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SPECTRAL ANALYSIS OF EIGENVALUE PROBLEM WITH THE $(P(x), Q(x))$–LAPLACIAN

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Abstract. In this study, a spectral problem that involves the Robin $(a(x), b(x))$-Laplacian on a bounded domain in $\mathbb{R}^n$ is analyzed. With the help of the Ljusternik-Schnirelmann principle, it is established that an increasing sequence of eigenvalues exists. Using the min-max principle, we also established that the infimum of the spectrum is zero, and all the eigenvalues are non-negative. It was further proved that there exists a principal eigenvalue for this problem and that the spectrum is open.

1. Introduction

In recent years, studies of different problems in Mathematics with variable exponents have attracted much concentration, with some of the latest studies including [3],[6] and [7]. Subsequently, eigenvalue problems for non-linear differential operators like $\nabla\left(|\nabla w|^{a(x)-2}\nabla w + |\nabla w|^{b(x)-2}\nabla w\right)$ whereby $a(x)$ and $b(x)$ are continuous positive functions, subject to different boundary conditions have come up as potential mathematical problems which are also of interest for physical applications. Here, we considered the eigenvalue problem having variable exponents $c(x), b(x)$ and $a(x)$, defined as;

$$\begin{cases} -\nabla\left(|\nabla w|^{a(x)-2}\nabla w + |\nabla w|^{b(x)-2}\nabla w\right) = \mu |w|^{c(x)-2} w, & x \in D \\ |\nabla w|^{a(x)-2}\frac{\partial w}{\partial \Gamma} + \beta(x)|w|^{b(x)-2} w = 0, & x \in \partial D \end{cases} \tag{1.1}$$

where D is a bounded domain in $\mathbb{R}^N$, $N \geq 3$ with a smooth boundary, $\mu \in \mathbb{R}^+$, a, b and c are continuous functions on D, $\frac{\partial w}{\partial \Gamma}$ is the outwardly perpendicular directional derivative of w relative to ∂D, $\beta : \partial D \to \mathbb{R}$ is a positive function which is bounded on ∂D, which means that for every $x \in \partial D, \beta(x) > 0$,

$$1 < b(x) < c_-(y) \leqslant c^+(y) < a(x), \text{for every } x, \text{and} \in \bar{D}$$

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and $c^+(y) < b^*(x)$, for every x and $y \in \bar{D}$ where by $c_-(y)$ is the inf $c(y)$, $c^+(y)$ is the sup $c(y)$ and $b^*(x)$ is a critical Sobolev exponent given by

$$b^*(x) = \begin{cases} \frac{mb(x)}{m-b(x)}, & if \quad b(x) < m \\ \infty, & if \quad b(x) \geq m \end{cases}$$

Using the Ljusternik-Schnirelman(L-S) theory, the non-linear eigenvalue problem $-\Delta_a u = \lambda|u|^{a-2}u$ was studied in [8]. It was proved that there is a positive sequence of non-decreasing eigenvalues, the initial eigenvalue is unique and separated, and the second eigenvalue concurs with the second-variational eigenvalue got using the L-S theory. It was also shown that the second eigenvalue obtained by way of L-S theory is the least eigenvalue in the spectrum which is bigger than the principal one. A related study with variable exponents $-\Delta_{a(x)}u = \lambda|u|^{b(x)-2}u$ was investigated in [3] with Robin boundary conditions, and ascertained that a non-decreasing sequence of non-negative eigenvalues exists, proved that a principle eigenvalue exists and is the smallest of the possible eigenvalues and that the spectrum is not closed.

In another study, about a nonlinear eigenvalue problem for the (a, b)- Laplacian with a positive weight $-\Delta_a u - \Delta_b u = \lambda g(x)|u|^{a-2}u$ in $\mathbb{R}^N$, it was proved in [2] by employing the Mountain -Pass Theorem that a continuous positive spectrum exists. Another related study was carried out in [4] on an eigenvalue problem $-\Delta_a u - \Delta_b u = \lambda|u|^{b-2}$ with, $b \in (2,\infty)$, $a \in (1,\infty)$, a is not equal to b subject to the associated homogeneous Neumann boundary on an open domain that is bounded, with smooth boundary from $\mathbb{R}^n$ and $n \geq 2$. They carefully analyzed the problem and came up with a complete description that the set of eigenvalues has a precise interval $(\lambda_1, +\infty)$ plus an isolated point $\lambda = 0$. This outcome is related strongly to the well-known case $a = b \neq 2$ in which a full description of the spectrum is still unavailable.

With the help of the concentration-compactness principle and the variational methods, [1], proved the existence of results for $-\Delta_a u - \Delta_b u = \lambda W(x)|u|^{l-2}u + L(x)|u|^{a^*-2}u$ in all of $\mathbb{R}^n$ for the (a, b)-Laplace operator type elliptic problem that involves a critical term, non-negative weights and the parameter λ which is positive. Particularly, with congruous conditions on exponents of non-linearity, they proved that there exist countless weak solutions with negative energy, so long as λ lies within some interval.

