughout this paper. For any real number $s \in \mathbb{R}$, we define s^+ as the non-negative part of s given by $s^+ := \max(s, 0)$. Additionally, let sign_0^+ be a function that assigns values as follows

$$\operatorname{sign}_0^+(s) = \begin{cases} 1 \text{ if } s > 0, \\ 0 \text{ if } s \leq 0. \end{cases}$$

For $\delta > 0$, we define the function $H_\delta^+ : \mathbb{R} \to \mathbb{R}$ by

$$H_\delta^+(r) = \begin{cases} 1 \quad \text{if } r > \delta \\ \\ \dfrac{r}{\delta} \quad \text{if } 0 \leq r \leq \delta \\ \\ 0 \quad \text{if } r < 0. \end{cases}$$

It is obvious that H_δ^+ is an approximation of sign_0^+.

Definition 2.9. Let $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. A collection $\mathcal{F} \subset L^1(\mu)$ is said to be uniformly integrable (or equi-integrable) if, for every $\alpha > 0$ there exists $\delta > 0$ such that

$$\int_A |f| d\mu < \alpha \text{ whenever } \mu(A) < \delta.$$

Proposition 2.10. [15] *Let $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. A collection $\mathcal{F} \subset L^1(\mu)$ is said to be uniformly integrable if one has*

$$\lim_{b \to \infty} \left(\sup_{g \in \mathcal{F}} \int_{\{|g| \geq b\}} |g| d\mu \right) = 0.$$

Proposition 2.11. [15] *If the measure μ is bounded, any subset $\mathcal{F}$ in L^1, bounded in L^∞, is uniformly integrable.*

Proposition 2.12. [15] *Let $\mathcal{F}$ be a subset of $L^1(\mu)$. If there exists a positive function $f \in L^1$ such that $|g| \leq f$ for any $g \in \mathcal{F}$, then, $\mathcal{F}$ is uniformly integrable.*

Theorem 2.13. [19] *(Dunford). A subset of $L^1(\mu)$ is relatively weakly compact if and only if it is both bounded and uniformly integrable.*

3. Assumptions and main results

Let p be in $C_+(\overline{\Omega})$ and let $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that for a.e. $x \in \Omega$, and for any $\varphi, \varphi' \in \mathbb{R}^N$, $s \in \mathbb{R}$.

$$a(x, s, \varphi).\varphi \geq \lambda |\varphi|^{p(x)}; \tag{3.1}$$

$$|a(x, s, \varphi)| \leq \Lambda(j(x) + |s|^{p(x)-1} + |\varphi|^{p(x)-1}); \tag{3.2}$$

$$(a(x, s, \varphi) - a(x, s, \varphi'))(\xi - \varphi') > 0 \text{ if } \varphi \neq \varphi'; \tag{3.3}$$

where λ, Λ are two positive constants and $j(.)$ a given non-negative function in $L^{p'(.)}(\Omega)$.

Here, $h : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying for almost every $x \in \Omega,\ s \in \mathbb{R},\ \xi \in \mathbb{R}^N$,

$$h(x, s, \varphi)s \geq 0; \tag{3.4}$$

$$|h(x, s, \varphi)| \leq d(|s|)(c(x) + |\varphi|^{p(x)}); \tag{3.5}$$

where $d : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $c(.)$ is a non-negative function in $L^1(\Omega)$.
There exist $\sigma > 0$ and $\rho > 0$ such that

$$|h(x, s, \varphi)| \geq \rho|\varphi|^{p(x)} \text{ for all } |s| \geq \sigma. \tag{3.6}$$

Lemma 3.1. [6] *Let $f \in L^{p(.)}(\Omega)$ and $f_m \in L^{p(.)}(\Omega)$ such that $\|f_m\|_{L^{z(.)}(\Omega)} \leq C$. If $f_m \rightharpoonup f$ a.e. in Ω, then $f_m \rightharpoonup f$ in $L^{p(.)}(\Omega)$, as $m \to \infty$.*

Lemma 3.2. [6] *Let (3.1)-(3.3) hold true. Let $(u_m)_{m\in\mathbb{N}}$ be a sequence in $W_0^{1,p(.)}(\Omega)$ such that $u_m \rightharpoonup u$ in $W_0^{1,p(.)}(\Omega)$ and*

$$\int_\Omega \Big[a(x, u_m, \nabla u_m) - a(x, u_m, \nabla u)\Big]\nabla(u_m - u)dx \to 0. \tag{3.7}$$

Then, $u_m \to u$ in $W_0^{1,p(.)}(\Omega)$, as $m \to \infty$.

4. Problem with $f \in L^\infty(\Omega)$

In this section, we aim to establish the existence of a weak solution for the problem (E, f) with f in $L^\infty(\Omega)$. The proof is carried out in several steps.

Theorem 4.1. *Suppose that $f \in L^\infty(\Omega)$. Then, the problem (E, f) admits at least one weak solution $(u, b) \in W_0^{1,p(.)}(\Omega) \times L^1(\Omega)$. Namely, $b(x) \in \beta(u(x))$ a.e. in Ω, $h(x, u, \nabla u) \in L^1(\Omega)$ and*

$$\int_\Omega b\varphi dx + \int_\Omega a(x, u, \nabla u)\nabla\varphi dx + \int_\Omega h(x, u, \nabla u)\varphi dx = \int_\Omega f\varphi dx, \tag{4.1}$$

for any $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

Proof. **step 1 Approximate problem**.
We consider the approximating problems

$$(E_\epsilon, f)\begin{cases} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - \mathrm{div}a(x, u_\epsilon, \nabla u_\epsilon) + h_\epsilon(x, u_\epsilon, \nabla u_\epsilon) = f \text{ in } \Omega \\ u_\epsilon = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where the function $\beta_\epsilon : \mathbb{R} \to \mathbb{R}$ is obtained via Yosida's approximation of β (see [13]), and $h_\epsilon(x, s, \varphi) = \dfrac{h(x, s, \varphi)}{1 + \epsilon|h(x, s, \varphi)|}$ for any $\epsilon \in]0, 1]$.
For all $u \in W_0^{1,p(.)}(\Omega)$, we can readily observe that

$$\langle\beta_\epsilon(u), u\rangle \geq 0,\ |\beta_\epsilon(u)| \leq \frac{1}{\epsilon}|u| \ \text{ and } \ \lim_{\epsilon\to 0}\beta_\epsilon(u) = \beta(u),$$

$$h_\epsilon(x, s, \xi)s \geq 0, \quad |h_\epsilon(x, s, \xi)| \leq |h(x, s, \xi)|\ ,\ |h_\epsilon(x, s, \xi)| \leq \frac{1}{\epsilon}$$

and

$$|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))| \leq \frac{1}{\epsilon^2}.$$

Proposition 4.2. *Let $f \in L^\infty(\Omega)$. Then, the problem (E_ϵ, f) has at least one weak solution $u_\epsilon \in W_0^{1,p(.)}(\Omega)$. Namely,*

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi dx + \int_\Omega a(x, u_\epsilon, \nabla u_\epsilon)\nabla\varphi dx + \int_\Omega h_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi dx = \int_\Omega f\varphi dx, \tag{4.2}$$

for all $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $A_\epsilon : W_0^{1,p(.)}(\Omega) \to W^{-1,p'(.)}(\Omega)$ be an operator defined by

$$\langle A_\epsilon(u), \varphi\rangle = \langle Au, \varphi\rangle + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\varphi dx + \int_\Omega h_\epsilon(x, u, \nabla u)\varphi dx \ \ \forall\, u, \varphi \in W_0^{1,p(.)}(\Omega),$$

where $\langle Au, \varphi\rangle = \displaystyle\int_\Omega a(x, u, \nabla u)\nabla\varphi dx$.
The existence of a weak solution is established based on the following lemma.

Lemma 4.3. *The operator A_ϵ is both pseudo-monotone and bounded. Additionally, A_ϵ satisfies a coercivity condition, expressed as follows:*

$$\frac{\langle A_\epsilon(u), u\rangle}{\|u\|_{1,p(.)}} \to \infty \ \text{ as } \|u\|_{1,p(.)} \to \infty.$$

Proof. There exists a constant $C > 0$ such that (see [33])

$$\left|\int_\Omega h_\epsilon(x, u, \nabla u)\varphi dx\right| \leq C\|\varphi\|_{1,p(.)}. \tag{4.3}$$

Since $\beta_\epsilon \circ T_{\frac{1}{\epsilon}}$ is bounded in $L^{p'(.)}(\Omega)$, there exists a constant $C' > 0$ such that, by applying a Hölder-type inequality, we obtain

$$\begin{aligned} \left|\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\varphi dx\right| &\leq \int_\Omega |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\varphi| dx \\ &\leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'}\right)\|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\|_{p'(.)}\|\varphi\|_{p(.)} \\ &\leq C'\|\varphi\|_{1,p(.)}. \end{aligned} \tag{4.4}$$

By applying the Hölder-type inequality along with the growth condition (3.5), we can prove that A is bounded. Consequently, from (3.2), (4.3) and (4.4), we conclude that A_ϵ is bounded. Regarding coercivity, one has for any $u \in W_0^{1,p(.)}(\Omega)$,

$$\begin{aligned}
\frac{\langle A_\epsilon(u), u\rangle}{\|u\|_{1,p(.)}} &= \frac{\langle Au, u\rangle + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))udx + \int_\Omega g_\epsilon(x,u,\nabla u)udx}{\|u\|_{1,p(.)}} \\
&\geq \frac{\int_\Omega a(x,u,\nabla u)\nabla udx}{\|u\|_{1,p(.)}} \qquad \text{(by neglecting positive terms)} \\
&\geq \lambda \frac{\int_\Omega |\nabla u|^{p(x)}dx}{\|u\|_{1,p(.)}} \to \infty \text{ as } \|u\|_{1,p(.)} \to \infty \text{ (since } p_- > 1).
\end{aligned}$$

