

THE INVARIANT SUBSPACE PROBLEM FOR BI-EXPANSIVE OPERATORS IN KREIN SPACES

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ABSTRACT. The invariant subspace problem is one of the most classical questions in operator theory and has attracted research studies for several years. The solutions for this question exist for compact operators. Recently, this question has been solved for bi-contractive operators defined on a Krein space. The authors established that every bi-contractive operators defined on a Krein space, has maximal semi-definite invariant sub-spaces. In this paper, we address this question for bi-expansive operators defined on a Krein space using Riesz projection and fixed fundamental decomposition. we establish that every bi-expansive operator defined on a Krein space has maximal invariant sub-spaces.

1. INTRODUCTION

Let G be a bounded linear operator defined on a normed linear space $\mathcal{K}$. If $\mathcal{T}$ is a subspace of $\mathcal{K}$, then $\mathcal{T}$ is an invariant subspace of $\mathcal{K}$ if

$$G\mathcal{T} \subseteq \mathcal{T}. \quad (1.1)$$

The invariant subspace classical question states as follows: Does there exist a closed subspace $\mathcal{T}$ of $\mathcal{K}$ besides the trivial ones $\mathcal{T} = \mathcal{K}$ and $\mathcal{T} = \{0\}$? Let's note here that this question is still unsolved on Hilbert space in the general and on a Krein space for bi-expansive operators. It is known that solutions for this question do exist for some classes of operators, such as compact operators and contractive operators defined on a Krein space $\mathcal{K}$. It is known that operators without invariant sub-spaces were independently discovered by [4] and [9] on unknown Banach space. Counter examples on the spaces l_1 and C_0 were also established by [9] and were later established by [3]. An equivalent account of the theory of invariant subspace written before these examples was found in [12]. The question of invariant subspace for bi-contractive operators defined on a Krein space $\mathcal{K}$ was addressed by [11]. The fixed fundamental decomposition of $\mathcal{K}$ is not unique in general however, the components $\mathcal{K}_+$ and $\mathcal{K}_-$ is be uniquely determined. This fixed fundamental decomposition is expressed as

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- \quad (1.2)$$

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A Krein space for which $\dim \mathcal{K}_- < \infty$ is called a Pontryagin space.

2. INDEFINITE INNER PRODUCT SPACES

Let $\mathcal{V}$ be a complex linear space. By an indefinite inner product on $\mathcal{V}$ we mean a complex valued function $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ defined on $\mathcal{V} \times \mathcal{V}$ which satisfies linearity and symmetry conditions.

- (i) For linearity, we have, $\langle \alpha x + \beta y, z \rangle_{\mathcal{V}} = \alpha \langle x, z \rangle_{\mathcal{V}} + \beta \langle y, z \rangle_{\mathcal{V}}$ for all $x, y, z \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$,
- (ii) For symmetry we have, $\langle x, y \rangle_{\mathcal{V}} = \overline{\langle y, x \rangle_{\mathcal{V}}}$ for all $x, y \in \mathcal{V}$.

From linearity and symmetry conditions, we obtain;

$$\langle z, \alpha x + \beta y \rangle_{\mathcal{V}} = \overline{\alpha} \langle z, x \rangle_{\mathcal{V}} + \overline{\beta} \langle z, y \rangle_{\mathcal{V}}$$

We call a linear space $\mathcal{V}$ together with an indefinite inner product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ defined on it, an indefinite inner product space. We denote this by $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$. Let $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$ be an indefinite inner product space. By the anti-space of $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$ we mean the space $(\mathcal{V}, -\langle \cdot, \cdot \rangle_{\mathcal{V}})$ which coincides with $\mathcal{V}$ as a vector space but with the sign of the indefinite inner product reversed. Let $\mathcal{V}$ be an indefinite product space and let $\mathcal{T}$ be its subspace. We say that $\mathcal{T}$ is:

- (i) *Positive* if $\langle f, f \rangle_{\mathcal{V}} > 0$ for all $f \in \mathcal{T}$,
- (ii) *Negative* if $\langle f, f \rangle_{\mathcal{V}} < 0$ for all $f \in \mathcal{T}$,
- (iii) *Definite* if it is either positive or negative,
- (iv) *Indefinite* if it is neither positive nor negative,
- (v) *Neutral* if it $\langle f, f \rangle_{\mathcal{V}} = 0$ all $f \in \mathcal{T}$.

