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CHARACTERIZATION OF SOME SUBLOOPS OF A BASARAB LOOP

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Abstract. A loop $(Q,\cdot)$ is called a Basarab loop if the identities: $(x\cdot yx^\rho)\cdot(xz)=x\cdot yz$ and $(yx)\cdot(x^\lambda z\cdot x)=yz\cdot x$ hold. A Basarab loop was shown to be a group if and only if it is a cross inverse property loop or its middle inner mapping is automorphic in action. Some algebraic properties of the left, right and middle inner mappings of a Basarab loop were explored. The algebraic properties of some bi-variate functions defined on a Basarab loop Q were established and these were used to explore the properties of some important mono-variate functions defined on Q. In addition, these bi-variate functions aided the establishment of fine-tuned necessary and sufficient conditions for a loop to be a left (right) Basarab loop, and Basarab loop in relations to left (right) CC-loop, and CC-loop, respectively. Some subloops of a Basarab loop were shown to be characterized by the mono-variate functions. New characterizations of the centrum of a Basarab loop were obtained.

1. Introduction

. Let G be a non-empty set. Define a binary operation $(\cdot)$ on G. If $x\cdot y\in G$ for all $x,y\in G$, then the pair $(G,\cdot)$ is called a *groupoid* or *Magma*.

If each of the equations:

$$a\cdot x=b \qquad \text{and} \qquad y\cdot a=b$$

has unique solutions in G for x and y respectively, then $(G,\cdot)$ is called a *quasigroup*.

If there exists a unique element $e\in G$ called the *identity element* such that for all $x\in G$, $x\cdot e=e\cdot x=x$, $(G,\cdot)$ is called a *loop*. We write xy instead of $x\cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x\cdot yz$ stands for $x(yz)$.

In a loop $(G,\cdot)$ with identity element e, the *left inverse element* of $x\in G$ is the element $xJ_\lambda=x^\lambda\in G$ such that

$$x^\lambda\cdot x=e$$

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while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \cdot x^\rho = e.$$

If $x^\lambda = x^\rho$ for any $x \in G$, then we simply write $x^\lambda = x^\rho = x^{-1}$ or $J_\lambda = J_\rho = J$. Let x be a fixed element in a groupoid $(G, \cdot)$. The left and right translation maps of G, L_x and R_x respectively can be defined by

$$yL_x = x \cdot y \qquad \text{and} \qquad yR_x = y \cdot x.$$

It can now be seen that a groupoid $(G, \cdot)$ is a quasigroup if its left and right translation mappings are bijections or permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist. Let

$$x\backslash y = yL_x^{-1} = xM_y \qquad \text{and} \qquad x/y = xR_y^{-1} = yM_x^{-1}$$

and note that

$$x\backslash y = z \Leftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Leftrightarrow z \cdot y = x.$$

Let a, b and c be three elements of a loop G. The loop associator of a, b and c is the unique element (a, b, c) of G which satisfies $(ab)c = \{a(bc)\}(a, b, c)$. The loop commutator of a and b is the unique element (a, b) of G which satisfies $(ab) = (ba)(a, b)$.

The right nucleus of G is defined by $N_\rho(G, \cdot) = \{a \in G \mid xy \cdot a = x \cdot ya \ \forall\ x, y \in G\}$. The left nucleus of G is defined by $N_\lambda(G, \cdot) = \{a \in G : ax \cdot y = a \cdot xy \ \forall\ x, y \in G\}$. The middle nucleus of G is defined by $N_\mu(G, \cdot) = \{a \in G : ya \cdot x = y \cdot ax \ \forall\ x, y \in G\}$. The nucleus of G is defined by $N(G, \cdot) = N_\lambda(G, \cdot) \cap N_\rho(G, \cdot) \cap N_\mu(G, \cdot)$. The centrum of G is defined by $C(G, \cdot) = \{a \in G : ax = xa \ \forall\ x \in G\}$. The center of G is defined by $Z(G, \cdot) = N(G, \cdot) \cap C(G, \cdot)$. $N_\rho(G, \cdot), N_\lambda(G, \cdot), N_\mu(G, \cdot), N(G, \cdot), Z(G, \cdot)$ are subgroups of $(G, \cdot)$.

A loop is called an automorphic inverse property loop (AIPL) if it obeys the automorphic inverse property

$$(xy)^\rho = x^\rho y^\rho \ \textit{or}\ (xy)^\lambda = x^\lambda y^\lambda.$$

A loop is called an antiautomorphic inverse property loop (AIPL) if it obeys the antiautomorphic inverse property

$$(xy)^\rho = y^\rho x^\rho \ \textit{or}\ (xy)^\lambda = y^\lambda x^\lambda.$$

A loop is a cross inverse property loop (CIPL) if it obeys the cross inverse property

$$xy \cdot x^\rho = y \qquad \text{or} \qquad x \cdot yx^\rho = y \qquad \text{or} \qquad x^\lambda \cdot (yx) = y \qquad \text{or} \qquad x^\lambda y \cdot x = y.$$

. The group of all permutations on G is called the permutation group of G and denoted by $SYM(G)$. The group $M(G, \cdot) = \Big\langle \{R_x, L_x, \ : \ x \in G\} \Big\rangle$ is called the multiplication group of $(G, \cdot)$ and $M(G, \cdot) \leq SYM(G)$.

If $e\alpha = e$ in a loop G such that $\alpha \in \mathcal{M}(G)$, then α is called an inner mapping and they form a group $Inn(G)$ called the inner mapping group. The right, left and middle inner mappings

$$R_{(x,y)} = R_x R_y R_{xy}^{-1},\ L_{(x,y)} = L_x L_y L_{yx}^{-1},\ T_x = R_x L_x^{-1} \text{ and } T_{(x,y)} = T_x T_y T_{xy}^{-1}$$

respectively generate the left inner mapping group $\mathrm{Inn}_\lambda(G)$, right inner mapping group $\mathrm{Inn}_\rho(G)$ and the middle inner mapping group $\mathrm{Inn}_\mu(G)$. The triple (A, B, C) of bijections of a loop $(G, \cdot)$ is called an autotopism if

$$xA \cdot yB = (x \cdot y)C \ \forall \ x, y \in G. \tag{1.1}$$

Such triples form a group $AUT(G, \cdot)$ called the autotopism group of $(G, \cdot)$. Furthermore, if $A = B = C$, then A is called an automorphism of $(G, \cdot)$. Such bijections form a group $AUM(G, \cdot)$ called the automorphism group of $(G, \cdot)$. If

$$\mathrm{Inn}_\lambda(G) \leq AUM(G),\ \mathrm{Inn}_\rho(G) \leq AUM(G),\ \text{and}\ \mathrm{Inn}_\mu(G) \leq AUM(G),$$

then G is called a left A-loop (A_λ-loop), right A-loop (A_ρ-loop)and middle A-loop (A_μ-loop) respectively. If $\mathrm{Inn}(G, \cdot) \leq AUM(G, \cdot)$, then $(G, \cdot)$ is called an automorphic loop (A-loop)

Definition 1.1. Let G be a non-empty set and let $A, B, C \in SYM(G)$.
(1) $[A, B]$ is called the commutator of A and B if $AB = BA[A, B]$.
(2) The left conjugate of B by A is represented by ${}^AB$ and defined by ${}^AB = A^{-1}BA$.
(3) The right conjugate of B by A is represented by B^A and defined by $B^A = ABA^{-1}$.