Therefore, A_ϵ is coercive.
It now remains to establish that A_ϵ is pseudo-monotone. Let $(u_m)_{m\in\mathbb{N}}$ be a sequence in $W_0^{1,p(.)}(\Omega)$ such that

$$\begin{cases}
u_m \rightharpoonup u \text{ in } W_0^{1,p(.)}(\Omega) \text{ as } m \to \infty, \\
A_\epsilon u_m \rightharpoonup \chi \text{ in } W^{-1,p'(.)}(\Omega) \text{ as } m \to \infty, \\
\lim_{m\to\infty} \sup\langle A_\epsilon u_m, u_m\rangle \leq \langle \chi, u\rangle.
\end{cases} \tag{4.5}$$

Let us prove that

$$\langle A_\epsilon u_m, u_m\rangle \longrightarrow \langle \chi, u\rangle \text{ as } m \longrightarrow \infty, \text{ where } \chi = A_\epsilon u.$$

Given the compact embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)}(\Omega)$, there exists a subsequence, still denoted $(u_m)_{m\in\mathbb{N}}$, such that $u_m \to u$ in $L^{p(.)}(\Omega)$ as $m \to \infty$. Since $(u_m)_{m\in\mathbb{N}}$ is bounded in $W_0^{1,p(.)}(\Omega)$, and by using the growth condition, it follows that $(a(x,u_m,\nabla u_m))_{m\in\mathbb{N}}$ is bounded in $(L^{p'(.)}(\Omega))^N$.
Therefore, there exists a function $\xi \in (L^{p'(.)}(\Omega))^N$ such that

$$a(x,u_m,\nabla u_m) \rightharpoonup \xi \text{ in } (L^{p'(.)}(\Omega))^N \text{ as } m \to \infty. \tag{4.6}$$

Since $(h_\epsilon(x,u_m,\nabla u_m))_{m\in\mathbb{N}}$ is bounded in $(L^{p'(.)}(\Omega))^N$, there exists a function $\psi_\epsilon \in (L^{p'(.)}(\Omega))^N$ such that

$$h_\epsilon(x,u_m,\nabla u_m)) \rightharpoonup \psi_\epsilon \text{ in } (L^{p'(.)}(\Omega))^N \text{ as } m \to \infty. \tag{4.7}$$

Given that $|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_m))| \leq \frac{1}{\epsilon^2}$ as $m \to \infty$, and $u_m \longrightarrow u$ a.e. in Ω, we can apply the Lebesgue dominated convergence theorem to conclude that

$$\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_m)) \longrightarrow \beta_\epsilon(T_{\frac{1}{\epsilon}}(u)) \text{ in } L^{(p_-)'}(\Omega) \text{ as } m \to \infty. \tag{4.8}$$

For all $v \in W_0^{1,p(.)}(\Omega)$, we have

$$\begin{aligned}\langle \chi, v\rangle &= \lim_{m\to\infty} \langle A_\epsilon u_m, v\rangle \\ &= \lim_{m\to\infty} \int_\Omega a(x, u_m, \nabla u_m)\nabla v dx + \lim_{m\to\infty} \int_\Omega h_\epsilon(x, u_m, \nabla u_m) v dx \\ &+ \lim_{m\to\infty} \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_m)) v dx \\ &= \int_\Omega \xi \nabla v dx + \int_\Omega \psi_\epsilon v dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u)) v dx. \end{aligned} \tag{4.9}$$

Applying (4.5) and (4.9), we obtain

$$\begin{aligned}&\lim_{m\to\infty} \sup \langle A_\epsilon u_m, u_m\rangle \\ &= \lim_{m\to\infty} \sup \left\{ \int_\Omega a(x, u_m, \nabla u_m)\nabla u_m dx + \int_\Omega h_\epsilon(x, u_m, \nabla u_m) u_m dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_m)) u_m dx \right\} \\ &\le \int_\Omega \xi \nabla u dx + \int_\Omega \psi_\epsilon u dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u)) u dx. \end{aligned} \tag{4.10}$$

Thanks to (4.7), one has

$$\int_\Omega h_\epsilon(x, u_m, \nabla u_m) u_m dx \longrightarrow \int_\Omega \psi_\epsilon u dx \text{ as } m \to \infty. \tag{4.11}$$

By using (4.8), one obtains

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_m)) u_m dx \longrightarrow \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u)) u dx \text{ as } m \to \infty. \tag{4.12}$$

It follows that

$$\lim_{m\to\infty} \sup \int_\Omega a(x, u_m, \nabla u_m)\nabla u_m dx \le \int_\Omega \xi \nabla u dx. \tag{4.13}$$

Using (3.3), we have

$$\int_\Omega (a(x, u_m, \nabla u_m) - a(x, u_m, \nabla u))(\nabla u_m - \nabla u) dx \ge 0,$$

then

$$\int_\Omega a(x, u_m, \nabla u_m)\nabla u_m dx \ge \int_\Omega a(x, u_m, \nabla u_m)\nabla u dx + \int_\Omega a(x, u_m, \nabla u)(\nabla u_m - \nabla u) dx.$$

Since $\nabla u_m \rightharpoonup \nabla u$ in $L^{p(.)}(\Omega)$ and using (4.6), we obtain

$$\lim_{m\to\infty} \inf \int_\Omega a(x, u_m, \nabla u_m)\nabla u_m dx \ge \int_\Omega \xi \nabla u dx.$$

Due to (4.13), one obtains

$$\lim_{m\to\infty} \int_\Omega a(x, u_m, \nabla u_m)\nabla u_m dx = \int_\Omega \xi \nabla u dx. \tag{4.14}$$

Therefore, combining (4.9), (4.11), (4.12) and (4.14), we obtain

$$\langle A_\epsilon u_m, u_m\rangle \to \langle \chi_\epsilon, u\rangle \text{ as } m\to\infty.$$

It remain to prove that $a(x,u_m,\nabla u_m) \rightharpoonup a(x,u,\nabla u)$ in $(L^{p'(.)}(\Omega))^N$ and

$$h_\epsilon(x,u_m,\nabla u_m) \rightharpoonup h_\epsilon(x,u,\nabla u) \text{ in } L^{p'(.)}(\Omega) \text{ as } m\to\infty.$$

From (4.14), we can prove that

$$\lim_{m\to\infty}\int_\Omega (a(x,u_m,\nabla u_m)-a(x,u_m,\nabla u))(\nabla u_m-\nabla u)dx = 0.$$

Thanks to Lemma 3.2, one obtains

$$u_m \to u \;\text{ in } W_0^{1,p(.)}(\Omega) \text{ and } \nabla u_m \to \nabla u \text{ a.e. in } \Omega \text{ as } m\to\infty.$$

Then, we conclude that

$$a(x,u_m,\nabla u_m) \rightharpoonup a(x,u,\nabla u) \text{ in } (L^{p'(.)}(\Omega))^N \text{ as } m\to\infty \tag{4.15}$$

and

$$h_\epsilon(x,u_m,\nabla u_m) \rightharpoonup h_\epsilon(x,u,\nabla u) \text{ in } L^{p'(.)}(\Omega) \text{ as } m\to\infty, \tag{4.16}$$

Thus, we can express $\chi_\epsilon = A_\epsilon u$, which ends the proof of Lemma 4.3. □

Since A_ϵ is bounded, coercive, and pseudo-monotone, it follows from Theorem 2.7 in [27] (see also [28]) that A_ϵ is surjective. Therefore, for any $f\in W^{-1,p'(.)}(\Omega)$, there exists at least one solution $u_\epsilon\in W_0^{1,p(.)}(\Omega)$ for the problem (E_ϵ,f), thereby completing the proof of Proposition 4.2. □

Step 2 A priori estimates.

Lemma 4.4. *Let $u_\epsilon\in W_0^{1,p(.)}(\Omega)$ be a weak solution of problem (E_ϵ,f) under assumption $f\in L^\infty(\Omega)$. Then, one has*
- *there exists a constant $C_1>0$ that is independent of ϵ such that*

$$\|u_\epsilon\|_{1,p(.)}\le C_1, \tag{4.17}$$

$$\|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\|_\infty \le \|f\|_\infty. \tag{4.18}$$

- *For any $k\ge 1$, there exists a constant $C_2>0$ not depending on k, such that*

$$\|\nabla T_k(u_\epsilon)\|_{p(.)} \le C_2 k^{\frac{1}{\gamma}} \tag{4.19}$$

and there exists a constant $C_3>0$ not depending on ϵ such that

$$\int_\Omega u_\epsilon h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)dx \le C_3, \tag{4.20}$$

where $\gamma = \begin{cases} p_+ \text{ if } \|\nabla T_k(u_\epsilon)\|_{p(.)}\le 1 \\ p_- \text{ if } \|\nabla T_k(u_\epsilon)\|_{p(.)} > 1. \end{cases}$

Proof. Taking u_ϵ as test function in (4.2), we obtain

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))u_\epsilon dx + \int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon dx + \int_\Omega h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)u_\epsilon dx$$

$$= \int_\Omega f u_\epsilon dx. \tag{4.21}$$

Since $\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))u_\epsilon dx \geq 0$, and h_ϵ satisfies the sign condition, one can infer from (3.1) that

$$\lambda \int_\Omega |\nabla u_\epsilon|^{p(x)} dx \leq \int_\Omega f u_\epsilon dx. \tag{4.22}$$

Thanks to Poincaré inequality, there exists a constant $C_0 > 0$ (depending on p_- and Ω) such that

$$\int_\Omega |u_\epsilon|^{p_-} dx \leq C_0^{p_-} \int_\Omega |\nabla u_\epsilon|^{p_-} dx.$$