3. BOUNDED LINEAR OPERATORS

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Krein spaces. By $G(\mathcal{K}_1, \mathcal{K}_2)$ we mean the space of all bounded linear operators from $\mathcal{K}_1$ to $\mathcal{K}_2$. For the case where $\mathcal{K}_1 = \mathcal{K}_2$, we write $G(\mathcal{K}_1)$ instead of $G(\mathcal{K}_1, \mathcal{K}_1)$. For every $A \in G(\mathcal{K}_1, \mathcal{K}_2)$, there is a unique $A^* \in G(\mathcal{K}_2, \mathcal{K}_1)$ such that $\langle Af, g \rangle_{\mathcal{K}_2} = \langle f, A^*g \rangle_{\mathcal{K}_1}$, $f \in \mathcal{K}_1, g \in \mathcal{K}_2$. The operator A^* is called an adjoint operator of A . The operator $A \in G(\mathcal{K}_1)$ is self adjoint if it satisfies the condition that $A^* = A$. A linear operator A in an inner product space $\mathcal{K}_1$ is said to be isometric if $\langle Ax, Ay \rangle = \langle x, y \rangle$, for every pair $x, y \in D(A)$.

4. KREIN SPACES

A Krein space is defined as an indefinite inner product space $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$ which can be expressed as a direct orthogonal sum

$$\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-, \quad (4.1)$$

where $\mathcal{V}_+$ is a Hilbert space and $\mathcal{V}_-$ is the anti space of a Hilbert space. The representation in (4.1) is called a fundamental decomposition. We note that this decomposition is not unique in general. However, the components $\mathcal{V}_+$ and $\mathcal{V}_-$ in (4.1) are uniquely determined. For a Krein space $\mathcal{V}$, the numbers $\text{ind}_{\pm} \mathcal{K} = \dim \mathcal{K}_{\pm}$ do not depend on the choice of the decomposition (4.1) above. We call $\text{ind}_{\pm} \mathcal{K}$ the positive and negative indices of the Krein space $\mathcal{K}$. A Krein space $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$ in which $\dim \mathcal{V}_- < \infty$ is called a Pontryagin space. The decomposition

(4.1) induces a topology on the Krein $\mathcal{K}$ space. This is done as follows: Firstly, we form the associated Hilbert space $|\mathcal{V}| := \mathcal{V}_+ \oplus |\mathcal{V}_-|$ where we replace $\mathcal{K}_-$ with its anti space $|\mathcal{V}_-|$ that is clearly a Hilbert space. The norm of a Hilbert space $\|\cdot\|_{|\mathcal{V}|}$ is called the norm for the Krein space $\mathcal{K}$. It is known that all notions of convergence and continuity are with respect to this norm topology. The operator $J_{\mathcal{K}}h = h_+ - h_-$ for which $h = h_+ + h_-$ with $f_{\pm} \in \mathcal{V}_{\pm}$ is called the fundamental symmetry or signature operator associated with decomposition (4.1). If $|\mathcal{V}|$ is a Hilbert space associated with decomposition (4.1) and $[\cdot, \cdot]_{|\mathcal{V}|}$ denotes the corresponding positive definite inner product then

$$[h, g]_{|\mathcal{V}|} = \langle J_{\mathcal{V}}h, g \rangle_{\mathcal{V}}$$

for any vectors $h, g \in \mathcal{V}$. This follows from

$$\begin{aligned} \langle J_{\mathcal{K}}h, g \rangle &= \langle h_+ - h_-, g_+ + g_- \rangle \\ &= \langle h_+, g_+ \rangle - \langle h_-, g_- \rangle \\ &= [h, g]_{\mathcal{K}}. \end{aligned}$$

Next, we give an example of a Krein space

Example 4.1. Consider the space $\mathbb{C}^3$ with an indefinite inner product $\langle \mathbf{z}, \mathbf{y} \rangle = z_1\bar{y}_1 + z_2\bar{y}_2 - z_3\bar{y}_3$ defined on it for which $\mathbf{z} = (z_1, z_2, z_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Then space $(\mathbb{C}^3, \langle \cdot, \cdot \rangle)$ is a Krein space with only one possible decomposition that is given by

$$\mathbb{C}^3 = \mathbb{C}_+^3 \oplus \mathbb{C}_-^3,$$

where $\mathbb{C}_+^3$ consists of those elements of $\mathbb{C}^3$ of the form $C_+^3 = (z_1, z_2, 0)$ while $\mathbb{C}_-^3$ consists of elements of $\mathbb{C}^3$ of the form $C_-^3 = (0, 0, z_3)$.