A related study on $-div\left(|\nabla w|^{a(x)-2}\nabla w + |\nabla w|^{b(x)-2}\nabla w\right) = \lambda|w|^{c(x)-2}w$ was carried out in [5]. By applying the Dirichlet boundary conditions they established that two positive constants, that is λ_0 and λ_1 exist, and that $\lambda_0 \leq \lambda_1$ where, every $\lambda \in [\lambda_1, \infty)$ becomes an eigenvalue, but a $\lambda \in (0, \lambda_0)$ fails. We however note that, a conclusion on if $\lambda_0 \leq \lambda_1$ was not made. This study thus covers a related study of $-\nabla\left(|\nabla w|^{a(x)-2}\nabla w + |\nabla w|^{b(x)-2}\nabla w\right) = \mu|w|^{c(x)-2}w$ on a Sobolev space using the Robin boundary conditions. Sobolev spaces allow the application of weak derivatives. The paper is organised into four sections that include introduction, preliminaries, main results and conclusion.

2. Preliminaries

Consider a domain $\Omega \in \mathbb{R}^N$ that is bounded with a smooth boundary $\partial\Omega$, where $\bar{\Omega}$ stands for closure of Ω. We present the following definitions and theories, most of which can be found in [3] and [5].

Definition 2.1. Let α be a multi-index and that $u, v \in L^1_{loc}(\Omega)$. Then v is referred to or termed as the α^{th} partial weak derivative for u, denoted by $D^\alpha u = v$, as long as for any $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v\phi dx.$$

Let $\mathbf{C}_+(\bar{\Omega}) = \{h \ such \ that \ h \in C(\bar{\Omega}) \ and \ h(x) > 1$, for every x in $\bar{\Omega}\}$.
For every $h \in C_+(\bar{\Omega})$, then we can define

$$h^+ = \sup_{x\in\Omega} h(x) \ and \ also \ that \ h^- = \inf_{x\in\Omega} h(x)$$

Definition 2.2. The Lebesgue space characterized by variable exponents is denoted and defined as
$L^{p(x)}(\Omega) =$

$$\left\{ v : v \text{ is the real-valued and measurable function with } \int_\Omega |v(x)|^{p(x)} < +\infty \right\}$$

having a norm

$$\|v\|_{L^{p(x)}(\Omega)} = \inf \left\{ \gamma > 0 : \int_\Omega |\frac{v(x)}{\gamma}|^{p(x)} dx < 1 \right\}$$

Definition 2.3. The Sobolev space characterized by variable exponents is denoted and defined as

$$W^{1,p(x)} = \{v \in L^{p(x)}(\Omega) \ such \ that \ |\nabla v| \in L^{p(x)}(\Omega)\}$$

having the norm

$$\|v\|_{W^{1,p(x)}(\Omega)} = inf \left\{ \gamma > 0 : \int_\Omega \left(|\frac{\nabla v}{\gamma}|^{p(x)} + |\frac{v}{\gamma}|^{p(x)} \right) dx < 1 \right\} \tag{2.1}$$

Definition 2.4. (Condition S_+) Let X be a real and reflexive Banach space. Then the operator B: $X \to X^*$ is said to be satisfying the condition S_+ if and only if as $n \to \infty, w_n \rightharpoonup w$ and $\lim_{n\to\infty}\langle Bw_n - Bw, w_n - w\rangle \leq 0$, then $w_n \to w$.

Definition 2.5. (Palais-Smule(PS) condition) We say that a continuously differentiable functional $h \in C^1(\mathcal{H}, \mathbb{R})$ from a Hilbert space $\mathcal{H}$ to the real numbers satisfies this PS condition in case any sequence $\{w_l\}_{l=1}^\infty \subset \mathcal{H}$ so that $\{h[w_l]\}_{l=1}^\infty$ is bounded and $h'[w_l] \to 0$ in $\mathcal{H}$.

The embedding $W^{1,a(x)} \hookrightarrow C(\bar{\Omega})$ is both continuous and compact if $a(x) \geqslant 1, \forall$ $x \in \Omega$[9].

Definition 2.6. Let $L^{b(x)}$ the be conjugate space of $L^{a(x)}(\Omega)$; $\frac{1}{a(x)} + \frac{1}{b(x)} = 1$. Then for every $u \in L^{a(x)}$ and $v \in L^{b(x)}$, the holders inequality is given by;

$$\int_\Omega |uv| dx \leqslant \left(\frac{1}{a(x)} + \frac{1}{b(x)} \right) \|u_{a(x)}\| \|v_{b(x)}\|.$$

Lemma 2.7. *Suppose that $a \in C_0(\bar{D})$ and $b \in L^{\infty}(D)$ with $b_- \geq 1$, then the embeding $W^{1,a(x)}(D) \hookrightarrow L^{b(x)}(D)$ becomes compact, as long as $b(x) \ll a^*(x)$.*

Refer to [3] for the proof.

Definition 2.8. [3, Definition 3.1] Let $g\colon X \to \mathbb{R}$ be any functional defined on a space X. Then a point w is said to be a critical point of g if $g'(w) = 0$. We say that μ is a critical value of g in case a critical point w exists, and $g(w) = \mu$.

