

UNIQUENESS PROBLEMS ON L-FUNCTIONS CONCERNING WITH WEIGHTED SHARING OF DIFFERENCE-DIFFERENTIAL POLYNOMIALS

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ABSTRACT. Relying on Nevanlinna's Value Distribution Theory and certain properties of L -functions in the extended Selberg class, we prove some unicity results on L -functions concerning with weighted sharing of difference-differential polynomials. This article extends the earlier works due to Nintu Mandal and Nirmal Kumar Datta.

1. INTRODUCTION AND PRELIMINARIES

The concept of Riemann hypothesis and its forwarded version to the general classes of L -function is the most speculative open world problem in today's pure Mathematics. L -functions find its applications in the modern number theory.

$L(z) = \sum_{n=1}^{\infty} \frac{a(n)}{n^z}$ is said to be an L -function in the Selberg class provided it satisfies the following conditions.

- (1) $a(n) \ll n^\varepsilon$ for every $\varepsilon > 0$.
- (2) There exists an integer $k \geq 0$ such that $(z-1)^k L(z)$ is a finite order entire function.
- (3) Every L -function satisfies the functional equation $\lambda_L(z) = \overline{\omega \lambda_L(1-\bar{z})}$,

where $\lambda_L(z) = L(z) Q^z \prod_{i=1}^k \Gamma(\gamma_i z + v_i)$ with positive real number Q , γ_i and complex numbers v_i , ω with $Re v_i \geq 0$ and $|\omega| = 1$.

- (4) $L(z) = \prod_p L_p(z)$, where $L_p(z) = \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{kz}}\right)$ with co-efficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$ and p denotes the prime number.

If L satisfies the conditions (1)-(3), then L is an L -function in the extended Selberg class. In our work we always mean an L -function in the extended Selberg class with $a(1) = 1$. Using Value Distribution Theory of Nevanlinna we study the uniqueness of difference-differential polynomials of an L -function determined

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in the extended Selberg class. The standard notations and definitions are taken from (see [7], [10], [14]). Let $\alpha \in \mathbb{C} \cup \{\infty\}$ and f, g be meromorphic functions in the complex plane. The set of all α points of f with multiplicities not exceeding l is denoted by $E_l(\alpha, f)(\overline{E}_l(\alpha, f))$, where l is a positive integer and we consider (ignore) the multiplicities of the α -points. The hyper order $\rho_2(f)$ of f is defined by

$$\rho_2(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

We denote $S(r, f)$ by any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \rightarrow \infty$, outside a possible exceptional set of finite linear measure. We say that f and g share α CM if they have the same set of α points with the same multiplicities and if we do not consider the multiplicity then we say that f and g share α IM.

2. MAIN RESULTS

In 2017, Liu, Li and Yi [1] proved the following uniqueness theorems.

Theorem 2.1. [1] *Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L -function and f be a non-constant meromorphic function. If $(f^j)^{(k)}$ and $(L^j)^{(k)}$ share $(1, \infty)$ then $f \equiv \alpha L$ for some non-constant α satisfying $\alpha^j = 1$.*

Theorem 2.2. [1] *Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L -function and f be a non-constant meromorphic function. If $(f^j)^{(k)}$ and $(L^j)^{(k)}$ share (z, ∞) then $f \equiv \alpha L$ for some non-constant α satisfying $\alpha^j = 1$.*

In 2018, W. J. Hao and J. F. Chen [2] proved the uniqueness results on L -function.

Theorem 2.3. [2] *Let f be a non-constant meromorphic function and L be an L -function such that $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share $(1, \infty)$, where n, m, k are positive integers. If $n > m + 3k + 6$ and $k \geq 2$ then $f \equiv L$ or $f^n(f-1)^m = L^n(L-1)^m$.*

Theorem 2.4. [2] *Let f be a non-constant meromorphic function and let L be an L -function such that $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share $(1, 0)$, where n, m, k are positive integers. If $n > 4m + 7k + 11$ and $k \geq 2$ then $f \equiv L$ or $f^n(f-1)^m = L^n(L-1)^m$.*

In 2019, W. Q. Zhu and J. F. Chen [3] by using the concept of truncated sharing proved the following.