For an overview of the theory of loops, readers may check [6, 7, 8, 9, 10, 12, 17, 18, 19].

Definition 1.2. A loop $(Q, \cdot)$ is called a Basarab loop (or K-loop), if the identities:

$$(x \cdot yx^\rho) \cdot xz = x \cdot yz, \quad yx \cdot (x^\lambda z \cdot x) = yz \cdot x \tag{1.2}$$

hold for all $x, y, z \in Q$.

. Some early results on the class of loop called Basarab loop are seen in the two prominent papers of Basarab [1, 2]. Basarab [4] used the result published by Belousov [5] to construct an example of a Basarab loop whose nucleus is an abelian group. In Basarab [3, 4], the author studied the relationship between a generalized Moufang loop, Osborn loop, VD-loop, and a Basarab loop, and a special type of a Basarab loop, known as IK-loop respectively. In this current study, an IK-loop is an automorphic inverse property Basarab loop. Jaiyéọlá and Effiong [13] investigated Basarab loop and its variance with inverse properties, and established some necessary and sufficient conditions for a Basarab loop to be of exponent 2 or a centrum square. In Jaiyéọlá and Effiong [14], it was shown that a Basarab loop is a totally automorphic loop if and only if it is a commutative and flexible loop.

Here are some other existing results on Basarab loop.

Theorem 1.1. (Basarab [4]) Let $(Q, \cdot)$ be a Basarab loop.
(1) $N(Q, \cdot)$ contains the associator of any three elements of Q.
(2) The quotient loop $Q/Z(Q, \cdot)$ is a group.

Theorem 1.2. (Basarab [2]) Let $(Q, \cdot)$ be a Basarab loop.
(1) $N(Q, \cdot)$ is a nontrivial normal subloop.

(2) The quotient loop $Q/N(Q,\cdot)$ is an abelian.

Theorem 1.3. (Basarab [3])

(1) Any Basarab loop (any VD-loop) is a G-loop.
(2) Any Basarab loop (VD-loop) is an Osborn loop.

Theorem 1.4. (Effiong et al. [11])

(1) A loop $(Q,\cdot)$ is a left Basarab loop if and only if it is an LCC-loop and satisfies
$xy = (x \cdot yx^{\rho})x$, for all $x, y \in Q$.
(2) A loop $(Q,\cdot)$ is a right Basarab loop if and only if it is an RCC-loop and satisfies $yx = x(x^{\lambda}y \cdot x)$, for all $x, y \in Q$.
(3) A loop $(Q,\cdot)$ is a Basarab loop if and only if it is a CC-loop and satisfies $xy = (x \cdot yx^{\rho})x,\ yx = x(x^{\lambda}y \cdot x)$, for all $x, y \in Q$.
(4) $(Q,\cdot)$ is a Basarab loop if and only if the following are true for all $x, y, z \in Q$:
   (a): $L_xR_{xz} = R^{-1}_{x^{\rho}}R_zL_x$.
   (b): $R_xL_{yx} = L^{-1}_{x^{\lambda}}L_yR_x$.

Motivation and Objective. In Kinyon and Kunen [16], the authors introduced some subsets of a CC-loop Q: set of weak inverse property elements ($W(Q)$); set of pseudo-Moufang elements ($P(Q)$); set of nuclear squared elements ($S(Q)$); set of extra elements ($Ex(Q)$) and established that $W(Q), P(Q), S(Q), Ex(Q)$ (see Theorem 3.4, Theorem 4.6, Lemma 4.11, Theorem 4,12 respectively) are normal subloops of Q. If Q is a power associative CC-loop, it was found that $P(Q) = Ex(Q) \leq W(Q)$. In general case, they showed that a subset X of a CC-loop Q is a subloop if and only if X is closed under product and either J_{ρ} or J_{λ}. Motivated by these, the main objective of the current study is to identify some subloops of a Basarab loop (as special CC-loop) which are characterized by some important permutations.

2. Main Results

. In the first subsection of this section, algebraic properties of some bi-variate functions defined on a Basarab loop Q will be established and these will then be used to explore the properties of some important mono-variate functions defined on Q. The subsequent subsections explore some subloops of a Basarab loop which will be shown to be characterized by the mono-variate functions.

2.1. Properties of Some functions in a Basarab loop.

Lemma 2.1. A loop $(Q,\cdot)$ is a Basarab loop if and only if there exist functions $g, h : Q \times Q \longrightarrow Q$ defined by $g(x,\ y) = x \cdot yx^{\rho}$ and $h(x, z) = x^{\lambda}z \cdot x$ such that $L^{-1}_xL_yL_x = L_{g(x,y)}$ and $R^{-1}_xR_yR_x = R_{h(x,y)}$.

Proof. Let $(Q,\cdot)$ be a Basarab loop. Consider the Basarab identity $x\cdot y(xJ_{\rho})\cdot xz = x \cdot yz, \forall\ x, y, z \in Q$. We have that

$$zL_xL_{(x\cdot y(xJ_{\rho}))} = zL_yL_x \Rightarrow zL_xL_{(yR_{xJ_{\rho}}L_x)} = zL_yL_x \Rightarrow L_xL_{(yR_{xJ_{\rho}}L_x)} = L_yL_x$$

$$\Rightarrow L_xL_g(x,y) = L_yL_x \Rightarrow L_{g(x,y)} = L_x^{-1}L_yL_x \Rightarrow L_{(x\cdot yx^\rho)} = L_x^{-1}L_yL_x.$$

Also, consider the Basarab identity $(yx)\cdot((xJ_\lambda z)\cdot x) = yz\cdot x$. This means

$$yR_xR_{((xJ_\lambda z)\cdot x)} = yR_zR_x \Rightarrow R_xR_{(zL_{xJ_\lambda}R_x)} = R_zR_x$$

$$\Rightarrow R_xR_{h(x,z)} = R_zR_x \Rightarrow R_{h(x,z)} = R_x^{-1}R_zR_x.$$

Thus, for all $x,y\in Q,\ R_{h(x,y)} = R_x^{-1}R_yR_x$. □

Lemma 2.2. Let $(Q,\ \cdot)$ be a Basarab loop. Then there are functions $g,h : Q\times Q\longrightarrow Q$ such that any of the following holds for all $x,y,z\in Q$:

(1) $x\cdot yz = g(x,y)\cdot xz$.
(2) $zy\cdot x = zx\cdot h(x,y)$.
(3) $x = g(x,y)\cdot xy^\rho$: $CIP \Leftrightarrow g(x,y) = y \Leftrightarrow$ commmutative or abelian group.
(4) $x\cdot y = g(x,y)\cdot x$: Q is commutative $\Leftrightarrow g(x,y) = y$.
(5) $x = y^\lambda x\cdot h(x,y)$: $CIP \Leftrightarrow h(x,y) = y$.
(6) $yx = x\cdot h(x,y)$: Q is commutative $\Leftrightarrow h(x,y) = y$.
(7) $x\cdot yg(x,y) = g(x,y)\cdot xg(x,y)$.
(8) $h(x,z) = g(x,z) \Leftrightarrow xh(x,y)\cdot g(x,z) = yz\cdot x$.
(9) $g(x,y) = h(x,y) \Leftrightarrow h(x,y)\cdot (g(x,z)\cdot x) = x\cdot yz$.
(10) $xh(x,y)\cdot((xz)/x) = yz\cdot x \Leftrightarrow g(x,z) = h(x,z)$.
(11) $(x\backslash(yx))\cdot(g(x,z)\cdot x) = x\cdot yz \Leftrightarrow g(x,y) = h(x,y)$.