Applying Young's inequality, one obtains

$$\begin{aligned}
\int_\Omega f u_\epsilon dx &\leq \int_\Omega |f||u_\epsilon| dx \\
&= \int_\Omega \left(\frac{2^{\frac{1}{p_-}} C_0}{(\lambda p_-)^{\frac{1}{p_-}}}|f|\right)\left(\frac{(\lambda p_-)^{\frac{1}{p_-}}}{2^{\frac{1}{p_-}} C_0}|u_\epsilon|\right) dx \\
&\leq \int_\Omega \frac{1}{(p_-)'}\left(\frac{2^{\frac{1}{p_-}} C_0}{(\lambda p_-)^{\frac{1}{p_-}}}\right)^{(p_-)'} |f|^{(p_-)'} dx + \int_\Omega \frac{1}{p_-}\left(\frac{(\lambda p_-)^{\frac{1}{p_-}}}{2^{\frac{1}{p_-}} C_0}\right)^{p_-} |u_\epsilon|^{p_-} dx \\
&\leq C_4 \int_\Omega |f|^{(p_-)'} dx + \frac{\lambda}{2C_0^{p_-}} \int_\Omega |u_\epsilon|^{p_-} dx \\
&\leq C_4 \|f\|^{(p_-)'}_{L^\infty(\Omega)} \text{meas}(\Omega) + \frac{\lambda}{2}\int_\Omega |\nabla u_\epsilon|^{p_-} dx \\
&= C_4 \|f\|^{(p_-)'}_{L^\infty(\Omega)} \text{meas}(\Omega) + \frac{\lambda}{2}\left(\int_{\{|\nabla u_\epsilon|<1\}} |\nabla u_\epsilon|^{p_-} dx + \int_{\{|\nabla u_\epsilon|\geq 1\}} |\nabla u_\epsilon|^{p_-} dx\right) \\
&\leq C_4 \|f\|^{(p_-)'}_{L^\infty(\Omega)} \text{meas}(\Omega) + \frac{\lambda}{2}\text{meas}(\Omega) + \frac{\lambda}{2}\int_\Omega |\nabla u_\epsilon|^{p(x)} dx.
\end{aligned}$$

According to (4.22), one obtains

$$\int_\Omega |\nabla u_\epsilon|^{p(x)} dx \leq \text{meas}(\Omega)\left(C_4 \|f\|^{(p_-)'}_{L^\infty(\Omega)} + \frac{\lambda}{2}\right). \tag{4.23}$$

Using Proposition 2.2, we get

$$\|\nabla u_\epsilon\|^{\alpha}_{p(.)} \leq \text{meas}(\Omega)\left(C_4 \|f\|^{(p_-)'}_{L^\infty(\Omega)} + \frac{\lambda}{2}\right),$$

where

$$\alpha = \begin{cases} p_+ \text{ if } \|\nabla u_\epsilon\|_{p(.)} \leq 1, \\ \\ p_- \text{ if } \|\nabla u_\epsilon\|_{p(.)} > 1. \end{cases}$$

Hence,

$$\|\nabla u_\epsilon\|_{p(.)} \leq C_5,$$

where $C_5 := \left[\text{meas}(\Omega)\left(C_4\|f\|_{L^\infty(\Omega)}^{(p_-)'} + \frac{\lambda}{2}\right)\right]^{\frac{1}{\alpha}}$.

Thanks to Remark 2.5, the two norms $\|\nabla u\|_{p(.)}$ and $\|u\|_{1,p(.)}$ are equivalent in $W_0^{1,p(.)}(\Omega)$. Therefore, one has

$$\|u_\epsilon\|_{1,p(x)} \leq C_1.$$

For $\delta > 0$, one applies $\varphi_{\delta,\epsilon} = \dfrac{1}{\delta}\left[T_{k+\delta}(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))) - T_k(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))\right]$ as a test function in (4.2) to obtain

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_{\delta,\epsilon}dx + \int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla\varphi_{\delta,\epsilon}dx + \int_\Omega h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi_{\delta,\epsilon}dx = \int_\Omega f\varphi_{\delta,\epsilon}dx. \tag{4.24}$$

Since $\nabla\varphi_{\delta,\epsilon} = \begin{cases} \frac{1}{\delta}(\beta'_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))\nabla u_\epsilon & \text{if } k \leq |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))| \leq k+\delta, \\ \\ 0 & \text{elsewhere,} \end{cases}$

and as β_ϵ is non-decreasing, by using (3.1), one gets

$$\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla\varphi_{\delta,\epsilon}dx = \frac{1}{\delta}\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)(\beta'_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\nabla u_\epsilon dx \geq 0.$$

Sine $\varphi_{\delta,\epsilon}$ has the same sign as u_ϵ, we infer from (3.4) that

$$\int_\Omega h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi_{\delta,\epsilon}dx \geq 0.$$

Then,

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_{\delta,\epsilon}dx \leq \int_\Omega f\varphi_{\delta,\epsilon}dx.$$

Therefore

$$\int_{\{k+\delta\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_{\delta,\epsilon}dx \leq \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_{\delta,\epsilon}dx \leq \int_\Omega f\varphi_{\delta,\epsilon}dx.$$

Since $\varphi_{\delta,\epsilon} = 0$ on the set $\{|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))| < k\}$ and $|\varphi_{\delta,\epsilon}| \leq 1$, one deduces that

$$\frac{1}{\delta}\int_{\{k+\delta\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_{\delta,\epsilon}dx \leq \int_{\{k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}} |f|dx. \tag{4.25}$$

Since $k \leq |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|$ on the set $\{k+\delta \leq |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}$, one has

$$\begin{aligned} k\text{meas}\{k+\delta \leq |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\} &\leq \int_{\{k+\delta\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}} |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|dx \\ &\leq \frac{1}{\delta}\int_{\{k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) \\ &\times \Big[T_{k+\delta}(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))) - T_k(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))\Big]dx \\ &\leq \int_{\{k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}} |f|dx \\ &\leq \|f\|_{L^\infty(\Omega)}\text{meas}\{k \leq |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}. \end{aligned}$$

Letting $\delta \to 0$ and choosing $k > \|f\|_{L^\infty(\Omega)}$, we obtain

$$k\text{meas}\{k \leq |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\} \leq \|f\|_{L^\infty(\Omega)}\text{meas}\{k \leq |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}.$$

It follows necessarily that $\text{meas}\{k \leq |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\} = 0$ for any $k > \|f\|_{L^\infty(\Omega)}$. Thus,

$$\|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}.$$

Taking $T_k(u_\epsilon)$ as a test function in (4.2), we obtain

$$\begin{aligned} \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))T_k(u_\epsilon)dx + \int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)dx + \int_\Omega h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)T_k(u_\epsilon)dx \\ = \int_\Omega fT_k(u_\epsilon)dx. \end{aligned}$$

Neglecting the positive terms $\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))T_k(u_\epsilon)dx$ and $\int_\Omega h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)T_k(u_\epsilon)dx$, we obtain

$$\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)dx = \int_{\{|u_\epsilon|\leq k\}} (a(x,u_\epsilon,\nabla u_\epsilon))\nabla T_k(u_\epsilon)dx \leq \int_\Omega fT_k(u_\epsilon)dx.$$

From (3.1), we deduce that

$$\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)}dx \leq \frac{k\|f\|_\infty}{\lambda}.$$

Using Proposition 2.2, we get

$$\|\nabla T_k(u_\epsilon)\|^\gamma_{p(.)} \leq \frac{k\|f\|_\infty}{\lambda}.$$

Therefore,

$$\|\nabla T_k(u_\epsilon)\|_{p(.)} \leq C_2 \text{ for all } k \geq 1,$$

where $C_2 := \left(\frac{k\|f\|_\infty}{\lambda}\right)^{\frac{1}{\gamma}}$.

It now remains to prove that (4.20) is satisfied.

According to (4.21) one obtains
$$\int_\Omega u_\epsilon h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)dx \le \int_\Omega f u_\epsilon dx.$$
Then, we apply (4.17) to get
$$\int_\Omega u_\epsilon h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)dx \le C_3.$$
□

Lemma 4.5. *Let u_ϵ be a solution of (E_ϵ, f) and let $k>0$. Then, we have*
$$\int_\Omega (|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)| - k)^+ dx \le \int_\Omega (|f|-k)^+ dx. \tag{4.26}$$

Proof. Using the test function $\varphi = H_\delta^+(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)$ in (4.2) we obtain
$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))H_\delta^+(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))-k)dx + \int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla H_\delta^+(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))-k)dx$$
$$+\int_\Omega h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)H_\delta^+(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))-k)dx = \int_\Omega f H_\delta^+(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))-k)dx.$$
Since β_ϵ is non-decreasing, using (3.1), one gets
$$\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)(H_\delta^+)'(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))-k)\beta_\epsilon'(T_{\frac{1}{\epsilon}}(u_\epsilon))\nabla u_\epsilon dx \ge 0.$$
Using the sign condition on h_ϵ we deduce that
$$\int_\Omega h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)H_\delta^+(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))-k)dx \ge 0.$$
Then, we have
$$\int_\Omega (\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)-k)H_\delta^+(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))-k)dx \le \int_\Omega (f-k)H_\delta^+(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))-k)dx.$$
Letting $\delta \to 0$ we obtain
$$\int_\Omega (\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)-k)^+ dx \le \int_\Omega (f-k)^+ dx. \tag{4.27}$$
A similar reasoning gives
$$\int_\Omega (\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)+k)^- dx \le \int_\Omega (f+k)^- dx. \tag{4.28}$$
By combining (4.27) and (4.28), it follows (4.26). □

Step 3 Convergence results.