Remark 2.9. For the functional:

$$H[u] = \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx, u \in X := W^{1,p(x)}(\Omega).$$

It is shown in that $H \in C^1(X, R)$ and that the p(x)-Laplacian represents the weak sense derivative-operator of H;

$$\Delta_{p(x)} u := H_1(u)$$

Thus $H' : X \to X^*$ and

$$\langle H_1(u), u \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx.$$

3. MAIN RESULTS

In this section, we establish the existence of an eigenvalue sequence using the Ljusternik-Schnirelmann principle. Let $\xi \in [0, \infty)$ be fixed, and consider a non-linear eigenvalue problem,

$$F'(v) = \lambda G'(v) \qquad v \in M_{\xi}, \qquad \lambda \in \mathbb{R} \tag{3.1}$$

where $M_{\xi} = \{v \in X\colon G(v) = \xi\}$ is a level set. Assuming that

- (A_1) A Banach space X is real, separable, and of infinite dimension and is reflexive with X^* as its dual space, whereas $F, G\colon X \to \mathbb{R}$ are two continuously Fr ́echet differentiable even functionals with also $F(0) = 0 = G(0)$.
- (A_2) $F'|X \to X^*$ is strong and continuous on X (i.e. if $v_n \rightharpoonup v$ then $F'(v_n) \to F'(v)$) and $F(v) \neq 0, v \in \bar{co}M_{\xi}$ imply $F'(v) \neq 0$, with $\bar{co}M_{\xi}$ denoting the closed and convex hull in M_{ξ} and the symbol $\to$ $(\rightharpoonup)$ stands for usual (weak) convergence.
- (A_3) $G'|X \to X^*$ is bounded and remains continous on a space X and it satisfies the condition that if

  $$v_n \rightharpoonup v, G'(v_n) \rightharpoonup v, \textit{ and } \langle G'(v_n), v_n \rangle \to \langle v, v \rangle,$$

  then $v_n \to v$, where $\langle ., . \rangle | X \times X^* \to \mathbb{R}$ represents the duality-pairing, involving the spaces X and its dual X^*.
- (A_4) M_{ξ}(a bounded level set) and $\langle G'(u), u \rangle > 0$ and

  $$\lim_{t \to \infty} G(tv) = \infty$$

  for all non zero vectors v and

  $$\inf_{v \in M_{\xi}} \langle G'(v), v \rangle > o$$

Let

$$c_m = \begin{cases} sup_{k\in k_m} inf_{u\in K} F(u), & for \;\; k_m \neq \emptyset \\ 0, & for \;\; k_m = \emptyset, \end{cases} \tag{3.2}$$

where $K_m = \{K \subseteq M_\xi$: K is compact, symmetric, gen $(K) \geqslant m$, and $F(v) > 0$ on $K\}$. Here, to each symmetric closed subset K of X which does not contain the origin O, we assign an integer, called the genus of K, denoted by gen K, in the following way:

(i) gen (K)= 0 if $K = \emptyset$.
(ii) In case $K \neq \emptyset$, gen (K)=inf$\{n \geq 1$: there is an odd/uneven continuous map f: $K \to \mathbb{R}^n \setminus \{0\}\}$
(iii) In case $K \neq \emptyset$ and such an n does not exist, then gen $K = \infty$. Note that by the Borsuk-Ulam theorem, the genus of spheres =dimension of the underlying space.

The following theorem is the Ljusternik-Schnirelmann principle.

Theorem 3.1. *Given assumptions (A_1)-(A_4), these statements hold:*

(i) *If $c_m > 0$, then (1.1) possesses an eigen pair of vectors, $(u_m, -u_m)$ with the nonzero eigenvalue λ_m and $F(u_m) = c_m$. If, in addition, F' and G' are positively homogeneous of order $r > 0$ ($F'(tu) = t^r F'(u)$ and $G'(tu) = t^r G'(u)$ $\forall$ $u \in X$ and $t > 0$), it means $c_m = \xi\lambda_m$.*
(ii) *We have $\infty > c_1 \geqslant c_2 \geqslant ... \geqslant 0$ and $c_m \to 0$ as $m \to \infty$.*
(iii) *Given $F(u) = 0, u \in \bar{c}oN_\xi$ implying that $u = 0$, then there is an eigen sequence of functions (u_m, λ_m) for (1.1):$u_m \rightharpoonup 0$ in X and $\lambda_m \to \infty$ and $\lambda_m \neq \emptyset$ $\forall$ $m \in \mathbb{N}$.*

More details regarding the L-S principle can be traced in [3] and [4].
Throughout the study, we also assume that;

(i) $a \in C_0(\bar{D})$ and $a_- > 1$,
(ii) $b \in L^\infty(D)$ with $b_- \geq 1$, and $b(x) \ll a^*(x)$,
(iii) $X = W^{1,a(x)}(D)$ with the norm defined as in equation (2.1)

Lemma 3.2. *A pair (w,μ) $\in X \times \mathbb{R}$ is called a weak-solution to our problem* (1.1) *in case*

$$\int_D |\nabla w|^{a(x)-2} \nabla w \nabla v dx + \int_{\partial D} \beta(x)|w|^{a(x)-2} wv dS + \int_D |\nabla w|^{b(x)-2} \nabla w \nabla v dx$$
$$+ \int_{\partial\Omega} \beta(x)|w|^{b(x)-2} wv dS = \mu \int_D |w|^{c(x)-2} wv dx,$$

for all $v \in X$.