Proof. (1) From Lemma 2.1, a loop $(Q,\cdot)$ is a Basarab loop if and only if there are functions $g,h : Q^2 \longrightarrow Q$ so that $L_x^{-1}L_yL_x = L_{g(x,y)}$ and $R_x^{-1}R_yR_x = R_{h(x,y)}$ where $xy = (x\cdot yx^\rho)x$ and $yx = x(x^\lambda y\cdot x)$ hold for all $x,y\in Q$. Thus, $L_x^{-1}L_yL_x = L_{g(x,y)}$ implies that $L_yL_x = L_xL_{g(x,y)}$, so that for all $z\in Q,\ zL_yL_x = zL_xL_{g(x,y)}$ gives $x\cdot yz = g(x,y)\cdot xz$ for all $x,y,z\in Q$.

(2) Also, $R_x^{-1}R_yR_x = R_{h(x,y)}$ implies $R_yR_x = R_xR_{h(x,y)}$. It follows that for every $z\in Q,\ zR_yR_x = zR_xR_{h(x,y)}$ implies $zy\cdot x = zx\cdot h(x,y)$.

(3) Substitute $z = y^\rho$ in 1.
(4) Substitute $z = e$ in 1.
(5) Substitute $z = y^\lambda$ in 2.
(6) Substitute $z = e$ in 2.
(7) Substitute $z = g(x,y)$ in 1.

(8) From the Basarab identity $(yx)\cdot(x^\lambda z\cdot x) = yz\cdot x$, it follows that $yx\cdot h(x,z) = yz\cdot x$ for all $x,y,z\in Q$. Replace $h(x,z)$ with $g(x,z)$ and consider yx using 6, then it follows that $h(x,z) = g(x,z)$ if and only if $xh(x,y)\cdot g(x,z) = yz\cdot x$ for all $x,y,z\in Q$.

(9) From the Basarab identity $(x\cdot yx^\rho)\cdot xz = x\cdot yz,\ g(x,y)\cdot xz = x\cdot yz$ holds for all $x,y,z\in Q$. Then, replacing $g(x,y)$ with $h(x,y)$ and applying 4, it follows that $g(x,y) = h(x,y)$ if and only if $h(x,y)\cdot(g(x,z)\cdot x) = x\cdot yz$ for all $x,y,z\in Q$.

(10) From 4, $xz = g(x,z)\cdot x$ implies $(xz)/x = g(x,z)$ for all $x,z\in Q$. Putting $g(x,z) = (xz)/x$ in 8 gives

$$xh(x,y)\cdot((xz)/x) = yz\cdot x \Leftrightarrow g(x,z) = h(x,z).$$

(11) Using 6, $yx = x \cdot h(x, y)$ implies $h(x, y) = x\backslash(yx)$ for every $x, y \in Q$. Then 9 becomes

$$(x\backslash(yx)) \cdot g(x, z) \cdot x = x \cdot yz \Leftrightarrow g(x, y) = h(x, y).$$

□

Lemma 2.3. Let $(Q,\ \cdot)$ be a Basarab loop. Then, there are functions $g, h, m, n : Q \times Q \longrightarrow Q$ defined by $h(x, y) = x^{\lambda} y \cdot x,\ g(x, y) = x \cdot xy^{\rho},\ m_z(x, y) = (zx)\backslash(zy \cdot x),\ n_z(x, y) = (x \cdot yz)/(xz)$ such that any of the following holds for all $x, y, z \in Q$:

(1) $h(x, z) = g(x, z) \Leftrightarrow yx \cdot ((xz)/x) = yz \cdot x \Leftrightarrow g(x, z) = m_y(x, z)$.
(2) $h(x, y) = g(x, y) \Leftrightarrow (x\backslash(yx)) \cdot xz = x \cdot yz \Leftrightarrow h(x, y) = n_z(x, y)$.
(3) If $h(x, z) = g(x, z)$, then $x \cdot ((xz)/x) = zx$.
(4) If $h(x, y) = g(x, y)$, $(x\backslash(yx)) \cdot x = xy$.
(5) If $h(x, z) = g(x, z)$, then $(xz)/x = x^{\lambda} z \cdot x$.
(6) If $h(x, y) = g(x, y)$, then $x\backslash(yx) = x \cdot yx^{\rho}$.
(7) If $h(x, y) = g(x, y)$, then $(xy)/x = x\backslash(yx)$ or $T_x = T_x^{-1}$ or $|T_x| = 2$.
(8) If $h(x, z) = g(x, z)$, then $x\backslash(zx) = x^{\lambda} z \cdot x$.
(9) If $h(x, y) = g(x, y)$, then $(xy)/x = x \cdot yx^{\rho}$.

Proof. (1) From 10 of Lemma 2.2,

$$h(x, z) = g(x, z) \Leftrightarrow xh(x, y) \cdot ((xz)/x) = yz \cdot x.$$

Now, $yx \cdot ((xz)/x) = yz \cdot x$ for all $x, y, z \in Q$ by 6 of Lemma 2.2. So, with

$$g(x, z) = ((xz)/x) \text{ and } m_y(x, z) = yx\backslash(yz \cdot x),$$

$$\text{we have } yx \cdot ((xz)/x) = yz \cdot x \Leftrightarrow g(x, z) = m_y(x, z).$$

(2) Applying 11 of Lemma 2.2,

$$g(x, y) = h(x, y) \Leftrightarrow (x\backslash(yx)) \cdot (g(x, z) \cdot x) = x \cdot yz.$$

Now, $(x\backslash(yx)) \cdot xz = x \cdot yz$ by 6 of Lemma 2.2, for all $x, y, z \in Q$. So, with $h(x, y) = (x\backslash(yx))$ and $n_z(x, y) = (x \cdot yz)/xz$. Then, we have

$$(x\backslash(yx)) \cdot xz = x \cdot yz \Leftrightarrow h(x, y) = n_z(x, y).$$

(3) With $h = g$, set $y = e$ in 1, then $x \cdot ((xz)/x) = zx$.
(4) With $h = g$, set $z = e$ in 2, then $(x\backslash(yx)) \cdot x = xy$.
(5) With $h = g$, set $y = x^{\lambda}$ in 1, then $(xz)/x = x^{\lambda} z \cdot x$.
(6) With $h = g$, set $z = x^{\rho}$ in 2, then $x\backslash(yx) = x \cdot yx^{\rho}$.
(7) From 3 or 4, $zL_xR_x^{-1}L_x = zR_x \Rightarrow L_xR_x^{-1} = R_xL_x^{-1} \Rightarrow yL_xR_x^{-1} = yR_xL_x^{-1}$ for all $x, y \in Q$. Then $(xy)/x = x\backslash(yx)$ for all $x, y \in Q$.
(8) Using 7 in 5, then $x\backslash(zx) = x^{\lambda} z \cdot x$.
(9) Using 7 in 6, then $(xy)/x = x \cdot yx^{\rho}$.