Theorem 2.5. [3] *Let L be an L -function and f be a transcendental meromorphic function defined in $\mathbb{C}$. Also let $n, k (\geq 2), l (\geq 2)$ be positive integers such that $n \geq 7k + 17$. If $\overline{E}_l(1, (f^n(f-1))^{(k)}) = \overline{E}_l(1, (L^n(L-1))^{(k)})$, then $f \equiv L$.*

In 2020, Nintu Mandal and Nirmal Kumar Datta [5] proved the following theorem by considering truncated sharing of small functions.

Theorem 2.6. [5] *Let L be a non-constant L -function and ρ be a small function of L such that $\rho (\not\equiv 0, \infty)$. If $\overline{E}_4(\rho; L) = \overline{E}_4(\rho; (L^m)^{(k)})$, $E_2(\rho; L) = E_2(\rho; (L^m)^{(k)})$ and $2N_{k+2}(r, 0; L^m) \leq (\sigma + o(1))T(r, L)$, where $m \geq 1, k \geq 1$ are integers and $0 < \sigma < 1$, then $L \equiv (L^m)^{(k)}$.*

In 2021, N. Mandal and N. K. Datta [6] proved the following two results by replacing differential polynomials by difference differential polynomials.

Theorem 2.7. [6] *Let L be a non-constant L -function and f be a transcendental meromorphic function. Let k, n, d, μ_j ($j = 1, 2, \dots, d$), $\lambda = \sum_{j=1}^d \mu_j$ be positive integers such that $n > \lambda + d(2k + 4) + 4$ and $\omega_j \in \mathbb{C} - \{0\}$, ($j = 1, 2, \dots, d$) be distinct constants. Also let $\rho_2(L) < 1$, $\rho_2(f) < 1$, $\left[L^n \prod_{j=1}^d L(z + \omega_j)^{\mu_j} \right]^{(k)}$*

and $\left[f^n \prod_{j=1}^d f(z + \omega_j)^{\mu_j} \right]^{(k)}$ share $(\rho(z), l)$ and f, L share $(\infty, 0)$ where $\rho(z)$ is a small function of f and L . If $l = 0$ and $n > \lambda + (d + 1)(5k + 7)$ or $l = 1$ and $n > \lambda + \frac{3}{2}(d + 1)(2k + 3)$, then one of the following holds.

- (i) $\left[L^n \prod_{j=1}^d L(z + \omega_j)^{\mu_j} \right]^{(k)} \equiv \left[f^n \prod_{j=1}^d f(z + \omega_j)^{\mu_j} \right]^{(k)},$
- (ii) $\left[L^n \prod_{j=1}^d L(z + \omega_j)^{\mu_j} \right]^{(k)} \left[f^n \prod_{j=1}^d f(z + \omega_j)^{\mu_j} \right]^{(k)} \equiv \rho^2.$

Theorem 2.8. [6] *Let L be a non-constant L -function and f be a transcendental meromorphic function. Let k, n, d, μ_j ($j = 1, 2, \dots, d$), $\lambda = \sum_{j=1}^d \mu_j$ be positive integers such that $n > \lambda + d(2k + 4) + 4$ and $\omega_j \in \mathbb{C} - \{0\}$, ($j = 1, 2, \dots, d$) be distinct constants. Also let $\rho_2(L) < 1$, $\rho_2(f) < 1$, $\left[L^n \prod_{j=1}^d L(z + \omega_j)^{\mu_j} \right]^{(k)}$*

and $\left[f^n \prod_{j=1}^d f(z + \omega_j)^{\mu_j} \right]^{(k)}$ share $(R(z), l)$ and f, L share $(\infty, 0)$ where $\rho(z)$ is a small function of f and L . If $l = 0$ and $n > \lambda + (d + 1)(5k + 7)$ or $l = 1$ and $n > \lambda + \frac{3}{2}(d + 1)(2k + 3)$, then one of the following holds.