Proof. If (w, μ) is a solution for (1.1) and that $w \in X\backslash\{0\}$, then μ is called an eigenvalue of (1.1) and w is an eigenfunction corresponding to μ. Indeed, if (w, μ) is a solution for (1.1), it means that for any v in X, we have

$$\langle -\Delta_{(a(x),b(x))} w, v\rangle_{L^2(D)} = \mu \langle |w|^{c(x)-2} wv\rangle_{L^2(D)},$$

implying that

$$-\int_D \nabla(|\nabla w|^{a(x)-2}\nabla w)v dx - \int_D \nabla(|\nabla w|^{b(x)-2}\nabla w)v dx = \mu \int_D |w|^{c(x)-2} w v dx.$$

On integration by parts and applying the boundary conditions to equation (1.1), we obtain

$$\int_D |\nabla w|^{a(x)-2}\nabla w \nabla v dx + \int_{\partial D} \beta(x)|w|^{a(x)-2} w v dS + \int_D |\nabla w|^{b(x)-2}\nabla w \nabla v dx$$
$$+ \int_{\partial D} \beta(x)|w|^{b(x)-2} w v dS = \mu \int_D |w|^{c(x)-2} w v dx.$$

□

From lemma 3, if we set $w \equiv v$, then we get

$$\int_D |\nabla w|^{a(x)-2}|\nabla w|^2 dx + \int_{\partial D} \beta(x)|w|^{a(x)-2}|w|^2 dS + \int_D |\nabla w|^{b(x)-2}|\nabla w|^2 dx$$
$$+ \int_{\partial D} \beta(x)|w|^{b(x)-2}|w|^2 dS = \mu \int_D |w|^{c(x)-2}|w|^2 dx,$$

from which we get

$$\int_D |\nabla w|^{a(x)} dx + \int_{\partial D} \beta(x)|w|^{a(x)} dS + \int_D |\nabla w|^{b(x)} dx$$
$$+ \int_{\partial D} \beta(x)|w|^{b(x)} dS = \mu \int_D |w|^{c(x)} dx.$$

This results into

$$\mu = \frac{\int_D |\nabla w|^{a(x)} dx + \int_{\partial D} \beta(x)|w|^{a(x)} dS + \int_D |\nabla w|^{b(x)} dx + \int_{\partial D} \beta(x)|w|^{b(x)} dS}{\int_D |w|^{c(x)} dx}.$$

Now, applying the variational principle, it then follows that $\mu_{min} =$

$$\min_{w \in X, w \neq 0} \left(\frac{\int_D |\nabla w|^{a(x)} dx + \int_{\partial D} \beta(x)|w|^{a(x)} dS + \int_D |\nabla w|^{b(x)} dx + \int_{\partial D} \beta(x)|w|^{b(x)} dS}{\int_D |w|^{c(x)} dx} \right).$$

Hence, we have $\mu > 0$, since $\beta(x) > 0$ for every x in ∂D. Consequently, this result follows.

Now, let g and h be functionals on X so that if $w \in X$,

$$g(w) = \int_D \frac{1}{c(x)}|w|^{c(x)} dx,$$

$$h(w) = \int_D \frac{1}{a(x)}|\nabla w|^{a(x)} dx + \int_{\partial D} \frac{1}{a(x)}\beta(x)|w|^{a(x)} dS$$
$$+ \int_D \frac{1}{b(x)}|\nabla w|^{b(x)} dx + \int_{\partial D} \frac{1}{b(x)}\beta(x)|w|^{b(x)} dS,$$

restricted to the level set $M_\xi = \{w \in X\colon g(w) = \xi\}$, for $\xi > 0$. Then $g, h \in C^1(X, \mathbb{R})$ and that $\forall$ v in X,

$$\langle g'(w), v\rangle = \int_D |w|^{c(x)-2} wv dx,$$

$$\langle h'(w), v\rangle = \int_D |\nabla w|^{a(x)-2} \nabla w \nabla v dx + \int_{\partial D} \beta(x)|w|^{a(x)-2} wv dS$$
$$+ \int_D |\nabla w|^{b(x)-2} \nabla w \nabla v dx + \int_{\partial D} \beta(x)|w|^{b(x)-2} wv dS,$$

whereby g' and h' are the respective weak derivatives for g and h, (see definition 2.1).

If $b(x) = b$, $a = a(x)$ and $c(x) = c$, then we can choose $\xi = 1$ since the functionals g' and h' are homogeneous. Note that if (w, μ) with $w \in M_\xi$ is a solution to (1.1), then given $r > 0, r^{\frac{1}{a}} u \in M_\xi$ and $(r^{\frac{1}{a}} u, r^{\frac{a-b}{a}} \mu)$ is a solution to (1.1). However, this is not possible when $a(x)$ and $b(x)$ are variable exponents due to the inhomogeneity of the functionals g and h. Let g_ξ be the restriction of g on M_ξ such that $g_\xi := f\ |_{M_\xi} \to \mathbb{R}$. Then, it is possible to show that $h'\colon X \to X^*$ is considered to be a monotone homeomorphism which is of S_+ type [see (2.4)] such that in case $w_m \rightharpoonup w \in X$ and $\lim_{m\to\infty}(h'(w), w_m - w) \le 0$, it implies, $w_m \to w \in X$.