Proposition 4.6. *Let $k>0$, and suppose that u_ϵ is a solution of (E_ϵ, f). Then, there exists a function b in $L^\infty(\Omega)$ such that*
$$\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) \rightharpoonup b \text{ weakly-* in } L^\infty(\Omega) \text{ as } \epsilon \to 0. \tag{4.29}$$

Proof. Thanks to (4.18), we can conclude that there exists a function $b \in L^\infty(\Omega)$ satisfying (4.29). □

Proposition 4.7. *There exists* $u \in W_0^{1,p(.)}(\Omega)$ *verifying* $u \in dom(\beta)$ *a.e. in* Ω,

$$u_\epsilon \text{ converges to } u \text{ in measure and almost everywhere in } \Omega \text{ as } \epsilon \to 0, \quad (4.30)$$

$$u_\epsilon \to u \quad \text{in } L^{p(.)}(\Omega) \text{ and } u_\epsilon \rightharpoonup u \text{ in } W_0^{1,p(.)}(\Omega) \text{ as } \epsilon \to 0, \quad (4.31)$$

and

$$T_k(u_\epsilon) \to T_k(u) \text{ in } L^{p(.)}(\Omega) \text{ and } T_k(u_\epsilon) \rightharpoonup T_k(u) \text{ in } W_0^{1,p(.)}(\Omega) \text{ as } \epsilon \to 0. \quad (4.32)$$

Proof. Thanks to (4.17) and (4.19) in Lemma 4.4, it follows that the sequences $(u_\epsilon)_{\epsilon>0}$ and $(T_k(u_\epsilon))_{\epsilon>0}$ are bounded in $W_0^{1,p(.)}(\Omega)$. Then, there exists a subsequence still denoted $(T_k(u_\epsilon))_{\epsilon>0}$, and a measurable function $\sigma_k \in W_0^{1,p(.)}(\Omega)$ such that

$$\begin{cases} T_k(u_\epsilon) \rightharpoonup \sigma_k \text{ in } W_0^{1,p(.)}(\Omega) \text{ as } \epsilon \to 0, \\ T_k(u_\epsilon) \to \sigma_k \text{ in } L^{p(.)}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \quad (4.33)$$

Let us now show that $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in measure on Ω.
Indeed, using a Hölder-type inequality and (4.19), one obtains

$$\begin{aligned} k\text{meas}\{|u_\epsilon| > k\} &= \int_{\{|u_\epsilon|>k\}} |T_k(u_\epsilon)|dx \le \int_\Omega |T_k(u_\epsilon)|dx \\ &\le \left(\frac{1}{p^-} + \frac{1}{p^+}\right)\|1\|_{p'(.)}\|T_k(u_\epsilon)\|_{p(x)} \\ &\le \left(\frac{1}{p^-} + \frac{1}{p^+}\right)(\text{meas}(\Omega)+1)^{\frac{1}{p'^-}}\|T_k(u_\epsilon)\|_{p(.)} \\ &\le C_5 k^{\frac{1}{\gamma}}. \end{aligned}$$

Consequently,

$$\text{meas}\{|u_\epsilon| > k\} \le C_5 \frac{1}{k^{1-\frac{1}{\gamma}}} \to 0 \text{ as } k \to \infty. \quad (4.34)$$

For all $\delta > 0$, one has
$\text{meas}\{|u_{\epsilon_1} - u_{\epsilon_2}| > \delta\} \le \text{meas}\{|u_{\epsilon_1}| > k\} + \text{meas}\{|u_{\epsilon_2}| > k\} + \text{meas}\{|T_k(u_{\epsilon_1}) + T_k(u_{\epsilon_2})| > k\}$.
Let $\theta > 0$, applying (4.34), we can choose $k = k(\theta)$ such that

$$\text{meas}\{|u_{\epsilon_1}| > k\} \le \frac{\theta}{3} \text{ and } \text{meas}\{|u_{\epsilon_2}| > k\} \le \frac{\theta}{3}. \quad (4.35)$$

The estimate (4.17) implies that the sequence $(u_\epsilon)_{\epsilon>0}$ is Cauchy in measure on Ω.
Thus, for any $k, \theta, \delta > 0$ there exists $n_0 = n_0(k, \delta, \theta)$ such that

$$\text{meas}\{|T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})| > \delta\} \le \frac{\theta}{3} \text{ for any } \epsilon_1, \epsilon_2 \ge n_0. \quad (4.36)$$

We can deduce from (4.35) and (4.36) that for every δ, θ, there exists $n_0 = n_0(\delta, \theta)$ such that

$$\text{meas}\{|u_{\epsilon_1} - u_{\epsilon_2}| > \delta\} \le \theta \text{ for any } \epsilon_1, \epsilon_2 \ge n_0,$$

which means that $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in measure, and there exists a subsequence still labeled as $(u_\epsilon)_{\epsilon>0}$, along with a measurable function u such that

$$u_\epsilon \longrightarrow u \text{ a.e. in } \Omega, \text{ as } \epsilon \to 0.$$

Hence $\sigma_k = T_k(u)$ a.e. in Ω and so $u \in \mathcal{T}_0^{1,p(.)}(\Omega)$. We can also conclude that

$$\begin{cases} u_\epsilon \rightharpoonup u \text{ in } W_0^{1,p(.)}(\Omega) \text{ as } \epsilon \to 0, \\ u_\epsilon \to u \text{ in } L^{p(.)}(\Omega) \text{ and a.e. in } \Omega \text{ as } \epsilon \to 0. \end{cases}$$

We have $T_k(u) \in \mathrm{dom}\beta$ a.e. in Ω for any $k > 0$. Consequently, $u \in \mathrm{dom}\beta$ a.e. in Ω (see [3]). □

Proposition 4.8. *Let u_ϵ be a solution for problem (E_ϵ, f) and $t > 0$. Then, up to a subsequence, there exists a measurable function u such that*

$$T_t(u_\epsilon) \to T_k(u) \text{ strongly in } W_0^{1,p(.)}(\Omega), \tag{4.37}$$

$$\nabla u_\epsilon \to \nabla u \text{ a.e. in } \Omega, \tag{4.38}$$

$$a(x, u_\epsilon, \nabla u_\epsilon) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L^{p'(.)}(\Omega))^N \tag{4.39}$$

and

$$h_\epsilon(x, u_\epsilon, \nabla u_\epsilon) \to h(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \tag{4.40}$$

Proof. For any $t > 0$, we will use the following as a test function

$$\varphi_\epsilon = \varphi(T_t(u_\epsilon) - T_t(u)),$$

where

$$\varphi(s) = se^{\alpha s^2} \text{ and } \alpha = \left(\frac{d(k)}{\lambda}\right)^2.$$

It is widely recognized (see [9]) that

$$\varphi'(s) - \frac{d(k)}{\lambda}|\varphi(s)| \geq \frac{1}{2} \quad \forall s \in \mathbb{R}. \tag{4.41}$$

From this point forward, we will denote various sequences of real numbers that converge to zero as $\eta^1(\epsilon)$, $\eta^2(\epsilon)$, ..., as ϵ approaches zero.
Choosing φ_ϵ as a test function in (4.2), one obtains

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_\epsilon dx + \int_\Omega a(x, u_\epsilon, \nabla u_\epsilon)\nabla\varphi_\epsilon dx + \int_\Omega h_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon dx = \int_\Omega f_\epsilon\varphi_\epsilon dx.$$

Since $h_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon \geq 0$ and $\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_\epsilon \geq 0$ on $\{|u_\epsilon| > t\}$, one deduces that

$$\int_{\{|u_\epsilon|\leq t\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_\epsilon dx + \int_\Omega a(x, u_\epsilon, \nabla u_\epsilon)\nabla\varphi_\epsilon dx + \int_{\{|u_\epsilon|\leq t\}} h_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon dx \leq \int_\Omega f_\epsilon\varphi_\epsilon dx. \tag{4.42}$$

According to Lebesgue generalized convergence theorem, one has

$$\int_\Omega f_\epsilon\varphi_\epsilon dx = \eta^1(\epsilon). \tag{4.43}$$

Since the sequence $(\chi_{\{|u_\epsilon|\leq t\}}\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))_{\epsilon>0}$ is uniformly bounded, one can apply the Lebesgue dominated converge theorem to conclude that

$$\int_{\{|u_\epsilon|\leq t\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_\epsilon dx = \eta^3(\epsilon). \tag{4.44}$$

We will now define $G_t(z) = z - T_t(z)$ for any z and t in $\mathbb{R}$ with $t \geq 0$. Then, we decompose the second term of (4.42) as follows.

$$\int_\Omega a(x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla(T_t(u_\epsilon) - T_t(u))\varphi'(T_t(u_\epsilon) - T_t(u))dx$$

$$= \int_\Omega a(x, T_t(u_\epsilon), \nabla T_t(u_\epsilon)) \cdot \nabla(T_t(u_\epsilon) - T_t(u))\varphi'(T_t(u_\epsilon) - T_t(u))dx$$

$$+ \int_\Omega a(x, u_\epsilon, \nabla G_t(u_\epsilon)) \cdot \nabla(T_t(u_\epsilon) - T_t(u))\varphi'(T_t(u_\epsilon) - T_t(u))dx. \tag{4.45}$$

Since $\nabla T_t(u_\epsilon)$ is zero where $\nabla G_t(u_\epsilon)$ is different to zero, and conversely, we have

$$\int_\Omega a(x, u_\epsilon, \nabla G_t(u_\epsilon)) \cdot \nabla(T_t(u_\epsilon) - T_t(u))\varphi'(T_t(u_\epsilon) - T_t(u))dx$$

$$= -\int_\Omega a(x, u_\epsilon, \nabla G_t(u_\epsilon)) \cdot \nabla T_t(u)\varphi'(T_t(u_\epsilon) - T_t(u))dx.$$

Since $\nabla T_t(u_\epsilon) \equiv 0$ on the set $\{|u_\epsilon| \geq t\}$, one has

$$\nabla T_t(u)\chi_{\{|u_\epsilon|\geq t\}} \to 0, \quad \text{a.e. in } \Omega \text{ as } \epsilon \to 0.$$

Using the fact that $\nabla T_t(u_\epsilon) \in (L^{p'(.)}(\Omega))^N$, we can apply the Lebesgue dominated convergence theorem to conclude that

$$\nabla T_t(u)\chi_{\{|u_\epsilon|\geq t\}} \to 0, \text{ strongly in } (L^{p'(.)}(\Omega))^N \text{ as } \epsilon \to 0.$$

Having in mind that $(a(x, u_\epsilon, \nabla G_t(u_\epsilon)))_{\epsilon>0}$ is bounded in $(L^{p'(.)}(\Omega))^N$, we obtain

$$\int_\Omega a(x, u_\epsilon, \nabla G_t(u_\epsilon)) \cdot \nabla(T_t(u_\epsilon) - T_t(u))\varphi'(T_t(u_\epsilon) - T_t(u))dx = \eta^3(\epsilon) \to 0, \text{ as } \epsilon \to 0. \tag{4.46}$$

We now decompose the second term of (4.45) in the following way.