□

Lemma 2.4. Let $(Q, \cdot)$ be a Basarab loop. There are mappings $G, H : Q \longrightarrow SYM(Q)$ such that the following hold: $T_x^{-1} = G_x = J_\rho L_x M_x^{-1} = R_{x^\rho} L_x = L_x R_x^{-1}$ and $T_x = H_x = J_\lambda R_x M_x = L_{x^\lambda} R_x = R_x L_x^{-1}$ for all $x \in Q$.

Proof. From 3 of Lemma 2.2,

$$x = g(x,y)\cdot xy^\rho \Rightarrow g(x,y) = x/(xy^\rho) \Rightarrow g(x,y) = (xy^\rho)M_x^{-1}$$

$$\Rightarrow g(x,y) = y^\rho L_x M_x^{-1} \Rightarrow g(x,y) = yJ_\rho L_x M_x^{-1} \Rightarrow x{\cdot}yx^\rho = yJ_\rho L_x M_x^{-1} \text{ (by Lemma 2.1)}$$

$$\Rightarrow yR_{x^\rho}L_x = yJ_\rho L_x M_x^{-1} \Rightarrow R_{x^\rho}L_x = J_\rho L_x M_x^{-1} := G_x.$$

Also, from 5 of Lemma 2.2,

$$x = y^\lambda x \cdot h(x,y) \Rightarrow h(x,y) = (y^\lambda x)\backslash x \Rightarrow h(x,y) = (y^\lambda x)M_x$$

$$\Rightarrow h(x,y) = y^\lambda R_x M_x \Rightarrow h(x,y) = yJ_\lambda R_x M_x \Rightarrow x^\lambda y{\cdot}x = yJ_\lambda R_x M_x \text{ (by Lemma 2.1)}$$

$$\Rightarrow yL_{x^\lambda}R_x = yJ_\lambda R_x M_x \Rightarrow L_{x^\lambda}R_x = J_\lambda R_x M_x := H_x.$$

□

Lemma 2.5. Let $(Q,\cdot)$ be a Basarab loop. Let $W_x = R_xM_x$, and $U'_x = L_xM_x^{-1}$. Then any two of the following implies the third:

(1) $J_\lambda = J_\rho$ (2) $G_x = H_x$ (3) $U'_x = W_x$.

Proof. Let Q be a Basarab loop. Consider $G_x = J_\rho L_x M_x^{-1}$, $H_x = J_\lambda R_x M_x$, $W_x = R_xM_x$ and $U'_x = L_xM_x^{-1}$.

(i) $J_\rho = J_\lambda \Leftrightarrow G_xM_xL_x^{-1} = H_xM_x^{-1}R_x^{-1} \Leftrightarrow G_x(L_xM_x^{-1})^{-1} = H_x(R_xM_x)^{-1} \Leftrightarrow G_xU_x'^{-1} = H_xW_x^{-1} \Leftrightarrow I = H_xW_x^{-1}U'_xG_x^{-1} \Leftrightarrow W_xH_x^{-1} = U'_xG_x^{-1} \Leftrightarrow W_xG_x = U'_xH_x$. Thus, $J_\rho = J_\lambda$ and $G_x = H_x$ implies $U'_x = W_x$. Also, $J_\rho = J_\lambda$ and $U'_x = W_x$ implies $G_x = H_x$.

(ii) $G_x = H_x \Leftrightarrow J_\rho L_xM_x^{-1} = J_\lambda R_xM_x \Leftrightarrow J_\rho U'_x = J_\lambda W_x$. It follows that, $G_x = H_x$ and $J_\rho = J_\lambda$ implies $U'_x = W_x$. Also, $G_x = H_x$ and $U'_x = W_x$ implies $J_\rho = J_\lambda$.

(iii) $U'_x = W_x \Leftrightarrow R_xM_x = L_xM_x^{-1} \Leftrightarrow J_\rho H_x = J_\lambda G_x$. Thus, U'_x and $J_\rho = J_\lambda$ implies $H_x = G_x$. Also, $U'_x = W_x$ and $H_x = G_x$ implies $J_\rho = J_\lambda$. □

Lemma 2.6. Let $(Q,\cdot)$ be a Basarab loop. There are functions $g,h : Q\times Q \longrightarrow Q$ such that the following hold: $h(x,y) = (y^\lambda x)\backslash x = x^\lambda y\cdot x = x\backslash(yx)$ and $g(x,y) = x/(xy^\rho) = x\cdot yx^\rho = (xy)/x$ for all $x,y\in Q$.

Proof. The proof follows from Lemma 2.1. □

Corollary 2.1. A loop is right Basarab if and only if it is RCC and $h(x,y) = m_z(x,y)$.

Proof. This holds by Theorem 1.4(2) and Lemma 2.3. □

Corollary 2.2. A loop is left Basarab if and only if it is LCC and $g(x,y) = n_z(x,y)$.

Proof. This holds by Theorem 1.4(1) and Lemma 2.3. □

Corollary 2.3. A loop $(Q,\cdot)$ is a Basarab loop if and only if it is a CC-loop and $g(x,y) = n_z(x,y)$ and $h(x,y) = m_z(x,y)$.

Proof. This is true by Theorem 1.4(3) and Lemma 2.3. □

Corollary 2.4. Let $(Q,\cdot)$ be a CC-loop. Then
$L_x^{-1}L_yL_x = L_{(xy)/x} = L_{n_e(x,y)}$ and $R_x^{-1}R_yR_x = R_{x\backslash(yx)} = R_{m_e(x,y)}$.

Proof. By CC-loop translations and Lemma 2.3 the result follows. □

Theorem 2.1. Let $(Q,\cdot)$ be a Basarab loop.