Definition 4.2. [11]. Let G be abounded linear operator acting on a normed linear space $\mathcal{A}$. A subspace $\mathcal{T}$ of $\mathcal{K}$ is called an invariant subspace if $G\mathcal{T} \subseteq \mathcal{T}$.

5. RIESZ PROJECTION AND SPECTRAL DECOMPOSITION

The development of spectral theory in the finite dimensional space suggests that a useful definition of the function $f(G)$ of an operator G is given by the Cauchy formula given by (5.1)

$$f(G) = \frac{1}{2\pi i} \int_d (G - \beta I)^{-1} d\beta. \quad (5.1)$$

where T is analytic scalar-valued function and d is a suitable contour. To understand this better, we need to study the existence of the properties of the function,

$$G(\beta) = (G - \beta I)^{-1}.$$

Let $G: \mathcal{K} \rightarrow \mathcal{K}$, be a bounded linear operator acting on a normed space $\mathcal{K}$, if s is contained in $\mathcal{K}$ the equation $Gs - \beta s = t$ has a unique solution $s \in \mathcal{K}$. Equivalently, β is in the resolvent set iff $G - \beta I$ is an operator that has inverse, that is there is a linear bounded operator $G(\beta)$ on $\mathcal{K}$ such that $G(\beta)(G - \beta I) = (G - \beta I)G(\beta) = I$. Let G be a bounded linear operator on a Banach space $\mathcal{Y}$. Given that $\mathcal{T}$ is a subspace of $\mathcal{Y}$ that is invariant under G , then $G|_{\mathcal{T}}$ represents the restriction of G to $\mathcal{T}$, which is termed as an operator from $\mathcal{T}$ to $\mathcal{T}$. Suppose

the spectrum of G is a union of two non-empty and closed subsets ϕ and Ω with $\phi \cap \Omega = \emptyset$. Then the corresponding decomposition of a direct sum $\mathcal{Y} = \mathcal{N} \oplus \mathcal{H}$ such that $\mathcal{N}$ and $\mathcal{H}$ are G -invariant subspaces of $\mathcal{Y}$, the spectrum of restriction $G|_{\mathcal{N}} = \Omega$ and that of $G|_{\mathcal{H}} = \phi$. To prove that such a spectral decomposition exists, we study the properties of the projector given below.

$$B_{\Omega} = -\frac{1}{2\pi i} \int_{\gamma} (G - \beta I)^{-1} d\beta. \quad (5.2)$$

We say B_{Ω} is a bounded linear operator on $\mathcal{Y}$ given by (5.2) on the right hand and γ which is in the resolvent set of G is a Cauchy contour separating $\phi = \sigma(\mathcal{G}) \setminus \Omega$ around Ω . The operator B_{ω} is called a projection operator and it satisfies the condition that $B_{\Omega} = B_{\Omega}^2$.

6. EXPANSIVE OPERATORS IN KREIN SPACES

Definition 6.1. Let G be an operator defined on a Krein space $\mathcal{V}$. Then G is expansive if $\langle Gx, Gx \rangle \geq \langle x, x \rangle \forall x \in \mathcal{V}$.