Using the same argument, (2.4), $g'\colon X \to X^*$ is weakly strongly continuous, showing that g_ξ satisfies the (PS) condition on M_ξ, (2.5).
Let $K_m = \{K \subseteq M_\xi\colon K$ is compact, symmetric, gen $(K) \geqslant m$, with $m = 1, 2, ...,$ gen (K) represents the genus of K, and gen (K)=inf$\{n \ge 1$: there is an odd continuous map $h\colon K \to \mathbb{R}^n \setminus \{0\}\}$.
Also, let

$$c_m = \sup_{k \in k_m} \inf_{u \in K} g(u), u = 1, 2, ...$$

Clearly, $c_m > 0$, and

$$c_1 \ge c_2 \ge ... c_n \ge c_{n+1} \ge ...$$

From the above assumptions, one gets this;

Lemma 3.3. *Assume $a \in C_0(\bar{D})$ and $a_- > 1$, $b \in L^\infty(D)$ and $b(x) \ll a^*(x)$. Then for all $n \in \mathbb{N}, c_m$ is a critical value of g_ξ on M_ξ and $c_m \to 0$ as $m \to \infty$.*

Proof. With $b \in L^\infty$ and $b \ll a^+$, c_m is a decreasing sequence with $c_m > 0$ for all m. □

In addition, if $\chi_m(\xi) = \{w \in M_\xi : w$ is a critical point of g_ξ and $g_\xi(w) = c_m\}$,

$$\Pi_m(\xi) = \{\mu(w) : w \in \chi_m(\xi)\} \, and$$

$$\lambda_m(\xi) = \frac{\xi}{c_m},$$

for any $m \in \mathbb{N}$ and $\xi > 0$, then this outcome follows.

Theorem 3.4. *Let $a \in C_0(\bar{D})$ and $a_- > 1$, $b \in L^\infty(D)$ and $b(x) \ll a^*(x)$. Then as $m \to \infty$, $\pi_m \to \infty$ for all $m \in \bar{N}$, and $\xi > 0$.*

Proof. This follows directly from the above assumptions.
If $\xi > 0$ and $\mu \in \Pi_m$, then there is a $w \in \chi_m(\xi)$ so that

$$\mu = \frac{\int_D |\nabla w|^{a(x)}dx + \int_{\partial D} \beta(x)|w|^{a(x)}dS + \int_D |\nabla w|^{b(x)}dx + \int_{\partial D} \beta(x)|w|^{b(x)}dS}{\int_D |w|^{c(x)}dx}.$$

The fact that
$a_+ =$es $\sup_{x\in D} a(x)$, $a_- =$es $\inf_{x\in D} a(x)$
$b_+ =$es $\sup_{x\in D} b(x)$, $b_- =$es $\inf_{x\in D} b(x)$
and
$c_+ =$es $\sup_{x\in D} c(x)$, $c_- =$es $\inf_{x\in D} c(x)$,
it then means that, for any $x \in D, a(x) \leq a_+, b(x) \leq b_+$ and this in turn shows that $\frac{a_+}{a(x)} \geq 1, \frac{b_+}{b(x)} \geq 1$ and $c_- \leq c(x)$ as well implies $\frac{c_-}{c(x)} \leq 1$. This in turn gives

$$\begin{aligned}
\mu &= \frac{\int_D |\nabla w|^{a(x)}dx + \int_{\partial D} \beta(x)|w|^{a(x)}dS + \int_D |\nabla w|^{b(x)}dx + \int_{\partial D} \beta(x)|w|^{b(x)}dS}{\int_D |w|^{c(x)}dx} \\
&\leq \frac{a_+[\int_D |\nabla w|^{a(x)}dx + \int_{\partial D} \beta(x)|w|^{a(x)}dS\} + b_-\{\int_D |\nabla w|^{b(x)}dx + \int_{\partial D} \beta(x)|w|^{b(x)}dS]}{c_- \int_D |w|^{c(x)}dx} \\
&= \frac{a_+\xi + b_+\xi}{c_- c_m} \\
&= \frac{a_+}{c_-}\lambda_m(\xi) + \frac{b_+}{c_-}\lambda_m(\xi) \\
&\leq \frac{a_+}{c_-}\lambda_m(\xi) + \frac{b_+}{c_-}\lambda_m(\xi).
\end{aligned} \tag{3.3}$$

In the same way, for all $x \in D, b_- \leq b(x), a_- \leq a(x)$ and this shows basically that $\frac{a_-}{a(x)} \leq 1, \frac{b_-}{b(x)} \leq 1$ and $c_+ \leq c(x)$ as well implies again that $\frac{c_+}{c(x)} \leq 1$. This in turn gives

$$\begin{aligned}
\mu &= \frac{\int_D |\nabla w|^{a(x)}dx + \int_{\partial D} \beta(x)|w|^{a(x)}dS + \int_D |\nabla w|^{b(x)}dx + \int_{\partial D} \beta(x)|w|^{b(x)}dS}{\int_D |w|^{c(x)}dx} \\
&\geq \frac{a_-\{\int_D |\nabla w|^{a(x)}dx + \int_{\partial D} \beta(x)|w|^{a(x)}dS\} + b_-\{\int_D |\nabla w|^{b(x)}dx + \int_{\partial D} \beta(x)|w|^{b(x)}dS\}}{c_+ \int_D |w|^{c(x)}dx} \\
&= \frac{a_-\xi + b_-\xi}{c_+ C_m} \\
&= \frac{a_-}{c_+}\lambda_m(\xi) + \frac{b_-}{c_+}\lambda_m(\xi) \\
&\geq \frac{a_-}{c_+}\lambda_m(\xi) + \frac{b_-}{c_+}\lambda_m(\xi).
\end{aligned} \tag{3.4}$$