(1) $(Q,\cdot)$ is associative if and only if $T_{xy}=T_xT_y$.
(2) $|L_{(x,y)}|=2$ if and only if $|R_{(x,y)}|=2$ if and only if $|T_{(x,y)}|=2$ for all $x,y\in Q$.
(3) $R_{(x,y)}=L_{(y,x)}$ if and only if $(T_xT_y)^2=T_{xy}^2$. Hence, $(T_xT_y)^{2n}=T_{xy}^{2n}$.
(4) Q is a CIPL if and only if Q is commutative or abelian group if and only if $g(x,y)=y$ if and only if $h(x,y)=y$.
(5) Right and left distributive laws with respect to $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$: (i) $h(x,z)=g(x,z)$ if and only if $yz{\cdot}x=xh(x,y){\cdot}g(x,z)$ (ii) $h(x,z)=g(x,z)$ if and only if $x\cdot yz=h(x,y)\cdot(g(x,z)\cdot x)$.
(6) $h(x,y)=g(x,y)$ or $m_e(x,y)=n_e(x,y)$ implies $|T_x|=2$.
(7) $T_xL_{(x,y)}=L_{(x,y)}T_x$ if and only if $L_{(x,y)}^{T_x}=L_{(x,y)}$ if and only if $T_x^{T_yT_x}=T_x^{T_{yx}}\Leftrightarrow[L_{(x,y)},T_x]=I$ if and only if $T_x^{T_y}=T_x^{T_{yx}}$ if and only if ${}^{T_x}T_y^{-1}T_{yx}=T_y^{-1}T_{yx}$. Hence, (i) $[T_x,T_y]=I\Leftrightarrow[T_x,T_{xy}]=I$ (ii) $[T_x,T_y]=I\Leftrightarrow[T_x,T_{yx}]=I$.
(8) Let $T_xL_{(y,x)}=L_{(y,x)}T_x$. Then: (i) $[T_x,T_{xy}]=I$ if and only if $[T_x,T_y^{-1}]=I$ if and only if $L_{(y,x)}=L_{(x,y)}$ (ii) $[T_x,T_y]=I\Leftrightarrow[T_x,T_{xy}]=I$.
(9) $T_xR_{(x,y)}=R_{(x,y)}T_x$ if and only if $R_{(x,y)}={}^{T_x}R_{(x,y)}\Leftrightarrow[R_{(x,y)},T_x]=I$ if and only if $T_yT_{xy}^{-1}{}^{T_x}=T_yT_{xy}^{-1}\Leftrightarrow{}^{T_y}T_x={}^{T_{xy}}T_x$. Hence, $|T_x|=I\Leftrightarrow{}^{T_y}T_x^{-1}={}^{T_{xy}}T_x$ and $[T_x,T_y]=I\Leftrightarrow[T_x,T_{xy}]=I$.
(10) $T_xR_{(y,x)}=R_{(y,x)}T_x$ if and only if $R_{(y,x)}={}^{T_x}R_{(y,x)}\Leftrightarrow{}^{T_y}T_xT_{yx}^{-1}{}^{T_x}=T_xT_{yx}^{-1}$. Hence, $[T_x,T_y]=I$ if and only if $[T_x,T_{yx}^{-1}]=I$.
(11) $T_xT_yT_{xy}^{-1}=T_yT_xT_{yx}^{-1}$, $T_xT_y=T_yT_x$ if and only if $T_{xy}=T_{yx}$ if and only if $[T_x,T_y]=I$.

Proof. Let $(Q,\cdot)$ be a Basarab loop, then by [11],

$$L_{(x,y)}=T_x^{-1}T_y^{-1}T_{yx},\ R_{(x,y)}=T_xT_yT_{xy}^{-1}.$$

(1) Q is associative if and only if

$$L_{(x,y)}=I\Leftrightarrow R_{(x,y)}=I\Leftrightarrow T_{xy}=T_xT_y.$$

(2)

$$|L_{(x,y)}|=2\Leftrightarrow L^2_{(x,y)}=I\Leftrightarrow T_x^{-1}T_y^{-1}T_{yx}=T_{yx}^{-1}T_yT_x\Leftrightarrow(T_yT_x)^{-1}T_{yx}=T_{yx}^{-1}T_yT_x$$
$$\Leftrightarrow T_{yx}=T_yT_xT_{yx}^{-1}T_yT_x\Leftrightarrow T_yT_xT_{yx}^{-1}T_xT_{yx}^{-1}=I\Leftrightarrow|T_yT_xT_{yx}^{-1}|=2\Leftrightarrow|T_{(x,y)}|=2.$$

$$|R_{(x,y)}|=2\Leftrightarrow R^2_{(x,y)}=I\Leftrightarrow T_xT_yT_{xy}^{-1}=T_{xy}T_y^{-1}T_x^{-1}\Leftrightarrow T_xT_yT_{xy}^{-1}T_xT_y=T_{xy}$$
$$\Leftrightarrow T_xT_yTxy^{-1}T_xT_yT_{xy}^{-1}=I\Leftrightarrow|T_xTyT_{xy}^{-1}|=2\Leftrightarrow|T_{(x,y)}|=2.$$

So, $|L_{(x,y)}|=2$ if and only if $|R_{(y,x)}|=2$ if and only if $|T_{(x,y)}|=2$.

(3) Furthermore,

$$R_{(x,y)}=L_{(y,x)}\Leftrightarrow T_xT_yT_{xy}^{-1}=T_y^{-1}T_x^{-1}T_{xy}\Leftrightarrow T_xT_y=T_y^{-1}T_x^{-1}T_{xy}^2$$
$$\Leftrightarrow T_xT_yT_xT_y=T_{xy}^{-2}\Leftrightarrow(T_xT_y)^2=T_{xy}^2.$$

So, $R_{(x,y)} = L_{(y,x)} \Leftrightarrow (T_xT_y)^2 = T_{xy}^2$. The remaining part of the proof follows from Lemma 2.2 and Lemma 2.3.

(4) Use 3,4,5,6 of Lemma 2.2.
(5) Use 8,9,10,11 of Lemma 2.2.
(6) Use 3 or 7 of Lemma 2.3.
(7) $T_xL_{(x,y)} = L_{(x,y)}T_x \Leftrightarrow T_x(T_yT_x)^{-1}T_{yx} = (T_yT_x)^{-1}T_{yx}T_x \Leftrightarrow (T_yT_x)T_x(T_yT_x)^{-1} = T_{yx}T_xT_{yx}^{-1} \Leftrightarrow T_x^{T_yT_x} = T_x^{T_{yx}}$. Let $T_xL_{(x,y)} = L_{(x,y)}T_x$, then $[T_x, T_{yx}] = I \Leftrightarrow T_x = T_{yx}T_xT_{xy}^{-1} \Leftrightarrow T_xT_{xy} = T_{yx}T_x \Leftrightarrow [T_x, T_{xy}] = I$. So, $[T_x, T_{yx}] = I \Leftrightarrow T_yT_xT_y^{-1} = T_x \Leftrightarrow T_yT_x = T_xT_y \Leftrightarrow [T_y, T_x] = I$. Hence, $T_xL_{(y,x)} = L_{(y,x)}T_x \Leftrightarrow L_{(y,x)} = {}^{T_x}L_{(y,x)} \Leftrightarrow [T_x, T_{xy}] = I$.
(8) $T_xL_{(y,x)} = L_{(y,x)}T_x \Leftrightarrow T_xT_y^{-1}T_x^{-1}T_{xy} = T_y^{-1}T_x^{-1}T_{xy}T_x \Leftrightarrow T_x(T_xT_y)^{-1}T_{xy} = (T_xT_y)^{-1}T_{xy}T_x \Leftrightarrow (T_xT_y)T_x(T_xT_y)^{-1} = T_{xy}T_xT_{xy}^{-1} \Leftrightarrow T_x^{T_xT_y} = T_x^{T_{xy}}$. Also, $T_xT_y^{-1}T_x^{-1} = T_y^{-1}T_x^{-1}T_{xy}T_xT_{xy}^{-1} \Leftrightarrow T_yT_xT_y^{-1}T_x^{-1} = T_x^{-1}T_{xy}T_xT_y^{-1} \Leftrightarrow T_yT_xT_y^{-1} = T_x^{-1}T_{xy}T_x(T_x^{-1}T_{xy})^{-1} \Leftrightarrow T_x^{T_y} = T_x^{T_x^{-1}T_{xy}}$.
(9) $T_xR_{(x,y)} = R_{y,x}T_x \Leftrightarrow T_xT_xT_yT_{xy}^{-1} = T_xT_yT_{xy}^{-1}T_x \Leftrightarrow T_xT_yT_{xy}^{-1} = T_yT_{xy}^{-1}T_x \Leftrightarrow T_y^{-1}T_xT_yT_{xy}^{-1} = T_{xy}^{-1}T_x \Leftrightarrow T_y^{-1}T_xT_y = T_{xy}^{-1}T_xT_{xy} \Leftrightarrow^{T_y} T_x^{=\ T_{xy}}T_x$. Hence, (i) $|T_x| = 2 \Leftrightarrow T_yT_{xy}^{-1} = T_xT_yT_{xy}^{-1}T_x \Leftrightarrow T_y = T_xT_yT_{xy}^{-1}T_xT_{xy} \Leftrightarrow T_y^{-1}T_x^{-1}T_y = T_{xy}^{-1}T_xT_{xy} \Leftrightarrow {}^{T_y}T_x^{-1} = {}^{T_{xy}}T_x$. (ii) $[T_x, T_y] = I \Leftrightarrow T_yT_xT_{xy}^{-1} = T_yT_{xy}^{-1}T_x \Leftrightarrow T_xT_{xy}^{-1} = T_{xy}^{-1}T_x \Leftrightarrow T_{xy}T_x = T_xT_{xy} \Leftrightarrow [T_x, T_{xy}] = I$.
(10) $T_xR_{(y,x)} = R_{(y,x)}T_x \Leftrightarrow T_xT_yT_xT_{yx}^{-1} = T_yT_xT_{yx}^{-1}T_x \Leftrightarrow T_y^{-1}T_xT_yT_xT_{yx}^{-1}T_x^{-1} = T_xT_{yx}^{-1} \Leftrightarrow {}^{T_y}T_xT_{yx}^{-1\,T_x} = T_xT_{yx}^{-1}$. So, $[T_x, T_y] = I \Leftrightarrow T_yT_{yx}^{-1}T_x = T_yT_xT_{yx}^{-1} \Leftrightarrow T_{yx}^{-1}T_x = T_xT_{yx}^{-1} \Leftrightarrow [T_x, T_{yx}^{-1}] = I$. Hence, $[T_x, T_y] = I$ if and only if $[T_x, T_{yx}^{-1}] = I$.
(11) A Basarab loop is a CCL. In a CCL, $R_{(x,y)} = R_{(y,x)}$. So, $T_xT_yT_{xy}^{-1} = T_yT_xT_{yx}^{-1}$.