Let $\mathcal{V}$ be a Krein space with expansive bounded linear operator G defined it. Then, the following holds for G :

- (i) *Strictly expansive if $\langle Gx, Gx \rangle > \langle x, x \rangle \forall x \in \mathcal{V}$*
- (ii) *Uniformly expansive if $\langle Gx, Gx \rangle \geq \langle x, x \rangle + \varepsilon \|x\|^2$ for some $\varepsilon \leq 1$.*
- (iii) *Bi-expansive if both G and its adjoint A^* are expansive.*

We characterize a Krein space $\mathcal{V}$ using the Riesz projection B that is $B^2 = B$, fixed fundamental decomposition and known results from spectral theory as follows: From spectral theory, if $\eta = 1$, for every $\eta \in \rho(G)$, the resolvent of set G then the operator $(G - \eta I)^{-1}$ exists, where I represents the identity operator on $\mathcal{V}$. Let $b \in \mathcal{V}$ such that $b \neq 0$. As G is bi-expansive, we have from the definition of Bi-expansive of G that the inequality (6.1) is true.

$$\langle G(G - \eta I)^{-1}b, G(G - \eta I)^{-1}b \rangle \geq \langle (G - \eta I)^{-1}b, (G - \eta I)^{-1}b \rangle \quad (6.1)$$

Taking $c = (G - \eta I)^{-1}b$.

Then $b = c(G - \eta I)$, for I an identity operator as earlier defined.

$b = Gc - \eta c$.

It implies that $b = cG - c\eta$

Inequality (6.1) is simplified in 6.2 as

$$\langle Gc, Gc \rangle \geq \langle c, c \rangle. \quad (6.2)$$

Since $Gc = b + \eta c$, inequality (6.2) can be stated as;

$$\begin{aligned} \langle b + \eta c, b + \eta c \rangle &\geq \langle c, c \rangle \\ \langle b + \eta c, b + \eta c \rangle - \langle c, c \rangle &\geq 0 \\ \langle b, b \rangle + \eta \langle c, b \rangle + \bar{\eta} \langle c, c \rangle &\geq 0. \end{aligned}$$

using $c = (G - \eta I)^{-1}b$ in the above inequality, we obtained the following:

$$\langle b, b \rangle + \eta \langle (G - \eta I)^{-1}b, b \rangle + \bar{\eta} \langle b, (G - \eta I)^{-1}b \rangle \geq 0 \quad (6.3)$$

We introduce the Riesz Projection (5.2) and Substitute $\eta = e^{i\alpha}$ in 6.3 to obtain

$$\langle b, b \rangle + e^{i\alpha} \langle (G - e^{i\alpha} I)^{-1}b, b \rangle + \overline{e^{i\alpha}} \langle (G - e^{i\alpha} I)^{-1}b, b \rangle \geq 0 \quad (6.4)$$

Multiplying (6.4) by $\frac{1}{2\pi}$ and integrate from 0 to 2π with respect to α to obtain

$$\langle b, b \rangle + \frac{1}{2\pi i} \int_0^{2\pi} e^{i\alpha} \langle (G - e^{i\alpha} I)^{-1} b, b \rangle d\alpha + \frac{1}{2\pi} \int_0^{2\pi} \overline{e^{i\alpha} \langle (G - e^{i\alpha} I)^{-1} b, b \rangle} d\alpha \geq 0. \quad (6.5)$$

We write inequality (6.5) as;

$$\langle b, b \rangle + \frac{1}{2\pi i} \int_0^{2\pi} e^{i\alpha} \langle (\mathcal{G} - e^{i\alpha} I)^{-1} b, b \rangle d\alpha + \overline{\frac{1}{2\pi i} \int_0^{2\pi} e^{i\alpha} \langle (\mathcal{G} - e^{i\alpha} I)^{-1} b, b \rangle d\alpha} \geq 0 \quad (6.6)$$

$$- \frac{1}{2\pi i} \int_0^{2\pi} i e^{i\alpha} \langle (G - e^{i\alpha} I)^{-1} b, b \rangle d\alpha - \overline{\frac{1}{2\pi i} \int_0^{2\pi} i e^{i\alpha} \langle (\mathcal{G} - e^{i\alpha} I)^{-1} b, b \rangle d\alpha} \geq \langle b, b \rangle \quad (6.7)$$

The above (6.7) can be stated as;

$$- \frac{1}{2\pi i} \int_{|\eta|=1} \langle (G - \eta I)^{-1} b, b \rangle d\eta - \overline{\frac{1}{2\pi i} \int_{|\eta|=1} \langle (G - \eta I)^{-1} b, b \rangle d\eta} \geq \langle b, b \rangle \quad (6.8)$$