$$\int_\Omega a(x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla(T_t(u_\epsilon) - T_t(u))\varphi'(T_t(u_\epsilon) - T_t(u))dx$$

$$= \int_\Omega [a(x, T_t(u_\epsilon), \nabla T_t(u_\epsilon)) - a(x, T_t(u_\epsilon), \nabla T_t(u))] \cdot \nabla(T_t(u_\epsilon) - T_t(u))\varphi'(T_t(u_\epsilon) - T_t(u))dx$$

$$+ \int_\Omega a(x, T_t(u_\epsilon), \nabla T_t(u)) \cdot \nabla(T_t(u_\epsilon) - T_t(u))\varphi'(T_t(u_\epsilon) - T_t(u))dx.$$

Let us observe that $T_t(u_\epsilon)$ converges weakly to $T_t(u)$ in $W_0^{1,p'(.)}(\Omega)$ as $\epsilon \to 0$, $\lim_{\epsilon\to 0} \varphi'(T_t(u_\epsilon) - T_t(u)) = 0$, and the sequence $(a(x, T_t(u_\epsilon), \nabla T_t(u))))_{\epsilon>0}$ is bounded in $(L^{p'(.)}(\Omega))^N$. From these observations, we can conclude that

$$\int_\Omega a(x, T_t(u_\epsilon), \nabla T_t(u)) \cdot \nabla(T_t(u_\epsilon) - T_t(u))\varphi'(T_t(u_\epsilon) - T_t(u))dx = \eta^4(\epsilon). \tag{4.47}$$

Thus, putting together (4.46) and (4.47), we obtain

$$\int_\Omega a(x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla(T_t(u_\epsilon) - T_t(u))\varphi'(T_t(u_\epsilon) - T_t(u))dx = \tag{4.48}$$

$$\int_\Omega [a(x,T_t(u_\epsilon),\nabla T_t(u_\epsilon))-a(x,T_t(u_\epsilon),\nabla T_t(u))]\cdot\nabla(T_t(u_\epsilon)-T_t(u))\varphi'(T_t(u_\epsilon)-T_t(u))dx$$
$$+\eta^5(\epsilon).$$

On the other hand, one has

$$\begin{aligned}\left|\int_{\{|u_\epsilon|\le t\}} h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi_\epsilon dx\right| &\le \int_{\{|u_\epsilon|\le t\}} |h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)||\varphi_\epsilon|dx\\ &\le d(t)\int_\Omega (c(x)+|\nabla T_t(u_\epsilon)|^{p(x)})|\varphi_\epsilon|dx.\end{aligned}$$

Since $c(.)$ belongs to $L^1(\Omega)$ and $\lim_{\epsilon\to 0}\varphi_\epsilon=\varphi(0)=0$, one has

$$\int_\Omega c(x)|\varphi_\epsilon|dx=\eta^5(\epsilon). \tag{4.49}$$

Then, using (3.1), it follows that

$$\left|\int_{\{|u_\epsilon|\le t\}} h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi_\epsilon dx\right| \le \frac{d(t)}{\lambda}\int_\Omega a(x,T_t(u_\epsilon),\nabla T_t(u_\epsilon))\cdot\nabla T_t(u_\epsilon)|\varphi_\epsilon|dx+\eta^5(\epsilon). \tag{4.50}$$

By adding and subtracting the term below in the right-hand side of the inequality above

$$\frac{d(t)}{\lambda}\int_\Omega a(x,T_t(u_\epsilon),\nabla T_t(u_\epsilon))\cdot\nabla(T_t(u_\epsilon)-T_t(u))|\varphi_\epsilon|dx,$$

we obtain

$$\left\{\begin{array}{l}\left|\int_{\{|u_\epsilon|\le t\}} h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi_\epsilon dx\right|\\ \le \frac{d(k)}{\lambda}\int_\Omega\Big(a(x,T_t(u_\epsilon),\nabla T_t(u_\epsilon)-a(x,T_t(u_\epsilon),\nabla T_t(u)\Big)\cdot\nabla(T_t(u_\epsilon)-T_t(u))|\varphi_\epsilon|dx\\ +\frac{d(t)}{\lambda}\int_\Omega a(x,T_t(u_\epsilon),\nabla T_t(u))\cdot\nabla T_t(u)|\varphi_\epsilon|dx+\eta^5(\epsilon).\end{array}\right. \tag{4.51}$$

Since $(a(x,T_t(u_\epsilon),\nabla T_t(u)))_{\epsilon>0}$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$ and φ_ϵ converges to zero as $\epsilon\to 0$, one has

$$\frac{d(t)}{\lambda}\int_\Omega a(x,T_t(u_\epsilon),\nabla T_t(u))\cdot\nabla T_t(u)|\varphi_\epsilon|dx=\eta^6(\epsilon).$$

Then, we deduce that

$$\left\{\begin{array}{l}\left|\int_{\{|u_\epsilon|\le t\}} h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi_\epsilon dx\right|\\ \le \frac{d(t)}{\lambda}\int_\Omega\Big(a(x,T_t(u_\epsilon),\nabla T_t(u_\epsilon))-a(x,T_t(u),\nabla T_t(u))\Big)\\ \times\nabla(T_t(u_\epsilon)-T_t(u))|\varphi_\epsilon|dx+\eta^7(\epsilon).\end{array}\right. \tag{4.52}$$

By combining (4.42) and (4.52), one deduces that

$$\int_\Omega [a(x,T_t(u_\epsilon),\nabla T_t(u_\epsilon))-a(x,T_t(u_\epsilon),\nabla T_t(u))]\cdot\nabla(T_t(u_\epsilon)-T_t(u))$$

$$\times [\varphi'_\epsilon - \frac{d(t)}{\lambda}|\varphi_\epsilon|]dx \leq \eta^7(\epsilon). \tag{4.53}$$

Then, using (4.41), we obtain

$$0 \leq \frac{1}{2}\int_\Omega [a(x,T_t(u_\epsilon),\nabla T_t(u_\epsilon)) - a(x,T_t(u_\epsilon),\nabla T_t(u))]\cdot\nabla(T_t(u_\epsilon)-T_t(u))dx \leq \eta^7(\epsilon).$$

Therefore

$$\lim_{\epsilon\to 0}\int_\Omega [a(x,T_t(u_\epsilon),\nabla T_t(u_\epsilon)) - a(x,T_t(u_\epsilon),\nabla T_t(u))]\cdot\nabla(T_t(u_\epsilon)-T_t(u))dx = 0. \tag{4.54}$$

Using Lemma 3.2, as $\epsilon \to 0$, one has

$$T_t(u_\epsilon) \to T_t(u) \text{ in } W_0^{1,p(.)}(\Omega), \tag{4.55}$$

then

$$\nabla u_\epsilon \to \nabla u \text{ a.e. in } \Omega.$$

We deduce that

$$a(x,u_\epsilon,\nabla u_\epsilon) \to a(x,u,\nabla u) \text{ a.e. in } \Omega \tag{4.56}$$

and

$$h_\epsilon(x,u_\epsilon,\nabla u_\epsilon) \to g(x,u,\nabla u) \text{ a.e. in } \Omega. \tag{4.57}$$

Since $(a(x,u_\epsilon,\nabla u_\epsilon))_{\epsilon>0}$ is bounded in $(L^{p'(.)}(\Omega))^N$, using (4.56) and Lemma 2.3, we have

$$a(x,u_\epsilon,\nabla u_\epsilon) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L^{p'(.)}(\Omega))^N \text{ as } \epsilon \to 0.$$

□

Proposition 4.9.

$$h_\epsilon(x,u_\epsilon,\nabla u_\epsilon) \to h(x,u,\nabla u) \textit{ strongly in } L^1(\Omega) \textit{ weakly in } (L^{p'(.)}(\Omega))^N \textit{ as } \epsilon \to 0. \tag{4.58}$$

Proof. Let us prove that $h(x,u_\epsilon,\nabla u_\epsilon)$ is uniformly equi-integrable. To achieve this, we apply a classical technique similar to that used in [3, 5, 8, 10].
For any measurable subset B of Ω and for any $\nu \in \mathbb{R}^+$, one has

$$\begin{aligned}\int_B |h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx &= \int_{B\cap\{|u_\epsilon|\leq\nu\}} |h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx + \int_{B\cap\{|u_\epsilon|>\nu\}} |h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx \\ &= \int_{B\cap\{|u_\epsilon|\leq\nu\}} |h_\epsilon(x,T_\nu(u_\epsilon),\nabla T_\nu(u_\epsilon))|dx \\ &\quad + \int_{B\cap\{|u_\epsilon|>\nu\}} |h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx.\end{aligned}$$

Since

$$\begin{aligned}\int_{\{|u_\epsilon|>\nu\}} |h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx &\leq \int_{\{|u_\epsilon|>\nu\}} \frac{1}{|u_\epsilon|} u_\epsilon h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)dx \\ &\leq \frac{1}{\nu}\int_\Omega u_\epsilon h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)dx,\end{aligned}$$

we deduce from (3.5) and (4.20) that

$$\int_E |h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx \leq d(\nu)\int_B (C(x)+|\nabla T_\nu(u_\epsilon)|^{p(x)})dx + \frac{C_3}{\nu}. \tag{4.59}$$

Thanks to (4.59) and the strong convergence of $\nabla T_\nu(u_\epsilon)$ in $(L^{p(\cdot)}(\Omega))^N$ as $\epsilon \to 0$, we deduce the equi-integrability of $h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)$ (see [3, 10]).
According to (4.57), we obtain

$$h_\epsilon(x,u_\epsilon,\nabla u_\epsilon) \to h(x,u,\nabla u) \text{ strongly in } L^1(\Omega) \text{ as } \epsilon \to 0.$$

□

Relying on the aforementioned convergence results, we take the limit in (4.2), as $\epsilon \to 0$, to get

$$\int_\Omega b\varphi dx + \int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega h(x,u,\nabla u)\varphi dx = \int_\Omega f\varphi dx.$$

Moreover, as $h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)u_\epsilon \geq 0$ a.e. in Ω, $h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)u_\epsilon \to h(x,u,\nabla u)u$ a.e. in Ω, as $\epsilon \to 0$ and

$$0 \leq \int_\Omega h_\epsilon(x,u_\epsilon,\nabla u_\epsilon)u_\epsilon dx \leq C_3,$$

by applying the Fatou's lemma, one obtains

$$h(x,u,\nabla u)u \in L^1(\Omega).$$

We complete the proof of Theorem 4.1 by expressing that $u \in D(\beta)$ and $b \in \beta(u)$ a.e. in Ω (see [3]). □

5. Problem with $f \in L^1(\Omega)$

This section is devoted to establish the existence of weak, renormalized, and entropy solution for problem (E,f) when the datum f belongs to $L^1(\Omega)$.