- (i) $\left[L^n \prod_{j=1}^d L(z + \omega_j)^{\mu_j} \right]^{(k)} \equiv \left[f^n \prod_{j=1}^d f(z + \omega_j)^{\mu_j} \right]^{(k)},$
- (ii) $\left[L^n \prod_{j=1}^d L(z + \omega_j)^{\mu_j} \right]^{(k)} \left[f^n \prod_{j=1}^d f(z + \omega_j)^{\mu_j} \right]^{(k)} \equiv R(z)^2.$

Theorem 2.9. *Let L be a non-constant L -function and f be a transcendental meromorphic function. Let k, n, d, s, m, μ_j ($j = 1, 2, \dots, d$), $\lambda = \sum_{j=1}^d \mu_j$ be non-negative integers and let α and β be two non-zero constants such that $|\alpha| + |\beta| \neq 0$ and let $n + ms > \lambda + d(2k + 4) + 4$ and $\omega_j \in \mathbb{C} - \{0\}$, such that*

($j = 1, 2, \dots, d$) be distinct constants. Also let $\rho_2(L) < 1$, $\rho_2(f) < 1$, such that $\left[L^n(\alpha L^m + \beta)^s \prod_{j=1}^d L(z + w_j)^{\mu_j} \right]^{(k)}$ and $\left[f^n(\alpha f^m + \beta)^s \prod_{j=1}^d f(z + w_j)^{\mu_j} \right]^{(k)}$ share $(\rho(z), l)$ and f, L share $(\infty, 0)$ where $\rho(z)$ is a small function of f and L . If $l = 0$ and $n + ms > \lambda + (5k+7)(1+ms+d)$ or $l = 1$ and $n + ms > \lambda + \frac{1}{2}(5k+9)(1+ms+d)$, then one of the following holds.

$$(i) \left[L^n(\alpha L^m + \beta)^s \prod_{j=1}^d L(z + w_j)^{\mu_j} \right]^{(k)} \equiv \left[f^n(\alpha f^m + \beta)^s \prod_{j=1}^d f(z + w_j)^{\mu_j} \right]^{(k)},$$

$$(ii) \left[L^n(\alpha L^m + \beta)^s \prod_{j=1}^d L(z + w_j)^{\mu_j} \right]^{(k)} \left[f^n(\alpha f^m + \beta)^s \prod_{j=1}^d f(z + w_j)^{\mu_j} \right]^{(k)} \equiv \rho^2.$$

Theorem 2.10. Let L be a non-constant L -function and f be a transcendental meromorphic function. Let k, n, d, s, m, μ_j ($j = 1, 2, \dots, d$), $\lambda = \sum_{j=1}^d \mu_j$

be non-negative integers and let α and β be two non-zero constants such that $|\alpha| + |\beta| \neq 0$ and let $n + ms > \lambda + d(2k+4) + 4$ and $\omega_j \in \mathbb{C} - \{0\}$, such that ($j = 1, 2, \dots, d$) be distinct constants. Also let $\rho_2(L) < 1$, $\rho_2(f) < 1$, such that $\left[L^n(\alpha L^m + \beta)^s \prod_{j=1}^d L(z + w_j)^{\mu_j} \right]^{(k)}$ and $\left[f^n(\alpha f^m + \beta)^s \prod_{j=1}^d f(z + w_j)^{\mu_j} \right]^{(k)}$ share $(R(z), l)$ and f, L share $(\infty, 0)$ where $\rho(z)$ is a small function of f and L . If $l = 0$ and $n + ms > \lambda + (5k+7)(1+ms+d)$ or $l = 1$ and $n + ms > \lambda + \frac{1}{2}(5k+9)(1+ms+d)$, then one of the following holds.

$$(i) \left[L^n(\alpha L^m + \beta)^s \prod_{j=1}^d L(z + w_j)^{\mu_j} \right]^{(k)} \equiv \left[f^n(\alpha f^m + \beta)^s \prod_{j=1}^d f(z + w_j)^{\mu_j} \right]^{(k)},$$

$$(ii) \left[L^n(\alpha L^m + \beta)^s \prod_{j=1}^d L(z + w_j)^{\mu_j} \right]^{(k)} \left[f^n(\alpha f^m + \beta)^s \prod_{j=1}^d f(z + w_j)^{\mu_j} \right]^{(k)} \equiv R(z)^2.$$

Remark 1. If $s = 0$ then Theorem 2.9-2.10 reduce to Theorem 2.7-2.8.