□

Remark 2.1. In an extra loop, the result in Theorem 2.1(2) is naturally true. (see [15])

2.2. **Characterization of a Subloop of a Basarab loop by Middle Inner Mapping.** Let $(Q,\cdot)$ be a Basarab loop and let $Q' := \{x \in Q : G_x = H_x\} = \{x \in Q : T_x = T_x^{-1}\} = \{x \in Q : |T_x| = 2\}$ and

$$Q'_* := \{x \in Q : U'_x = W_x,\ U'_x = L_xM_x^{-1},\ W_x = R_xM_x\}.$$

From Lemma 2.4,

$$G_x = J_\rho L_xM_x^{-1} = R_{x^\rho}L_x = L_xR_x^{-1} \tag{2.1}$$

$$\text{and}\quad H_x = J_\lambda R_xM_x = L_{x^\lambda}R_x = R_xL_x^{-1}. \tag{2.2}$$

Theorem 2.2. Let $(Q,\cdot)$ be a Basarab loop. Then Q' is a subloop of Q, if
(1) $R_{(x,y)} = L_{(y,x)}$ or $(T_xT_y)^2 = T_{xy}^2$ for all $x,y \in Q'$ and
(2) $[T_x, T_y] = I$ for all $x,y \in Q'$.

Proof. $Q' \neq \emptyset$ because $T_e^2 = I \Rightarrow T_e = T_e^{-1} \Rightarrow e \in Q'$. Let $x,y \in Q'$, then $T_x^2 = T_y^2 = I$, and $T_{xy}^2 = (T_xT_y)^2 = T_xT_yT_xT_y = T_x^2T_y^2 = II = I$. So, $xy \in Q'$. Therefore, Q' is a subloop of Q. □

Theorem 2.3. Let $(Q,\cdot)$ be a Basarab loop. Then for every $x\in Q$, x is contained in Q' or $|T_x|=2$ if and only if any of the following is true:

(1) $L_xM_x^{-1}L_{yx}=J_\lambda L_yR_x$
(2) $M_x^{-1}R_x^{-1}=L_x^{-1}J_\lambda L_{x^\lambda}$ or $W_x^{-1}=L_x^{-1}J_\lambda L_{x^\lambda}$
(3) $J_\rho L_xM_x^{-1}=L_yR_xL_{yx}^{-1}$
(4) $T_x^{-1}=J_\lambda T_xU_x$.
Hence,
(5) ${}^{R_x}L_{x^\lambda}={}^{T_x}L_{x^\lambda}$
(6) $L_{x^\lambda}J_\rho=M_x^{T_x}$.

Proof. Let $(Q,\cdot)$ be a Basarab loop. Using Lemma 2.4, we apply equations 2.1 and 2.2 as follows:

(1) For every $x\in Q,\ x\in Q'\Leftrightarrow J_\rho L_xM_x^{-1}=L_{x^\lambda}R_x$

$$\Leftrightarrow L_xM_x^{-1}=J_\rho^{-1}L_{x^\lambda}R_x\Leftrightarrow L_xM_x^{-1}=J_\lambda L_{x^\lambda}R_x.$$

Then, by Theorem 1.4(4), $L_xM_x^{-1}=J_\lambda L_{x^\lambda}R_x$

$$\Leftrightarrow L_xM_x^{-1}=J_\lambda L_yR_xL_{yx}^{-1}\Leftrightarrow L_xM_x^{-1}L_{yx}=J_\lambda L_yR_x,$$

for all $y\in Q$.

(2) For every $x\in Q,\ x\in Q'\Leftrightarrow J_\rho L_xM_x^{-1}=L_{x^\lambda}R_x$

$$\Leftrightarrow L_xM_x^{-1}=J_\lambda L_{x^\lambda}R_x\Leftrightarrow M_x^{-1}=L_x^{-1}J_\lambda L_{x^\lambda}R_x\Leftrightarrow M_x^{-1}R_x^{-1}=L_x^{-1}J_\lambda L_{x^\lambda}$$

$$\Leftrightarrow (R_xM_x)^{-1}=L_x^{-1}J_\lambda L_{x^\lambda}\Leftrightarrow W_x^{-1}=L_x^{-1}J_\lambda L_{x^\lambda}.$$

(3) For every $x\in Q,\ x\in Q'\Leftrightarrow J_\rho L_xM_x^{-1}=L_{x^\lambda}R_x$. Then, by Theorem 1.4(4),

$$J_\rho L_xM_x^{-1}=L_{x^\lambda}R_x\Leftrightarrow J_\rho L_xM_x^{-1}=L_yR_xL_{yx}^{-1}R_x^{-1}R_x$$

$$\Leftrightarrow J_\rho L_xM_x^{-1}=L_yR_xL_{yx}^{-1}.$$

(4) For every $x\in Q,\ x\in Q'\Leftrightarrow J_\rho L_xM_x^{-1}=R_xL_x^{-1}$

$$\Leftrightarrow L_x=J_\lambda T_xM_x\Leftrightarrow L_xR_x^{-1}=J_\lambda T_xM_xR_x^{-1}\Leftrightarrow T_x^{-1}=J_\lambda T_xU_x.$$