5.1. Existence of weak solution.

Theorem 5.1. *Suppose that $f \in L^1(\Omega)$. Then, the problem (E,f) has at least one weak solution $(u,b) \in W_0^{1,p(\cdot)}(\Omega) \times L^1(\Omega)$. Namely, $b(x) \in \beta(u(x))$ a.e. in Ω, $h(x,u,\nabla u) \in L^1(\Omega)$ and*

$$\int_\Omega b\varphi dx + \int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega h(x,u,\nabla u)\varphi dx = \int_\Omega f\varphi dx, \tag{5.1}$$

for any $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Proof. **Step 1 The approximated problem**
For each $n \in \mathbb{N}$, we consider the following approximated problem

$$(E,f_n)\begin{cases} \beta(u_n) - div(a(x,u_n,\nabla u_n)) + h(x,u_n,\nabla u_n) \ni f_n & \text{in } \Omega,\\ \\ u_n = 0 & \text{on } \partial\Omega,\end{cases}$$

where f_n is a sequence of L^∞-functions which converges strongly to f in $L^1(\Omega)$ and $|f_n| \leq |f|$. For example, one can choose $f_n = T_n(f)$.

Thanks to Section 4, there exists a solution $(u_n, b_n) \in W_0^{1,p(.)}(\Omega) \times L^\infty(\Omega)$ of (E, f_n) such that

$$\int_\Omega b_n \varphi dx + \int_\Omega a(x, u_n, \nabla u_n)\nabla\varphi dx + \int_\Omega h(x, u_n, \nabla u_n)\varphi dx = \int_\Omega f_n \varphi dx, \quad (5.2)$$

for all $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

Step 2 A priori estimates

Lemma 5.2. *Let (u_n, b_n) be a solution to (E, f_n), with $t \geq 1$. Then, there exists a constant $C_2 > 0$ not independent of t such that*

$$\|\nabla T_t(u_n)\|_{p(x)} \leq C_2 t^{\frac{1}{\gamma}} \quad (5.3)$$

and

$$\|b_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \quad (5.4)$$

where $\gamma = \begin{cases} p_+ \text{ if } \|\nabla T_k(u_\epsilon)\|_{p(.)} \leq 1 \\ p_- \text{ if } \|\nabla T_k(u_\epsilon)\|_{p(.)} > 1. \end{cases}$

Proof. Taking $T_t(u_n)$ as a test function in (5.2), we obtain

$$\int_\Omega b_n T_t(u_n) dx + \int_\Omega a(x, u_n, \nabla u_n)\nabla T_t(u_n) dx + \int_\Omega h(x, u_n, \nabla u_n) T_t(u_n) dx$$
$$= \int_\Omega f T_t(u_n) dx. \quad (5.5)$$

Given that the first term on the left hand side of (5.5) is non-negative and g satisfies the sign condition, it follows from (3.1) that

$$\int_\Omega |\nabla T_t(u_n)|^{p(x)} dx \leq \frac{k\|f\|_{L^1(\Omega)}}{\lambda}.$$

Applying Proposition 2.2, we obtain

$$\|\nabla T_t(u_n)\|_{p(.)}^\gamma \leq \frac{k\|f\|_{L^1(\Omega)}}{\lambda}.$$

Then,

$$\|\nabla T_t(u_n)\|_{p(x)} \leq C_6 t^{\frac{1}{\gamma}} \text{ for all } t \geq 1,$$

where $C_6 := \left(\frac{\|f\|_{L^1(\Omega)}}{\lambda}\right)^{\frac{1}{\gamma}}$.

Since $\int_\Omega b_n T_t(u_n) dx \geq 0$ and $\int_\Omega h(x, u_n, \nabla u_n) T_t(u_n) dx \geq 0$, we can infer from (5.5) that

$$\int_\Omega b_n T_t(u_n) dx \leq \int_\Omega f T_t(u_n) dx \leq k\|f\|_{L^1(\Omega)}.$$

Dividing the above inequality by $t > 1$, we obtain

$$\int_\Omega b_n \frac{1}{t} T_t(u_n) dx \leq \|f\|_{L^1(\Omega)}.$$

Since $b_n \in \beta(u_n)$ a.e. in Ω and $\lim_{t\to\infty} \frac{1}{t} T_t(u_n) = \text{sign}_0(u_n)$, we pass to the limit in the above inequality as $t \to \infty$ to obtain

$$\int_\Omega |b_n| dx \le \|f\|_{L^1(\Omega)}.$$

$\square$

Step 3 Convergence results and passage to the limit

Lemma 5.3. *Let (u_n, b_n) be a solution to (E, f_n). Then, as $n \to \infty$, one has*

$$b_n \rightharpoonup b \text{ weakly in } L^1(\Omega). \tag{5.6}$$

Proof. Let $(u_n^\epsilon, b_n^\epsilon)$ be a solution of the following problem

$$\begin{cases} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_n^\epsilon)) - \text{div} a(x, u_n^\epsilon, \nabla u_n^\epsilon) + h(x, u_n^\epsilon, \nabla u_n^\epsilon) = f_n & \text{in } \Omega \\ \\ u_n^\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof of (5.6) follows the same line as in [3].
Let $k > 0$, then from Lemma 4.5, we have

$$\int_\Omega (|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_n^\epsilon)| - k)^+ dx \le \int_\Omega (|f_n| - k)^+ dx. \tag{5.7}$$

Since $\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_n^\epsilon) \rightharpoonup b_n$ in $L^\infty(\Omega)$ as ϵ goes to 0, we obtain

$$\int_\Omega (|b_n| - k)^+ dx \le \int_\Omega (|f_n| - k)^+ dx. \tag{5.8}$$

Remark 5.4. We now make the following claim

$$\lim_{k\to\infty} \text{meas}(\{|f_n| \ge k\}) = 0, \tag{5.9}$$

$$\lim_{k\to\infty} \int_\Omega (|f_n| - k)^+ dx = 0 \tag{5.10}$$

and

$$\lim_{k\to\infty} k\text{meas}(\{|b_n| \ge k\}) = 0. \tag{5.11}$$

Indeed, since $\int_\Omega |f_n| dx \le \int_\Omega |f| dx = \|f\|_1$, taking the limit as $k \to \infty$ in the estimate $\text{meas}(\{|f_n| \ge k\}) \le \frac{1}{k} \int_\Omega |f_n| dx$ yields (5.9).
Since $A := (f_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ and $|f_n| \le |f| \in L^1(\Omega)$ for every $n \in \mathbb{N}$, it follows from Proposition 2.12 that the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly integrable.

$$\begin{aligned} \int_\Omega (|f_n| - k)^+ dx &= \int_{\{|f_n|\ge k\}} |f_n| dx - k\text{meas}(\{|f_n| \ge k\}) \\ &\le \int_{\{|f_n|\ge k\}} |f_n| dx \\ &\le \sup_{f_n\in A} \int_{\{|f_n|\ge k\}} |f_n| dx. \end{aligned}$$

Then, passing to the limit as $k \to \infty$, we obtain (5.10).

Claim: $b_n \rightharpoonup b$ in $L^1(\Omega)$ as $n \to \infty$.
Indeed,

$$\int_{\{|f_n|\geq k\}} |b_n| dx = \int_\Omega (|b_n| - k)^+ dx + k\text{meas}(\{|b_n| \geq k\})$$
$$\leq \int_\Omega (|f_n| - k)^+ dx + k\text{meas}(\{|b_n| \geq k\}).$$

Using (5.11), we have

$$\lim_{k\to\infty} \sup_{f_n \in \mathcal{F}} \left(\int_{\{|f_n|\geq k\}} |b_n| dx \right) = 0.$$

Therefore, the sequence $(b_n)_{n\in\mathbb{N}}$ is uniformly integrable. Consequently, it follows from Theorem 2.13 that it is relatively weakly compact in $L^1(\Omega)$.
Now, we can deduce that there exists a subsequence, still denoted $(b_n)_{n\in\mathbb{N}}$ such that

$$b_n \rightharpoonup b \text{ weakly in } L^1(\Omega) \text{ as } n \to \infty.$$

□

Remark 5.5. The estimate (5.3) leads to the following (see the proof of Proposition 4.7)

$$u_n \to u \text{ a.e. in } \Omega, \tag{5.12}$$
$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,p(.)}(\Omega), \tag{5.13}$$
$$T_k(u_n) \to T_k(u) \text{ in } L^{p(.)}(\Omega) \text{ and a.e. in } \Omega. \tag{5.14}$$

Lemma 5.6. *Let (u_n, b_n) be a solution of the problem (E, f_n), with $k > 0$. Then, letting $n \to \infty$, it follows that*

$$T_k(u_n) \to T_k(u) \textit{ strongly in } W_0^{1,p(.)}(\Omega), \tag{5.15}$$
$$\nabla u_n \to \nabla u \textit{ a.e. in } \Omega, \tag{5.16}$$
$$a(x, u_n, \nabla u_n) \to a(x, u, \nabla u) \textit{ a.e. in } \Omega, \tag{5.17}$$
$$h_n(x, u_n, \nabla u_n) \to h(x, u, \nabla u) \textit{ a.e. in } \Omega \tag{5.18}$$

and

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \textit{ weakly in } (L^{p'(.)}(\Omega))^N. \tag{5.19}$$

Proof. For any $k > 0$, we define the function

$$\varphi_n = \varphi(T_k(u_n) - T_k(u)),$$

where

$$\varphi(s) = se^{\alpha s^2} \text{ and } \alpha = \left(\frac{d(k)}{\lambda}\right)^2.$$