3. SOME DEFINITIONS

Definition 3.1. [11] Let f be a meromorphic function defined in the complex plane. Let n be a positive integer and $\alpha \in \mathbb{C} \cup \{\infty\}$. By $N(r, \alpha; f| \leq n)$ we denote the counting function of the α -points of f with multiplicity less than or equal to n and by $\overline{N}(r, \alpha; f| \leq n)$ the reduced counting function. Also by $N(r, \alpha; f| \geq n)$ we denote the counting function of the α -points of f with multiplicity greater than or equal to n and by $\overline{N}(r, \alpha; f| \geq n)$ the reduced counting function. We define

$$N_n(r, \alpha; f) = \overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; f| \geq 2) + \dots + \overline{N}(r, \alpha; f| \geq k).$$

Definition 3.2. [11] Let f be a meromorphic function in defined in $\mathbb{C}$ and $p(z)$ be a small function of f or a rational function. Then we denote the notations by

$N(r, p; f| \leq m)$, $\overline{N}(r, p; f| \leq m)$, $N(r, p; f| \geq m)$, $\overline{N}(r, p; f| \geq m)$, $N_m(r, p; f)$ etc., the counting functions $N(r, 0; f - p| \leq m)$, $\overline{N}(r, 0; f - p| \leq m)$, $N(r, 0; f - p| \geq m)$, $\overline{N}(r, 0; f - p| \geq m)$, $N_m(r, 0; f - p)$ etc., respectively.

Definition 3.3. [12] Let f and g be two meromorphic functions defined in the complex plane and n be an integer (≥ 0) or infinity. We denote by $E_n(\alpha; f)$ the set of all zeros of $f - \alpha$ and $\alpha \in \mathbb{C} \cup \{\infty\}$ and a zero of multiplicity k is counted k times if $k \leq n$ and $n + 1$ times if $k > n$, we say that f and g share α with weight n if $E_n(\alpha; f) = E_n(\alpha; g)$. We say that f and g share (α, n) to mean that f, g share α with weight n . Clearly, f, g share α IM or CM if and only if f and g share $(\alpha, 0)$ or (α, ∞) respectively.

Definition 3.4. [5] Let f be a meromorphic function defined in the complex plane and $p(z)$ be a rational function or a small function of f . Then we denote by $E_m(p; f)$, $\overline{E}_m(p; f)$ and $E_m(p; f)$ the sets $E_m(0; f - p)$, $\overline{E}_m(0; f - p)$ and $E_m(0; f - p)$ respectively. We write f, g share (p, n) to mean that $f - p$ and $g - p$ share the value 0 with weight n . Clearly, if f, g share (p, n) then f, g share (p, m) for all integers m , $0 \leq m < n$. Also, we note that f, g share p IM or CM if and only if f, g share $(p, 0)$ or (p, ∞) respectively.

Definition 3.5. [13] Let f and g be two non-constant meromorphic functions share a value α IM. Denote by $\overline{N}_*(r, \alpha; f, g)$ the counting function of the α -points of f and g with different multiplicities, where each α -point is counted only once.

Definition 3.6. [6] Let f and g be two non-constant meromorphic functions share a value α IM. We denote by $\overline{N}(r, \alpha; f| > g)$ the counting function of the α -points of f and g with multiplicities with respect to f is greater than the multiplicities with respect to g , where each α -point is counted only once.

Definition 3.7. [6] Let f and g be two non-constant meromorphic functions share a value α IM. We denote by $\overline{N}_E(r, \alpha; f, g| > m)$ the counting function of the α -points of f and g with multiplicities greater than m and the multiplicities with respect to f is equal to the multiplicities with respect to g , where each α -point is counted only once.