Using the above two inequalities (3.3) and (3.4), it can be noted that

$$\frac{a_-}{c_+}\lambda_m(\xi) + \frac{b_-}{c_+}\lambda_m(\xi) \leq \Pi_m(\xi) \leq \frac{a_+}{c_-}\lambda_m(\xi) + \frac{b_+}{c_-}\lambda_m(\xi). \tag{3.5}$$

The fact that as $m \to \infty, \lambda_m(\xi) \to \infty$, it follows that by the squeeze theorem for limits of sequences, $\Pi_m(\xi) \to \infty$ as $m \to \infty$ for every $\xi > 0$.
Therefore, there exists a sequence or pattern of eigenvalues which is infinite for our eigenvalue problem. □

Definition 3.5. An eigenvalue $\mu \in \Pi$ is said to be a principle eigenvalue of (1.1) if there is a $w > 0$ so that the pair (w, μ) is a solution for (1.1).

Let us assume that

$$\Pi = \{\mu : \mu \textit{ is an eigenvalue of } (1.1)\} \tag{3.6}$$

Suppose that $b, a \in c_0(\bar{D})$ and that $a_- > 1$ and $b_- > 1$, $c \in L^\infty(D)$, with $c_- > 1$ and $c(x) \ll c^*(x)$, then it is seen from c_m that for every $\xi > 0$, $c_1 = sup_{w\in M_\xi} g(w)$. Let us now define $\Pi^+ = \{\mu | \mu \textit{ is an eigenvalue of } (1.1)\}$, then this outcome follows.

Theorem 3.6. *For every* $\xi > 0$, $\Pi_1(\xi) \subset \Pi^+$.

Proof. Suppose that $\xi > 0$ and $\mu \in \Pi_1(\xi)$. It means that there is a $w \in M_\xi$ so that $g(w) = c_1 = \sup_{w\in M(\xi)} g(w)$ with

$$\mu = \frac{\int_D |\nabla w|^{a(x)}dx + \int_{\partial D}\beta(x)|w|^{a(x)}dS + \int_D |\nabla w|^{b(x)}dx + \int_{\partial D}\beta(x)|w|^{b(x)}dS}{\int_D |w|^{c(x)}dx}.$$

Also assume that for any x in D, $v(x) = |w(x)|$. Then it basically implies that $h(v) = \xi = h(w)$ and $g(v) = g(w) = C_1$.

$$\begin{aligned}&\int_D |\nabla v|^{a(x)}dx + \int_{\partial D}\beta(x)|v|^{a(x)}dS + \int_D |\nabla v|^{b(x)}dx + \int_{\partial D}\beta(x)|v|^{b(x)}dS\\ =&\int_D |\nabla w|^{a(x)}dx + \int_{\partial D}\beta(x)|w|^{a(x)}dS + \int_D |\nabla w|^{b(x)}dx + \int_{\partial D}\beta(x)|w|^{b(x)}dS\end{aligned}$$

and $\int_D |\nabla v|^{c(x)}dx = \int_D |\nabla w|^{c(x)}dx$. $\therefore v > 0, v \in M_\xi, v \in \chi_1(\xi)$ results into

$$\mu = \frac{\int_D |\nabla w|^{a(x)}dx + \int_{\partial D}\beta(x)|w|^{a(x)}dS + \int_D |\nabla w|^{b(x)}dx + \int_{\partial D}\beta(x)|w|^{b(x)}dS}{\int_D |w|^{c(x)}dx}.$$

This implies that (v, μ) is a solution to (1.1) and $\mu \subset \Pi^+$ completes the proof.
Consequently, the first eigenvalue is the principle Eigenvalue. □

Finally, we determine whether the the spectrum of (1.1) is closed or not. This is achieved by showing that the infimum of the set of eigenvalues of (1.1) is non-zero. The following theorem is the main result.

Theorem 3.7. *Assume there is a* $T \subset D$*(an open subset) and a compact subset* $F \subset T$ *with a Lebesgue measure(positive) so that* $b_+(F) < a_-(\partial T)$, $x \in \partial T$. *Also, suppose there are* $x_0 \in D$ *and an open neighborhood* J *of* x_0 *so that* $b \in C_o(J)$ *and* $b(x_0) < a(x_0)$. *Then as* $\xi \to 0, \mu_1(\xi) \to 0$ *and* $\Pi_1(\xi) \to 0$. $\therefore \inf \Pi(\mu_1) = 0$.