By taking φ_n as test function in (5.2), we get

$$\int_\Omega b_n \varphi_n dx + \int_\Omega a(x, u_n, \nabla u_n) \nabla \varphi_n dx + \int_\Omega h(x, u_n, \nabla u_n) \varphi_n dx = \int_\Omega f_n \varphi_n dx. \tag{5.20}$$

Let us denote by $\eta^1(n),\ \eta^2(n),...,$ various sequences of real numbers which converge to zero when $n\to\infty$.
Since $\varphi_n \overset{*}{\rightharpoonup} 0$ in $L^\infty(\Omega)$ and $f_n\to f$ in $L^1(\Omega)$ as $n\to\infty$, one has

$$\int_\Omega f_n\varphi_n dx=\eta^1(n).$$

As

$$\int_\Omega b_n\varphi_n dx=\int_{\{|u_n|\le k\}} b_n\varphi_n dx+\int_{\{|u_n|>k\}} b_n\varphi_n dx \tag{5.21}$$

and $b_n\in\beta(u_n)$, the second term of (5.21) is non-negative.
Notice that the function $\chi_{\{|u_n|\le k\}}b_n$ is uniformly bounded; thus, we can apply the Lebesgue dominated convergence theorem to obtain

$$\int_{\{|u_n|\le k\}} b_n\varphi_n dx=\eta^2(n).$$

Therefore, we deduce from (5.20) that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\varphi_n dx+\int_\Omega h(x,u_n,\nabla u_n)\varphi_n dx\le\eta^3(n). \tag{5.22}$$

Reasoning similarly as in the proof of Proposition 4.8, one deduces

$$\lim_{n\to\infty}\int_\Omega[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))dx=0. \tag{5.23}$$

Using Lemma 3.2, one has

$$T_k(u_n)\to T_k(u) \text{ in } W_0^{1,p(.)}(\Omega).$$

Thus

$$\nabla u_n\to\nabla u \text{ a.e. in } \Omega.$$

We can also conclude that

$$a(x,u_n,\nabla u_n)\to a(x,u,\nabla u) \text{ a.e. in } \Omega$$

and

$$h_n(x,u_n,\nabla u_n)\to h(x,u,\nabla u) \text{ a.e. in } \Omega.$$

Since the sequence $(a(x,u_n,\nabla u_n))_{n\in\mathbb{N}}$ is bounded in $(L^{p'(.)}(\Omega))^N$, one has

$$a(x,u_n,\nabla u_n)\rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L^{p'(.)}(\Omega))^N.$$

□

Proposition 5.7. *If u_n is a solution of (E,f_n), then we have*

$$h_n(x,u_n,\nabla u_n)\to h(x,u,\nabla u) \text{ strongly in } L^1(\Omega). \tag{5.24}$$

Proof. Since $h_n(x,u_n,\nabla u_n) \longrightarrow h(x,u,\nabla u)$ a.e. in Ω as $n\to\infty$, and given (3.5), it is sufficient to show that $g(x,u_n,\nabla u_n)$ is uniformly equi-integrable.
For any measurable subset B of Ω and any $\delta\in\mathbb{R}^+$, it holds that

$$\begin{aligned}\int_B |h_n(x,u_n,\nabla u_n|dx &= \int_{E\cap\{|u_n|\le\delta\}} |h_n(x,u_n,\nabla u_n)|dx \\ &\quad + \int_{B\cap\{|u_n|>\delta\}} |h_n(x,u_n,\nabla u_n)|dx \\ &= \int_{B\cap\{|u_n|\le\delta\}} |h_n(x,T_\delta(u_n),\nabla T_\delta(u_n))|dx \\ &\quad + \int_{B\cap\{|u_n|>\delta\}} |h_n(x,u_n,\nabla u_n)|dx \\ &\le d(\delta)\int_B (C(x)+|\nabla T_\delta(u_n)|^{p(x)})dx \\ &\quad + \int_{B\cap\{|u_n|>\delta\}} |h_n(x,u_n,\nabla u_n)|dx \\ &= L_1+L_2. \end{aligned} \tag{5.25}$$

For fixed δ in $\mathbb{N}$, one has

$$L_1 \le d(\delta)\int_B (C(x)+|\nabla T_\delta(u_n)|^{p(x)})dx,$$

which is small, uniformly in n for fixed δ when the measure of B is small (remark that $(\nabla T_\delta(u_n))_{n\in\mathbb{N}}$ converges strongly in $(L^{p(\cdot)}(\Omega))^N$ as $n\to\infty$).
We treat the second term of (5.25) by using the test function $S_\delta(u_n)$ in (5.2), where for $\delta>1$, S_δ is defined as follows.

$$\begin{cases} S_\delta(s)=0, & \text{if } |s|\le\delta-1, \\ S_\delta(s)=\frac{|s|}{s}, & \text{if } |s|\ge\delta, \\ S'_\delta(s)=1, & \text{if } \delta-1\le|s|\le\delta. \end{cases}$$

It follows that

$$\int_\Omega b_n S_\delta(u_n)dx + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n S'_\delta(u_n)dx + \int_\Omega h(x,u_n,\nabla u_n)S_\delta(u_n)dx = \int_\Omega f S_\delta(u_n)dx.$$

Using the fact that the function $\beta_n\circ T_{\frac{1}{n}}$ is non-decreasing, S_δ is increasing on $\{\delta-1\le|u_n|\le\delta\}$, and h_n verifies the sign condition, we deduce that

$$\int_{\{\delta-1\le|u_n|\le\delta\}} a(x,u_n,\nabla u_n)\nabla u_n dx + \int_{\{|u_n|>\delta-1\}} |h(x,u_n,\nabla u_n)|dx \le \int_{\{|u_n|>\delta-1\}} |f_n|dx.$$

Using (3.1), we obtain

$$\begin{cases} \lambda \displaystyle\int_{\{\delta-1\leq|u_n|\leq\delta\}} |\nabla u_n|^{p(x)}dx + \int_{\{|u_n|>\delta-1\}} |h(x,u_n,\nabla u_n)|dx \\ \leq \displaystyle\int_{\{\delta-1\leq|u_n|\leq\delta\}} a(x,u_n,\nabla u_n)\nabla u_n dx + \int_{\{|u_n|>\delta-1\}} |h(x,u_n,\nabla u_n)|dx \\ \leq \displaystyle\int_{\{|u_n|>\delta-1\}} |f_n|dx. \end{cases} \tag{5.26}$$

It follows that

$$\int_{\{|u_n|>\delta-1\}} |h(x,u_n,\nabla u_n)|dx \leq \int_{\{|u_n|\geq\delta-1\}} |f|dx,$$

then

$$\limsup_{\delta\to\infty} \sup_{n\in\mathbb{N}} \int_{\{|u_n|>\delta-1\}} |h(x,u_n,\nabla u_n)|dx \leq \int_{\{|u_n|>\delta-1\}} |f|dx.$$

Consequently, L_2 remains small, uniformly with respect to n and B when δ is sufficiently large. This allows us to conclude that $h_\epsilon(x,u_n,\nabla u_n)$ is uniformly equi-integrable in Ω.
By virtue of Vitali's theorem, we have

$$h_n(x,u_n,\nabla u_n) \to h(x,u,\nabla u) \text{ strongly in } L^1(\Omega).$$

□

Given the convergence results outlined above, we can take the limit as $n\to\infty$ in (5.2) to obtain

$$\int_\Omega b\varphi dx + \int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega h(x,u,\nabla u)\varphi dx = \int_\Omega f\varphi dx,$$

for any $\varphi \in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$. □

5.2. **Entropy solution.**
Here we establish the existence of entropy solution of the problem (E,f) for any datum f in $L^1(\Omega)$.

Theorem 5.8. *Let f be a function in $L^1(\Omega)$ and $t>0$. Then, the problem (E,f) admits at least one entropy solution $(u,b)\in \mathcal{T}_0^{1,p(.)}(\Omega)\times L^1(\Omega)$ in the sense that $b(x)\in\beta(u(x))$ a.e. in Ω, $h(x,u,\nabla u)\in L^1(\Omega)$ and*

$$\int_\Omega bT_t(u-\varphi)dx + \int_\Omega a(x,u,\nabla u)\nabla T_t(u-\varphi)dx + \int_\Omega h(x,u,\nabla u)T_t(u-\varphi)dx \leq \int_\Omega fT_t(u-\varphi)dx, \tag{5.27}$$

for any $\varphi\in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$.

Proof.
Let (u_n, b_n) be a to problem (E, f_n) and $t > 0$. For any $v \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, taking $T_t(u_n - v)$ as test function in (5.2) and setting $L = t + \|v\|_\infty$, we obtain

$$\int_\Omega b_n T_t(u_n-v)dx+\int_\Omega a(x,u_n,\nabla u_n)\nabla T_t(u_n-v)dx+\int_\Omega h_n(x,u_n,\nabla u_n)T_t(u_n-v)dx$$

$$= \int_\Omega f_n T_t(u_n - v)dx. \tag{5.28}$$

Notice that if $|u_n| \geq L$, then $|u_n - v| \geq |u_n| - \|v\|_\infty > t$. Therefore, $\{|u_n - v| \leq t\} \subseteq \{|u_n| \leq L\}$, which gives

$$\begin{cases} \displaystyle\int_\Omega a(x, u_n, \nabla u_n)\nabla T_t(u_n - v)dx \\ \displaystyle= \int_\Omega a(x, T_L(u_n), \nabla T_L(u_n))(\nabla T_L(u_n) - \nabla v)\chi_{\{u_n - v| \leq t\}}dx \\ \displaystyle= \int_\Omega (a(x, T_L(u_n), \nabla T_L(u_n)) - a(x, T_L(u_n), \nabla v))(\nabla T_L(u_n) - \nabla v)\chi_{\{u_n - v| \leq t\}}dx \\ \displaystyle+ \int_\Omega a(x, T_L(u_n), \nabla v)(\nabla T_L(u_n) - \nabla v)\chi_{\{u_n - v| \leq t\}}dx. \end{cases}$$