4. AUXILIARY LEMMAS

Denote H by the following function.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{L''}{L'} - \frac{2L'}{L-1} \right).$$

Lemma 4.1. [4] Let L be an L -function with degree q . Then

$$T(r, L) = \frac{q}{\pi} r \log r + O(1).$$

Lemma 4.2. [5] Let L be an L -function. Then

$$N(r, \infty; L) = S(r, L) = O(\log r).$$

Lemma 4.3. [6] *Let f be a non-constant meromorphic function and L be an L -function. If f and L share $(\infty, 0)$ then*

$$\overline{N}(r, \infty; f) = S(r, L) = O(\log r).$$

Lemma 4.4. [7] *Let $f(z) = \frac{\alpha_0 + \alpha_1 z + \dots + \alpha_n z^n}{\beta_0 + \beta_1 z + \dots + \beta_m z^m}$ be a non-constant rational function defined in the complex plane $\mathbb{C}$ where $\alpha_0, \alpha_1, \dots, \alpha_n (\neq 0)$ and $\beta_0, \beta_1, \dots, \beta_m (\neq 0)$ are complex constants. Then*

$$T(r, f) = \max\{m, n\} \log r + O(1).$$

Lemma 4.5. [8] *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$. Then for any $\alpha \in \mathbb{C} - \{0\}$*

$$T(r, f(z + \alpha)) = T(r, f) + S(r, f),$$

$$N(r, \infty; f(z + \alpha)) = N(r, \infty; f) + S(r, f),$$

$$N(r, 0; f(z + \alpha)) = N(r, 0; f) + S(r, f)$$

Lemma 4.6. [9] *Let F and G be two non-constant meromorphic functions sharing $(1, 1)$ and $(\infty, 0)$. If $H \neq 0$, then*

$$\begin{aligned} T(r, F) &\leq N_2(r, 0; F) + N_2(r, 0; G) + \frac{3}{2} \overline{N}(r, F) + \overline{N}(r, G) \\ &\quad + \overline{N}_*(r, F, G) + \frac{1}{2} \overline{N}(r, 0; F) + S(r, F) + S(r, G). \end{aligned}$$

Lemma 4.7. [9] *Let F and G be two non-constant meromorphic functions sharing $(1, 0)$ and $(\infty, 0)$. If $H \neq 0$. Then*

$$\begin{aligned} T(r, F) &\leq N_2(r, 0; F) + N_2(r, 0; G) + 3\overline{N}(r, F) + 2\overline{N}(r, G) + \overline{N}_*(r, F, G) + 2\overline{N}(r, 0; F) \\ &\quad + \overline{N}(r, 0; G) + S(r, F) + S(r, G). \end{aligned}$$

$$\begin{aligned} T(r, G) &\leq N_2(r, 0; F) + N_2(r, 0; G) + 3\overline{N}(r, G) + 2\overline{N}(r, F) + \overline{N}_*(r, F, G) + 2\overline{N}(r, 0; G) \\ &\quad + \overline{N}(r, 0; F) + S(r, F) + S(r, G). \end{aligned}$$

Lemma 4.8. [10] *Let F be a non-constant meromorphic function and k, p be two positive integers. Then*

$$T(r, F^{(k)}) \leq T(r, F) + k\overline{N}(r, F) + S(r, F).$$

$$N_p(r, 0; F^{(k)}) \leq T(r, F^{(k)}) - T(r, F) + N_{p+k}(r, 0; F) + S(r, F).$$

$$N_p(r, 0; F^{(k)}) \leq N_{p+k}(r, 0; F) + k\overline{N}(r, F) + S(r, F).$$

$$N(r, 0; F^{(k)}) \leq N(r, 0; F) + k\overline{N}(r, F) + S(r, F).$$

Lemma 4.9. [10] *Let f be a non-constant meromorphic function, define the polynomial $P(f) = a_0 + a_1 f + \dots + a_n f^n$, where $a_0, \dots, a_n$ are complex constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 4.10. *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and define the function $F(z) = f^n(\alpha f^m + \beta)^s \prod_{j=1}^d f(z + \omega_j)^{\mu_j}$ where $\omega_j \in \mathbb{C} - \{0\}$, $(j = 1, 2, \dots, d)$ are constants, n, m, d, s, μ_j ($j = 1, 2, \dots, d$) are non-negative integers and α and β are non-zero constants such that $|\alpha| + |\beta| \neq 0$ and $\lambda = \sum_{j=1}^d \mu_j$ then*

$$(n + ms - \lambda)T(r, f) + S(r, f) \leq T(r, F) \leq (n + ms + \lambda)T(r, f) + S(r, f).$$