Proof. Assume that $T_\delta = \{x \in D | d(x, \partial T) < \delta\}$, for $\delta > 0$. Then there must exist a $\delta > 0$, which is small enough to make $F \bigcap T_\delta = \emptyset$ and $b_+(x_0) < a_-(x)$. Thus, for if $\delta_0 = a_-(x) - b_+(x_0)$, it means $\delta_0 > 0$. Let the functional $w_0 \in C^\infty(D)$, so for any $x \in D$,

$$w_0(x) = \begin{cases} 1, & for \quad x \in D - \{T\}, \\ 0, & for \quad x \in D - \{T_\delta\}. \end{cases}$$

If we are given that $\xi > 0$, then there is an $r > 0 : rw_0 \in M_\xi$, that is $h(rw_0) = \xi$. Besides, $r \to 0$ as $\delta \to 0$. Indeed,

$$h(rw_0) = \int_D \frac{1}{a(x)}|\nabla rw_0|^{a(x)}dx + \int_{\partial D} \frac{1}{a(x)}\beta(x)|rw_0|^{a(x)}dS + \int_D \frac{1}{b(x)}|\nabla rw_0|^{b(x)}dx$$

$$+ \int_{\partial D} \frac{1}{b(x)}\beta(x)|rw_0|^{b(x)}dS = \xi$$

$$= \int_D \frac{1}{a(x)}|\nabla w|^{a(x)}dx + \int_{\partial D} \frac{1}{a(x)}\beta(x)|w|^{a(x)}dS + \int_D \frac{1}{b(x)}|\nabla w|^{b(x)}dx$$

$$+ \int_{\partial D} \frac{1}{b(x)}\beta(x)|w|^{b(x)}dS.$$

Thus,

$$r^{a(x)} \left(\int_D \frac{1}{a(x)}|\nabla w_0|^{a(x)}dx + \int_{\partial D} \frac{1}{a(x)}\beta(x)|w_0|^{a(x)}dS \right) +$$

$$r^{a(x)} \left(\int_D \frac{1}{b(x)}|\nabla w_0|^{b(x)}dx + \int_{\partial D} \frac{1}{b(x)}\beta(x)|w_0|^{b(x)}dS \right)$$

$$= \int_D \frac{1}{a(x)}|\nabla w|^{a(x)}dx + \int_{\partial D} \frac{1}{a(x)}\beta(x)|w|^{a(x)}dS + \int_D \frac{1}{b(x)}|\nabla w|^{b(x)}dx$$

$$+ \int_{\partial D} \frac{1}{b(x)}\beta(x)|w|^{b(x)}dS,$$

which, after making $r^{a(x)}$ the subject finally gives

$$r^{a(x)} = \frac{\int_D \frac{1}{a(x)}K_3dx + \int_{\partial D} \frac{1}{a(x)}\beta(x)L_3dS + \int_D \frac{1}{b(x)}K_4dx + \int_{\partial D} \frac{1}{b(x)}\beta(x)L_4dS}{\int_D \frac{1}{a(x)}K_1dx + \int_{\partial D} \frac{1}{a(x)}\beta(x)L_1dS + \int_D \frac{1}{b(x)}K_2dx + \int_{\partial D} \frac{1}{b(x)}\beta(x)L_2dS}$$

$$= \frac{\xi}{h(w_0)},$$

where $K_1 = |\nabla w_0|^{a(x)}$, $K_2 = |\nabla w_0|^{b(x)}$, $L_1 = |w_0|^{a(x)}$, $L_2 = |w_0|^{b(x)}$, $K_3 = |\nabla w|^{a(x)}$, $K_4 = |\nabla w|^{b(x)}$, $L_3 = |w|^{a(x)}$, $L_4 = |w|^{b(x)}$. We can now get $r = \left\{\frac{\xi}{h(w_0)}\right\}^{\frac{1}{a(x)}}$ and now applying limits gives $\lim_{\xi\to 0}(r) = \lim_{\xi\to 0}\left\{\frac{\xi}{h(w_0)}\right\}^{\frac{1}{a(x)}} = 0$. Suppose now that

$\xi > 0$ is so small to the extent that $0 < r < 1$. It then means that,

$$\begin{aligned}
\mu_1(\xi) &= \frac{\xi}{C_m} \\
&= \frac{h(rw_0)}{g(rw_0)} \\
&= \frac{\int_D \frac{1}{a(x)}|\nabla rw_0|^{a(x)}dx + \int_{\partial D} \frac{1}{a(x)}\beta(x)|rw_0|^{a(x)}dS}{\int_D \frac{1}{c(x)}|rw_0|^{c(x)}dx} \\
&+ \frac{\int_D \frac{1}{b(x)}|\nabla rw_0|^{b(x)}dx + \int_{\partial D} \frac{1}{b(x)}\beta(x)|rw_0|^{b(x)}dS}{\int_D \frac{1}{c(x)}|rw_0|^{c(x)}dx} \\
&= \frac{\int_D \frac{r^{a(x)}}{a(x)}K_1dx + \int_{\partial D} \frac{r^{a(x)}}{a(x)}\beta(x)L_1dS + \int_D \frac{r^{b(x)}}{b(x)}K_2dx + \int_{\partial D} \frac{r^{a(x)}}{b(x)}\beta(x)L_2dS}{\int_D \frac{r^{c(x)}}{c(x)}|w_0|^{c(x)}dx} \\
&= \frac{\int_{T\setminus T_\delta} \frac{r^{a(x)}}{a(x)}|\nabla w_0|^{a(x)}dx + \int_{T_\delta} \frac{r^{a(x)}}{a(x)}|\nabla w_0|^{a(x)}dx + \int_{\partial T_\delta} \frac{r^{a(x)\beta(x)}}{a(x)}|w_0|^{a(x)}dS}{\int_F \frac{r^{c(x)}}{c(x)}|w_0|^{c(x)}dx + \int_{T\setminus F} \frac{r^{c(x)}}{c(x)}|w_0|^{c(x)}dx} \\
&+ \frac{\int_{T\setminus T_\delta} \frac{r^{b(x)}}{b(x)}|\nabla w_0|^{b(x)}dx + \int_{T_\delta} \frac{r^{b(x)}}{b(x)}|\nabla w_0|^{b(x)}dx + \int_{\partial T_\delta} \frac{r^{b(x)\beta(x)}}{b(x)}|w_0|^{b(x)}dS}{\int_F \frac{r^{c(x)}}{c(x)}|w_0|^{c(x)}dx + \int_{T\setminus F} \frac{r^{c(x)}}{c(x)}|w_0|^{c(x)}dx} \\
&\le \frac{\int_{T_\delta} \frac{r^{a(x)}}{a(x)}|\nabla w_0|^{a(x)}dx + \int_{\partial T_\delta} \frac{r^{a(x)\beta(x)}}{a(x)}|w_0|^{a(x)}dS}{\int_F \frac{r^{c(x)}}{c(x)}|w_0|^{c(x)}dx} \\
&\quad + \frac{\int_{T_\delta} \frac{r^{b(x)}}{b(x)}|\nabla w_0|^{b(x)}dx + \int_{\partial T_\delta} \frac{r^{b(x)\beta(x)}}{b(x)}|w_0|^{b(x)}dS}{\int_F \frac{r^{c(x)}}{c(x)}|w_0|^{c(x)}dx}
\end{aligned} \tag{3.7}$$