(5) For every $x\in Q,\ x\in Q'\Leftrightarrow J_\rho L_xM_x^{-1}=R_xL_x^{-1}$. Then, by Theorem 1.4(4),

$$J_\rho L_xM_x^{-1}=R_xL_x^{-1}\Leftrightarrow J_\rho L_xM_x^{-1}=L_{x^\lambda}^{-1}L_yR_xL_{yx}^{-1}L_x^{-1}$$

$$\Leftrightarrow J_\rho L_xM_x^{-1}L_x=L_{x^\lambda}^{-1}L_yR_xL_{yx}^{-1}\Leftrightarrow L_yR_xL_{yx}^{-1}L_x=L_{x^\lambda}^{-1}L_yR_xL_{yx}^{-1},\text{ by 3.}$$

Also, from proof of 1, $L_xM_x^{-1}=J_\lambda L_yR_xL_{yx}^{-1}$. So,

$$L_yR_xL_{yx}^{-1}L_x=L_{x^\lambda}^{-1}L_yR_xL_{yx}^{-1}\Leftrightarrow L_yR_xL_{yx}^{-1}L_x=L_{x^\lambda}^{-1}J_\rho L_xM_x^{-1}.$$

Now, from 3, $L_yR_xL_{yx}^{-1}=J_\rho L_xM_x^{-1}$. So, $J_\rho L_xM_x^{-1}L_x=L_{x^\lambda}^{-1}J_\rho L_xM_x^{-1}$. Since, $J_\rho L_xM_x^{-1}=T_x$ by proof of 4, it follows that

$$J_\rho L_xM_x^{-1}L_x=L_{x^\lambda}^{-1}J_\rho L_xM_x^{-1}\Leftrightarrow T_xL_x=L_{x^\lambda}^{-1}T_x.$$

By Lemma 2.4, $L_x=R_x^{-1}L_{x^\lambda}^{-1}R_x$. Then, $T_xL_x=L_{x^\lambda}^{-1}T_x\Leftrightarrow T_xR_x^{-1}L_{x^\lambda}^{-1}R_x=L_{x^\lambda}^{-1}T_x$

$$\Leftrightarrow R_x^{-1}L_{x^\lambda}^{-1}R_x=T_x^{-1}L_{x^\lambda}^{-1}T_x\Leftrightarrow {}^{R_x}L_{x^\lambda}={}^{T_x}L_{x^\lambda}.$$

(6) For every $x \in Q,\ x \in Q' \Leftrightarrow J_\rho L_x M_x^{-1} = J_\lambda R_x M_x$. It follows that

$$J_\rho L_x M_x^{-1} = J_\lambda R_x M_x \Leftrightarrow T_x = J_\lambda R_x M_x \Leftrightarrow L_y R_x L_{yx}^{-1} = J_\lambda R_x M_x$$

by proof of 3. Next, we apply Theorem 1.4(4) on $L_y R_x L_{yx}^{-1} = J_\lambda R_x M_x$ and get

$$L_{x^\lambda} R_x L_{yx} L_{yx}^{-1} = J_\lambda R_x M_x \Leftrightarrow L_{x^\lambda} R_x = J_\lambda R_x M_x \Leftrightarrow J_\rho L_{x^\lambda} R_x = R_x M_x.$$

Thus, by Lemma 2.4, $R_x = L_{x^\lambda}^{-1} R_x L_x^{-1}$. So, $J_\rho L_{x^\lambda} R_x = R_x M_x$

$$\Leftrightarrow J_\rho L_{x^\lambda} R_x = L_{x^\lambda}^{-1} R_x L_x^{-1} M_x \Leftrightarrow L_{x^\lambda} J_\rho = T_x M_x T_x^{-1} \Leftrightarrow L_{x^\lambda} J_\rho = M_x^{T_x}.$$

□

Theorem 2.4. Let $(Q, \cdot)$ be a Basarab loop. Then, for every $x \in Q$, x is contained in Q' or $|T_x| = 2$ if and only if any of the following is true:

(1) $W_x R_{xz} = J_\rho R_z L_x$
(2) $U_x T_x = R_x^{-1} J_\rho R_{x^\rho}$
(3) $(L_x M_x)^{-1} T_x^{-1} J_\rho = R_{xz}(R_z L_x)^{-1}$
(4) $W_x = J_\rho T_x^{-1}$

Hence,
(5) ${}^{L_x} R_{x^\rho} = R_{x^\rho}^{T_x}$
(6) ${}^{T_x} M_x = J_\rho R_{x^\rho}^{-1}$.

Proof. Let $(Q, \cdot)$ be a Basarab loop. Using Lemma 2.4, we apply equation 2.1 as follows:

(1) For every $x \in Q,\ x \in Q' \Leftrightarrow J_\lambda R_x M_x = R_{x^\rho} L_x$

$$\Leftrightarrow R_x M_x = J_\lambda^{-1} R_{x^\rho} L_x \Leftrightarrow R_x M_x = J_\rho R_{x^\rho} L_x.$$

Then, by Theorem 1.4(4),

$$R_x M_x = J_\rho R_{x^\rho} L_x \Leftrightarrow R_x M_x = J_\rho R_z L_x R_{xz}^{-1} \Leftrightarrow R_x M_x R_{xz} = J_\rho R_z L_x,$$

for all $x, z \in Q \Leftrightarrow W_x R_{xz} = J_\rho R_z L_x$.

(2) For every $x \in Q,\ x \in Q' \Leftrightarrow J_\lambda R_x M_x = R_{x^\rho} L_x$

$$\Leftrightarrow R_x M_x = J_\rho R_{x^\rho} L_x \Leftrightarrow M_x = R_x^{-1} J_\rho R_{x^\rho} L_x \Leftrightarrow M_x L_x^{-1} = R_x^{-1} J_\rho R_{x^\rho},$$

for all $x \in Q \Leftrightarrow M_x R_x^{-1} R_x L_x^{-1} = R_x^{-1} J_\rho R_{x^\rho} \Leftrightarrow U_x T_x = R_x^{-1} J_\rho R_{x^\rho}$.

(3) For every $x \in Q,\ x \in Q' \Leftrightarrow J_\lambda R_x M_x = R_{x^\rho} L_x$. Then, by Theorem 1.4(4),

$$R_{x^\rho} = R_z L_x R_{xz}^{-1} L_x^{-1}.$$

Thus, $J_\lambda R_x M_x = R_{x^\rho} L_x \Leftrightarrow J_\lambda R_x M_x = R_z L_x R_{xz}^{-1} L_x^{-1} L_x \Leftrightarrow J_\lambda R_x M_x = R_z L_x R_{xz}^{-1}$

$$\Leftrightarrow M_x^{-1} R_x^{-1} J_\rho = R_{xz} L_x^{-1} R_z^{-1} \Leftrightarrow M_x^{-1} L_x^{-1} L_x R_x^{-1} J_\rho = R_{xz} L_x^{-1} R_z^{-1}$$

$$\Leftrightarrow (L_x M_x)^{-1} (R_x L_x^{-1})^{-1} J_\rho = R_{xz} L_x^{-1} R_z^{-1}$$

$$\Leftrightarrow (L_x M_x)^{-1} T_x^{-1} J_\rho = R_{xz}(R_z L_x)^{-1}.$$

(4) For every $x \in Q,\ x \in Q' \Leftrightarrow J_\lambda R_x M_x = L_x R_x^{-1} \Leftrightarrow R_x M_x = J_\lambda^{-1} L_x R_x^{-1}$

$$\Leftrightarrow R_x M_x = J_\rho L_x R_x^{-1} \Leftrightarrow W_x = J_\rho T_x^{-1}.$$