Proof. Since f is a transcendental meromorphic function of hyper order $\rho_2(f) < 1$. Then from Lemma 4.5 and Lemma 4.9 we have

$$\begin{aligned} T(r, F) &\leq T(r, f^n) + T(r, (\alpha f^m + \beta)^s) + T\left(r, \prod_{j=1}^d f(z + \omega_j)^{\mu_j}\right) \\ &\leq (n + ms + \lambda)T(r, f) + S(r, f). \end{aligned} \quad (4.1)$$

On the other hand from Lemma 4.5 and Lemma 4.9 we have

$$\begin{aligned} (n + ms + \lambda)T(r, f) &\leq T(r, f^n f^{ms} f^\lambda) + S(r, f) \\ &\leq m(r, f^n f^{ms} f^\lambda) + N(r, f^n f^{ms} f^\lambda) + S(r, f) \\ &\leq m\left(r, \frac{F(z) f^\lambda}{\prod_{j=1}^d f(z + \omega_j)^{\mu_j}}\right) + N\left(r, \frac{F(z) f^\lambda}{\prod_{j=1}^d f(z + \omega_j)^{\mu_j}}\right) + S(r, f) \\ &\leq T(r, F) + 2\lambda T(r, f) + S(r, f). \end{aligned}$$

$$(n + ms + \lambda)T(r, f) \leq T(r, F) + 2\lambda T(r, f) + S(r, f). \quad (4.2)$$

From inequalities (4.1) and (4.2) we get

$$(n + ms - \lambda)T(r, f) + S(r, f) \leq T(r, F) \leq (n + ms + \lambda)T(r, f) + S(r, f).$$

□

5. PROOF OF MAIN RESULTS

Proof of Theorem 2.9.

Proof. Denote $F = \frac{F_1^{(k)}}{\rho(z)}$ and $L = \frac{L_1^{(k)}}{\rho(z)}$ where $F_1 = f^n(\alpha f^m + \beta)^s \prod_{j=1}^d f(z + w_j)^{\mu_j}$

and $L_1 = L^n(\alpha L^m + \beta)^s \prod_{j=1}^d L(z + w_j)^{\mu_j}$ respectively. Then F, L share $(1, l)$ and share $(\infty, 0)$ except for zeros and poles of $\rho(z)$. Clearly by Lemma 4.1 L is a

transcendental meromorphic function. We have by Lemma 4.8 and 4.10

$$\begin{aligned}
N_2(r, 0; F) &\leq N_2(r, 0; F_1^{(k)}) + S(r, f) \\
&\leq T(r, F_1^{(k)}) - T(r, F_1) + N_{k+2}(r, 0; F_1) + S(r, f) \\
&\leq T\left(r, \frac{F_1^{(k)}}{\rho(z)}\right) - (n + ms - \lambda)T(r, f) + N_{k+2}(r, 0; F_1) \\
&\quad + S(r, f).
\end{aligned} \tag{5.1}$$

Hence from inequality (5.1) we get

$$(n + ms - \lambda)T(r, f) \leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f). \tag{5.2}$$

Similarly,

$$(n + ms - \lambda)T(r, L) \leq T(r, L) - N_2(r, 0; L) + N_{k+2}(r, 0; L_1) + S(r, f). \tag{5.3}$$

Now we have to consider the following two cases.

Case 1. Let $H \neq 0$. In this case we have to consider the following two subcases.

Subcase 1. Let $l = 0$. Hence by Lemmas 4.2, 4.3 and 4.7 and inequality 5.2 we have

$$\begin{aligned}
(n + ms - \lambda)T(r, f) &\leq N_2(r, 0; F) + N_2(r, 0; L) + 2\overline{N}(r, \infty; L) + 3\overline{N}(r, \infty; F) + \overline{N}(r, \infty; f, L) \\
&\quad + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; L) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, L) \\
&\leq N_2(r, 0; F) + N_2(r, 0; L) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; L) - N_2(r, 0; F) \\
&\quad + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, L) \\
&\leq N_2(r, 0; F_1^{(k)}) + N_2(r, 0; L_1^{(k)}) + 2\overline{N}(r, 0; F_1^{(k)}) + N(r, 0; L_1^{(k)}) - N_2(r, 0; F_1^{(k)}) \\
&\quad + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, L) \\
&\leq N_2(r, 0; L_1^{(k)}) + 2\overline{N}(r, 0; F_1^{(k)}) + \overline{N}(r, 0; L_1^{(k)}) + N_{k+2}(r, 0; F_1) \\
&\quad + S(r, f) + S(r, L) \\
&\leq N_{k+2}(r, 0; L_1) + 2N_{k+1}(r, 0; F_1) + N_{k+1}(r, 0; L_1) + N_{k+2}(r, 0; F_1) \\
&\quad + S(r, f) + S(r, L) \\
&\leq (3 + 2k)(1 + ms + d)T(r, L) + (3k + 4)(1 + ms + d)T(r, f) \\
&\quad + S(r, f) + S(r, L).
\end{aligned}$$