Still,
$r^{a(x)} \le r^{a_-(\bar{T}_\delta)}$, for every $x \in T_\delta$ and $r^{b_+(F)} \le r^{a(x)}$ for every $x \in F$.
Accordingly,

$$\begin{aligned}
\mu_1(\xi) &\le \frac{r^{a_-(\bar{T}_\delta)}\left(\int_{T_\delta} \frac{1}{a(x)}|\nabla w_0|^{a(x)}dx + \int_{\partial T_\delta} \frac{\beta(x)}{a(x)}|w_0|^{a(x)}dS\right)}{r^{c_+(F)}\int_F \frac{r^{c(x)}}{c(x)}|w_0|^{c(x)}dx} \\
&+ \frac{r^{b_-(\bar{T}_\delta)}\left(\int_{T_\delta} \frac{1}{b(x)}|\nabla w_0|^{b(x)}dx + \int_{\partial T_\delta} \frac{\beta(x)}{b(x)}|w_0|^{b(x)}dS\right)}{r^{c_+(F)}\int_F \frac{r^{c(x)}}{c(x)}|w_0|^{c(x)}dx} \\
&\le \frac{r^{a_-(T_\delta)}\left(\int_{T_\delta} \frac{1}{a(x)}K_1dx + \int_{\partial T_\delta} \frac{\beta(x)}{a(x)}L_1dS + \int_{T_\delta} \frac{1}{b(x)}K_2dx + \int_{\partial T_\delta} \frac{\beta(x)}{b(x)}L_2dS\right)}{r^{c_+(F)}\int_F \frac{1}{c(x)}|w_0|^{c(x)}dx} \\
&\le \frac{r^{\delta_0}\left(\int_{T_\delta} \frac{1}{a(x)}K_1dx + \int_{\partial T_\delta} \frac{\beta(x)}{a(x)}L_1dS + \int_{T_\delta} \frac{1}{b(x)}K_2dx + \int_{\partial T_\delta} \frac{\beta(x)}{b(x)}L_2dS\right)}{\int_F \frac{1}{c(x)}|w_0|^{c(x)}dx}
\end{aligned} \tag{3.8}$$

where $\delta_0 = a_-(T_\delta) - c_+(F), K_1 = |\nabla w_0|^{a(x)}, K_2 = |\nabla w_0|^{b(x)}, L_1 = |w_0|^{a(x)}, L_2 = |w_0|^{b(x)}, K_3 = |\nabla w|^{a(x)}, K_4 = |\nabla w|^{b(x)}, L_3 = |w|^{a(x)}, L_4 = |w|^{b(x)}$. and as a result, it can be concluded that $\mu_1(\xi) \to 0$ as $\xi \to 0$. Thus, by (3.5), $\Pi_1(\mu) \to 0$ as $\mu \to 0$. Hence $\mu_\star = \inf \Pi = 0$. □

Therefore, from the theorem above, it can easily be noticed that if an open set $T \subset D$ exists, and a compact set $F \subset D$ so that $b_+(F) < a_-(\partial T)$ for $x \in \partial T$, or if there is an open set $T \subset D$ and a compact set $F \subset T$ so that $b_-(F) < a_+(\partial T)$, then $\mu_1 = 0$. Thus, the infimum of the eigenvalue set for (1.1) equals zero, which shows that the set of all the eigenvalues is open. This is because zero is not an eigenvalue.

4. Conclusion

By applying the Ljusternik-Schnirelman(L-S) principle as well as the variational (min-max) principle, it is shown that for $\mu > 0$ and $\beta > 0$, problem 1.1 has an increasing sequence $\{\mu_m(\xi)\}$ of eigenvalues, the infimum of Π is zero, the principle eigenvalue, $\mu_1 > 0$, all eigenvalues are non-negative and the set of all the eigenvalues of 1.1 is open.

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