Using Fatou's Lemma, we obtain

$$\begin{cases} \displaystyle\liminf_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n)\nabla T_t(u_n - v)dx \\ \displaystyle\geq \int_\Omega (a(x, T_L(u), \nabla T_L(u_) - a(x, T_L(u), \nabla v))(\nabla T_L(u) - \nabla v)\chi_{\{u - v| \leq t\}}dx \\ \displaystyle+ \lim_{n\to\infty} \int_\Omega a(x, T_L(u_n), \nabla v)(\nabla T_L(u_n) - \nabla v)\chi_{\{u_n - v| \leq t\}}dx. \end{cases} \tag{5.29}$$

Since

$$\begin{cases} \displaystyle\lim_{n\to\infty} \int_\Omega a(x, T_L(u_n), \nabla v)(\nabla T_L(u_n) - \nabla v)\chi_{\{u_n - v| \leq t\}}dx \\ \displaystyle= \int_\Omega a(x, T_L(u), \nabla v)(\nabla T_L(u) - \nabla v)\chi_{\{u - v| \leq t\}}dx \end{cases}$$

we deduce from (5.29), that

$$\begin{cases} \liminf_{n\to\infty} \int_\Omega a(x,u_n,\nabla u_n)\nabla T_t(u_n-v)dx \\ \geq \int_\Omega a(x,T_L(u),\nabla v)(\nabla T_L(u)-\nabla v)\chi_{\{u-v|\leq t\}}dx \\ = \int_\Omega a(x,T_L(u),\nabla v)(\nabla T_L(u)-\nabla v)\chi_{\{u-v|\leq t\}}dx \\ = \int_\Omega a(x,T_L(u),\nabla v)\nabla T_t(u-v)dx. \end{cases}$$

Since $T_t(u_n-v) \overset{*}{\rightharpoonup} T_t(u-v)$ in $L^\infty(\Omega)$ and $h_n(x,u_n,\nabla u_n) \to h(x,u,\nabla u)$ in $L^1(\Omega)$ as $n\to\infty$, one deduces that

$$\int_\Omega h_n(x,u_n,\nabla u_n)T_t(u_n-v)dx \longrightarrow \int_\Omega h(x,u,\nabla u)T_t(u-v)dx. \tag{5.30}$$

By using Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega f_nT_t(u_n-v)dx \to \int_\Omega fT_t(u-v)dx. \tag{5.31}$$

Since $b_n \rightharpoonup b$ weakly in $L^1(\Omega)$ and $T_t(u_n-v) \overset{*}{\rightharpoonup} T_t(u-v)$ in $L^\infty(\Omega)$ as $n\to\infty$, we obtain

$$\int_\Omega b_nT_t(u_n-v)dx \to \int_\Omega bT_t(u-v)dx. \tag{5.32}$$

By passing to the limit in (5.2) as $n\to\infty$, we obtain the entropy inequality (5.27). □

5.3. **Renormalized solution.**

In this subsection, we prove that an entropy solution is also a renormalized solution of problem (E,f).

Theorem 5.9. *Let $f\in L^1(\Omega)$. then, the problem (E,f) admits at least one renormalized solution $(u,b)\in W_0^{1,p(.)}(\Omega)\times L^1(\Omega)$ in the sense that $b(x)\in\beta(u(x))$ a.e. in Ω, $h(x,u,\nabla u)\in L^1(\Omega)$ and*

$$\int_\Omega bS(u)\varphi dx + \int_\Omega a(x,u,\nabla u)(S'(u)\varphi\nabla u + S(u)\nabla\varphi)dx + \int_\Omega h(x,u,\nabla u)S(u)\varphi dx$$
$$= \int_\Omega fS(u)\varphi dx \tag{5.33}$$

and

$$\lim_{l\to+\infty}\int_{\{l\leq|u|\leq l+1\}} |\nabla u|^{p(x)}dx = 0, \tag{5.34}$$

for any $\varphi\in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$ and any function $S\in W^{1,\infty}(\Omega)$ with compact support.

Proof. Let us start by proving that (5.34) holds.
Let u be an entropy solution of (E, f).
According to (5.26) one has

$$\lambda \int_{\{l\leq|u_n|\leq l+1\}} |\nabla u_n|^{p(x)}dx + \int_{\{|u_n|>l\}} |h(x,u_n,\nabla u_n)|dx \leq \int_{\{|u_n|>l\}} |f_n|dx,$$

thus

$$\lambda \int_{\{l\leq|u_n|\leq l+1\}} |\nabla u_n|^{p(x)}dx \leq \int_{\{|u_n|>l\}} |f_n|dx.$$

Then, letting $n \to \infty$ and $l \to \infty$ successively we obtain (5.34).
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of weak solutions of (5.2) and $S \in W^{1,\infty}(\Omega)$ such that supp$S \subset [-L, L]$ for some $L > 0$. We already know from Lemma 5.6 that

$$\text{for any } t > 0,\ T_t(u_n) \to T_t(u) \text{ strongly in } W_0^{1,p(.)}(\Omega) \text{ as } n \to \infty.$$

For any $v \in C_0^\infty(\Omega)$, we choose $S(u_n)v \in W_0^{1,p(.)}(\Omega)$ as test function in (5.2) to obtain

$$\int_\Omega b_n S(u_n)vdx + \int_\Omega a(x,u_n,\nabla u_n)(S'(u_n)v\nabla u_n + S(u_n)\nabla v)dx$$
$$+ \int_\Omega h_n(x,u_n,\nabla u_n)S(u_n)vdx = \int_\Omega f_n S(u_n)vdx. \tag{5.35}$$

Since $S(u_n)v \overset{*}{\rightharpoonup} S(u)v$ in $L^\infty(\Omega)$ as $n \to \infty$, one deduces that

$$\int_\Omega b_n S(u_n)vdx \to \int_\Omega bS(u)vdx, \tag{5.36}$$

$$\int_\Omega h(x,u_n,\nabla u_n)S(u_n)vdx \to \int_\Omega h(x,u,\nabla u)S(u)vdx \tag{5.37}$$

and

$$\int_\Omega f_n S(u_n)vdx \to \int_\Omega fS(u)vdx. \tag{5.38}$$

It remain to treat the second term of the left on side of (5.35).
Indeed, one has

$$\int_\Omega a(x,u_n,\nabla u_n)(S'(u_n)v\nabla u_n + S(u_n)\nabla v)dx$$
$$= \int_\Omega a(x,T_L(u_n),\nabla T_L(u_n))(S'(u_n)v\nabla T_L(u_n) + S(u_n)\nabla v)dx.$$

Thanks to (3.2), $(a(x,T_L(u_n),\nabla T_L(u_n)))_{n\in\mathbb{N}}$ is bounded in $(L^{p'(.)}(\Omega))^N$ and

$$a(x,T_L(u_n),\nabla T_L(u_n)) \to a(x,T_L(u),\nabla T_L(u)) \text{ a.e. in } \Omega.$$

Then, using Lemma 3.1, we obtain

$$a(x,T_L(u_n),\nabla T_L(u_n)) \rightharpoonup a(x,T_L(u),\nabla T_L(u)) \text{ weakly in } (L^{p'(.)}(\Omega))^N.$$

As $n \to \infty$, one has

$$(S'(u_n)v\nabla T_L(u_n)+S(u_n)\nabla v) \to (S'(u)v\nabla T_L(u)+S(u)\nabla v) \text{ strongly in } (L^{p'(.)}(\Omega))^N,$$

then

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega a(x,T_L(u_n),\nabla T_L(u_n))(S'(u_n)v\nabla T_L(u_n)+S(u_n)\nabla v)dx\\
&=\int_\Omega a(x,T_L(u),\nabla T_L(u))(S'(u)v\nabla T_L(u)+S(u)\nabla v)dx\\
&\qquad=\int_\Omega a(x,u,\nabla u)(S'(u)v\nabla u+S(u)\nabla v)dx. \qquad (5.39)
\end{aligned}$$

Since (5.36)-(5.39) hold, passing to the limit in (5.35) as $n\to\infty$, we obtain the renormalized equality (5.33)

$$\int_\Omega bS(u)vdx+\int_\Omega a(x,u,\nabla u)(S'(u)v\nabla u+S(u)\nabla v)dx+\int_\Omega h(x,u,\nabla u)S(u)vdx = \int_\Omega fS(u)vdx.$$

Let us prove the last step which consists to show that u belongs to $W_0^{1,p(.)}(\Omega)$. Indeed, for any $t>0$, choosing $T_t(u_n)$ as a test function in (5.2), we deduce from the proof of Lemma 5.2 that

$$\int_{\{|u_n|\le t\}}|\nabla u_n|^{p(x)}dx=\int_\Omega|\nabla T_t(u_n)|^{p(x)}dx\le\frac{t\|f\|_{L^1(\Omega)}}{\lambda} \qquad (5.40)$$

and

$$\int_{\{|u_n|>t\}}|h(x,u_n,\nabla u_n)|dx\le t\|f\|_{L^1(\Omega)}. \qquad (5.41)$$

Taking t large enough ($t\ge\sigma$) and using (3.6), we get

$$\rho\int_{\{|u_n|>t\}}|\nabla u_n|^{p(x)}dx\le\int_{\{|u_n|>t\}}|g(x,u_n,\nabla u_n)|dx. \qquad (5.42)$$

Adding (5.40) and (5.42), we get

$$\int_{\{|u_n|\le t\}}|\nabla u_n|^{p(x)}dx+\rho\int_{\{|u_n|>t\}}|\nabla u_n|^{p(x)}dx\le\left(1+\frac{1}{\lambda}\right)\|f\|_{L^1(\Omega)},$$

then

$$\min\{1,\rho\}\int_\Omega|\nabla u_n|^{p(x)}dx\le\left(1+\frac{1}{\lambda}\right)\|f\|_{L^1(\Omega)},$$

thus

$$\int_\Omega|\nabla u_n|^{p(x)}dx\le C_7,$$

where $C_7:=\dfrac{1+\frac{1}{\lambda}}{\min\{1,\rho\}}\|f\|_{L^1(\Omega)}$.

We deduce from the Poincaré inequality, the existence of a constant $C_8>0$ not depending of n such that

$$\|u_n\|_{1,p(.)}\le C_8.$$

Hence,

$$\|u\|_{1,p(.)}\le C_8.$$

Thus, the proof of Theorem 5.9 is complete. □

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