Hence

$$\begin{aligned}
(n + ms - \lambda)T(r, f) &\leq (3 + 2k)(1 + ms + d)T(r, L) + (3k + 4)(1 + ms + d)T(r, f) \\
&\quad + S(r, f) + S(r, L).
\end{aligned} \tag{5.4}$$

Similarly,

$$\begin{aligned}
(n + ms - \lambda)T(r, L) &\leq (3 + 2k)(1 + ms + d)T(r, f) + (3k + 4)(1 + ms + d)T(r, L) \\
&\quad + S(r, f) + S(r, L).
\end{aligned} \tag{5.5}$$

From inequalities (5.4) and (5.5) we get

$$(n + ms - \lambda)\{T(r, f) + T(r, L)\} \leq (7 + 5k)(1 + ms + d)\{T(r, f) + T(r, L)\} + S(r, f) + S(r, L). \quad (5.6)$$

which is a contradiction from (5.6) as $n + ms > \lambda + (7 + 5k)(1 + ms + d)$.

Subcase 2. By Lemmas 4.2, 4.3 and 4.6 and inequality (5.2) we have

$$\begin{aligned} (n + ms - \lambda)T(r, f) &\leq N_2(r, 0; L) + \frac{3}{2}\overline{N}(r, F) + \overline{N}(r, L) + \overline{N}_*(r, \infty; F, L) + \frac{1}{2}\overline{N}(r, 0; F) \\ &\quad + N_{k+2}(r, 0; F_1) + S(r, F) + S(r, L) \\ &\leq N_2(r, 0; L_1^{(k)}) + \frac{1}{2}N_{k+1}(r, 0; F_1) + \overline{N}(r, 0; L_1^{(k)}) + N_{k+2}(r, 0; F_1) \\ &\quad + S(r, F) + S(r, L) \\ &\leq (k + 2)(1 + ms + d)T(r, L) + \frac{1}{2}(3k + 5)(1 + ms + d)T(r, f) \\ &\quad + S(r, f) + S(r, L). \end{aligned}$$

Hence

$$(n + ms - \lambda)T(r, f) \leq (k + 2)(1 + ms + d)T(r, L) + \frac{1}{2}(3k + 5)(1 + ms + d)T(r, f) + S(r, f) + S(r, L). \quad (5.7)$$

Similarly

$$(n + ms - \lambda)T(r, L) \leq (k + 2)(1 + ms + d)T(r, f) + \frac{1}{2}(3k + 5)(1 + ms + d)T(r, L) + S(r, f) + S(r, L). \quad (5.8)$$

From inequalities (5.7) and (5.8) we arrive at a contradiction as $n + ms > \lambda + \frac{1}{2}(9 + 5k)(1 + ms + d)$.

Case 2. Let $H \equiv 0$. Then

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{L''}{L'} - \frac{2L'}{L-1} \right) \equiv 0.$$

Integrating both sides we get

$$F - 1 = \frac{L - 1}{b - c(L - 1)}, \quad (5.9)$$

where $b \neq 0$ and c are constants. Now we have to consider the following subcases.