The function $\langle \cdot, b \rangle$ is linear functional on a Krein space $\mathcal{V}$. Using this fact we obtain;

$$\langle -\frac{1}{2\pi i} \int_{|\eta|=1} (\mathcal{G} - \eta I)^{-1} b d\eta, b \rangle + \overline{\langle -\frac{1}{2\pi i} \int_{|\eta|=1} (\mathcal{G} - \eta I)^{-1} b d\eta, b \rangle} \geq \langle b, b \rangle \quad (6.9)$$

We write inequality (6.9) as;

$$\langle -\frac{1}{2\pi i} \int_{|\eta|=1} (G - \eta I)^{-1} b d\eta, b \rangle + \overline{\langle -\frac{1}{2\pi i} \int_{|\eta|=1} (G - \eta I)^{-1} b d\eta, b \rangle} \geq \langle b, b \rangle \quad (6.10)$$

Then we obtain;

$$\langle Bb, b \rangle + \langle b, Bb \rangle \geq \langle b, b \rangle$$

Hence, from complex value function, we obtain the following expansive condition under the operator B .

$$2\Re \langle Bb, b \rangle \geq \langle b, b \rangle. \quad (6.11)$$

From the above, we obtain the following theorem.

Theorem 6.2. *Let G be a bi-expansive operator acting on Krein space $\mathcal{V}$. For $|\mu| = 1$ for $\mu \in \sigma(G)$, $\mathcal{V}$ is the direct sum $\mathcal{V} = \mathcal{R}_+ \oplus \mathcal{R}_-$ of two spaces $\mathcal{R}_+$ and $\mathcal{R}_-$ invariant under G such that a maximal non-negative is $\mathcal{R}_+$ and maximal non-positive is $\mathcal{R}_-$.*

Corollary 6.3. *Every bi-expansive operator defined on a Krein space has maximal non-positive and maximal non-negative invariant sub-spaces.*

7. CONCLUSION

Suppose that B is a bi-expansive operator acting on a Krein space $\mathcal{V}$. Then operator B and its adjoint B^* are both expansive. If the resolvent sets B and B^* intersect such that the circle whose radius $|\eta| = 1$ lies in this intersection, then B and B^* have maximal invariant sub-spaces.

REFERENCES

1. Aronszajn, N., & Smith, K. T. (1954). Invariant subspaces of completely continuous operators. *Annals of Mathematics*, 345–350. <https://doi.org/10.2307/1969637>
2. Bezire, W. A characterization of 2-summing operators in Krein spaces. *Annals of mathematics and computer science* Vol. 11(2023) 11 - 18.
3. Per Enflo, (2003). On the invariant subspace problem in hilbert spaces. arXiv preprint arXiv:2305.15442 .
4. Enflo, P. (1976). On the invariant subspace problem in banach spaces. *S ´eminaire d’Analyse fonctionnelle (dit” Maurey-Schwartz”)*, 1–6.
5. Gamal, M. F. (2023). On expansive operators that are quasisimilar to the unilateral shift of finite multiplicity. arXiv preprint arXiv:2303.16300 . <https://doi.org/10.48550/arXiv.2303.16300>
6. Halmos, P. R. (1966). Invariant subspaces of polynomially compact operators. *Pacific journal of Mathematics*, 16 (3), 433–437. <http://dx.doi.org/10.2140/pjm.1966.16.433>
7. Kreyszig, E. (1991). *Introductory functional analysis with applications* (Vol. 17). John Wiley & Sons.
8. Read, C. J. (1984). A solution to the invariant subspace problem. *Bulletin of the London Mathematical Society*, 16 (4), 337–401. <http://https://doi.org/10.1112/blms/16.4.337>
9. Read, (1985). A solution to the invariant subspace problem on the space l_1 . *Bulletin of the London Mathematical Society*, 17 (4), 305–317. <http://https://https://doi.org/10.1112/blms/17.4.305>
10. Wanjala, G. (2012). The invariant subspace problem for absolutely p -summing operators in krein spaces. *Journal of Inequalities and Applications*, 2012 , 1–13. <https://doi.org/10.1186/1029-242X-2012-254>
11. Wilbroad, B. vol 7 (2022), 32 - 44. The invariant subspace problem for contractive operators in krein spaces. *Annals of Mathematics and computer science*.
12. Yadav, B. (2005). The invariant subspace problem. *Nieuw. Arch. Wisk*, 5–6.

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