Subcase 3. Let $c = 0$. Then from (5.9) we have

$$F - 1 = \frac{L - 1}{b}. \quad (5.10)$$

If $b \neq 1$ then from (5.10)

$$\overline{N}(r, 0; F) = \overline{N}(r, 1 - b; L) \quad (5.11)$$

By Lemma 4.2 and 4.8, using second fundamental theorem of Nevanlinna and from inequality (5.3) we have

$$\begin{aligned}
(n + ms - \lambda)T(r, L) &\leq T(r, L) - N_2(r, 0; L) + N_{k+2}(r, 0; L_1^{(k)}) + S(r, L) \\
&\leq \overline{N}(r, 0; L) + \overline{N}(r, 1 - b; L) + \overline{N}(r, L) - N_2(r, 0; L) \\
&\quad + N_{k+2}(r, 0; L_1^{(k)}) + S(r, L) \\
&\leq \overline{N}(r, 0; L) + \overline{N}(r, 0; F) - N_2(r, 0; L) + N_{k+2}(r, 0; L_1) + S(r, L) \\
&\leq \overline{N}(r, 0; F_1^{(k)}) + N(r, 0; L_1^{(k)}) + N_{k+2}(r, 0; L_1) + S(r, L) \\
&\leq N_{k+1}(r, 0; F_1) + N_{k+1}(r, 0; G_1) + N_{k+2}(r, 0; L_1) + S(r, L) \\
&\leq (2k + 3)(1 + ms + d)T(r, L) + (k + 1)(1 + ms + d)T(r, f) \\
&\quad + S(r, f) + S(r, L).
\end{aligned}$$

Hence

$$\begin{aligned}
(n + ms - \lambda)T(r, L) &\leq (2k + 3)(1 + ms + d)T(r, L) + (k + 1)(1 + ms + d)T(r, f) \\
&\quad + S(r, f) + S(r, L).
\end{aligned} \tag{5.12}$$

Similarly

$$\begin{aligned}
(n + ms - \lambda)T(r, f) &\leq (2k + 3)(1 + ms + d)T(r, f) + (k + 1)(1 + ms + d)T(r, L) \\
&\quad + S(r, f) + S(r, L).
\end{aligned} \tag{5.13}$$

From the inequalities (5.12) and (5.13) we arrive at a contradiction as $n + ms > \lambda + (3k + 4)(1 + ms + d)$. Hence $b = 1$ and therefore we get from (5.10)

$$\left[L^n(\alpha L^m + \beta)^s \prod_{j=1}^d L(z + w_j)^{\mu_j} \right]^{(k)} \equiv \left[L^n(\alpha f^m + \beta)^s \prod_{j=1}^d f(z + w_j)^{\mu_j} \right]^{(k)}.$$

Subcase 4. Let $c \neq 0$ and $b = -c$. If $c = 1$, then from (5.9) we have $FL \equiv 1$.

Hence

$$\left[L^n(\alpha L^m + \beta)^s \prod_{j=1}^d L(z + w_j)^{\mu_j} \right]^{(k)} \left[L^n(\alpha f^m + \beta)^s \prod_{j=1}^d f(z + w_j)^{\mu_j} \right]^{(k)} = \rho^2.$$

If $c \neq 1$, then from (5.9) we have

$$\frac{1}{F} = \frac{-cL}{(1 - c)L - 1}.$$

Hence $\overline{N}(r, 0; F) = N\left(r, \frac{1}{1-c}; L\right)$. Now proceeding as in subcase 3 we arrive at a contradiction. If $c = 1$, then from (5.9) we have

$$F \equiv \frac{-b}{L - b - 1}. \tag{5.14}$$

Hence by Lemma 4.3 we have from (5.14)

$$\overline{N}(r, c + 1; L) = \overline{N}(r, F) = S(r, L).$$

Now proceeding as in subcase 3 we arrive at a contradiction. If $c \neq 1$, then from (5.9) we have

$$F - \left(1 - \frac{1}{c}\right) \equiv \frac{-b}{c^2 \left(L - \frac{b+c}{c}\right)}.$$

Therefore by Lemma 4.3 we have

$$\overline{N}\left(r, \frac{b+c}{c}\right) = \overline{N}(r, F) = S(r, L).$$

Proceeding as in subcase 3 we arrive at a contradiction. This completes the proof of the Theorem. □

Proof of Theorem 2.10.

Proof. Since f and L are transcendental meromorphic functions and $R(z)$ is a rational function therefore $R(z)$ is a small function of f and L . Hence by Theorem 2.9 we get required result. □

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