(5) For every $x \in Q,\ x \in Q' \Leftrightarrow J_\lambda R_x M_x = L_x R_x^{-1}$. By Theorem 1.4(4),

$$L_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}.$$

$$\text{Thus, } J_\lambda R_x M_x = L_x R_x^{-1} \Leftrightarrow J_\lambda R_x M_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1} R_x^{-1}$$
$$\Leftrightarrow J_\lambda R_x M_x R_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}.$$

By 3, it follows that,

$$J_\lambda R_x M_x R_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1} \Leftrightarrow R_z L_x R_{xz}^{-1} R_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}.$$

From 1, $J_\lambda R_x M_x = R_z L_x R_{xz}^{-1}$. So, $R_z L_x R_{xz}^{-1} R_x = R_{x^\rho}^{-1} R_z L_x R_{xz}^{-1}$

$$\Leftrightarrow R_z L_x R_{xz}^{-1} R_x = R_{x^\rho}^{-1} J_\lambda R_x M_x.$$

Using 3, $R_z L_x R_{xz}^{-1} R_x = R_{x^\rho}^{-1} J_\lambda R_x M_x \Leftrightarrow J_\lambda R_x M_x R_x = R_{x^\rho}^{-1} J_\lambda R_x M_x$

$$\Leftrightarrow T_x^{-1} R_x = R_{x^\rho}^{-1} T_x^{-1}$$

since $J_\lambda R_x M_x = T_x^{-1}$ by Lemma 2.4. Then, $T_x^{-1} R_x = R_{x^\rho}^{-1} T_x^{-1} \Leftrightarrow T_x^{-1} L_x^{-1} R_{x^\rho}^{-1} L_x = R_{x^\rho}^{-1} T_x^{-1}$, by Lemma 2.4. Hence,

$$T_x^{-1} L_x^{-1} R_{x^\rho}^{-1} L_x = R_{x^\rho}^{-1} T_x^{-1} \Leftrightarrow L_x^{-1} R_{x^\rho}^{-1} L_x = T_x R_{x^\rho}^{-1} T_x^{-1} \Leftrightarrow {}^{L_x}R_{x^\rho} = R_{x^\rho}^{T_x}.$$

(6) For every $x \in Q,\ x \in Q' \Leftrightarrow J_\rho L_x M_x^{-1} = J_\lambda R_x M_x$. By Lemma 2.4, $J_\lambda R_x M_x = T_x^{-1}$. It follows that

$$J_\rho L_x M_x^{-1} = J_\lambda R_x M_x \Leftrightarrow J_\rho L_x M_x^{-1} = T_x^{-1} \Leftrightarrow J_\rho L_x M_x^{-1} = R_z L_x R_{xz}^{-1},$$

by using $T_x^{-1} = R_z L_x R_{xz}^{-1}$ in proof of 3. Going by Theorem 1.4(4), $R_{x^\rho} L_x R_{xz} = R_z L_x$. It follows that

$$J_\rho L_x M_x^{-1} = R_z L_x R_{xz}^{-1} \Leftrightarrow J_\rho L_x M_x^{-1} = R_{x^\rho} L_x R_{xz} R_{xz}^{-1} \Leftrightarrow J_\rho L_x M_x^{-1} = R_{x^\rho} L_x$$
$$\Leftrightarrow L_x M_x^{-1} = J_\lambda R_{x^\rho} L_x.$$

By Lemma 2.4, $L_x M_x^{-1} = J_\lambda R_{x^\rho} L_x \Leftrightarrow R_{x^\rho}^{-1} L_x R_x^{-1} M_x^{-1} = J_\lambda R_{x^\rho} L_x$

$$\Leftrightarrow R_{x^\rho}^{-1} T_x^{-1} M_x^{-1} = J_\lambda T_x^{-1} \Leftrightarrow M_x T_x R_{x^\rho} = T_x J_\rho \Leftrightarrow T_x^{-1} M_x T_x = J_\rho R_{x^\rho}^{-1} \Leftrightarrow {}^{T_x}M_x = J_\rho R_{x^\rho}^{-1}.$$

□

Theorem 2.5. Let $(Q, \cdot)$ be a Basarab loop. Then $Q'_* \neq \emptyset$ if and only if $J_\lambda = J_\rho$.

Proof. Given $Q'_* := \{x \in Q : U'_x = W_x,\ U'_x = L_x M_x^{-1},\ W_x = R_x M_x\}$,

$$Q'_* \neq \emptyset \Leftrightarrow e \in Q'_* \Leftrightarrow U'_e = W_e$$
$$\Leftrightarrow L_e M_e^{-1} = R_e M_e \Leftrightarrow M_e^{-1} = M_e \Leftrightarrow J_\lambda = J_\rho.$$

Therefore, Q'_* is a subloop of Q. □

Theorem 2.6. Let $(Q, \cdot)$ be a Basarab loop. Let $Q'_* \neq \emptyset$, then $x \in Q'_*$ if and only if any of the following is true:

(1) $W_x R_{xz} = J_\lambda R_z L_x$
(2) $J_\lambda G_x = W_x$ or $U'_x = J_\rho H_x$
(3) $M_x = R_x^{-1} J_\lambda T_x^{-1}$ or $W_x = J_\lambda T_x^{-1}$
(4) $J_\rho L_y R_x L_{yx}^{-1} = U'_x$
(5) $U'_x = J_\rho L_{x^\lambda} R_x$ or $W_x = J_\lambda^{-1} L_x R_x^{-1}$
(6) $M_x^{-1} = L_x^{-1} J_\rho T_x$
(7) $R_{x^\rho} = L_{x^\rho} T_x$
(8) $L_y R_x = J_\lambda L_x R_x^{-1} L_{yx} \Leftrightarrow T_x^{-1} = L_y R_x L_{yx}^{-1}$
(9) $R_x M_x L_x^{-1} = J_\rho R_{x^\rho} \Leftrightarrow W_x T_x = J_\lambda$.

Proof. Let $(Q,\cdot)$ be a Basarab loop, by applying the mappings $U'_x = L_xM_x^{-1}$ and $W_x = R_xM_x$ to Lemma 2.4, it follows that:
$G_x = J_\rho U'_x = R_{x^\rho}L_x = L_xR_x^{-1}$ and $H_x = J_\lambda W_x = L_{x^\lambda}R_x = R_xL_x^{-1}$. This implies that

$$J_\lambda G_x = U'_x = J_\lambda R_{x^\rho}L_x = J_\lambda L_xR_x^{-1} \tag{2.3}$$

$$\text{and } J_\rho H_x = W_x = J_\rho L_{x^\lambda}R_x = J_\rho R_xL_x^{-1}. \tag{2.4}$$

By equations (2.3) and (2.4), it follows that:

(1) For every $x \in Q,\ x \in Q'_* \Leftrightarrow J_\rho H_x = J_\lambda R_{x^\rho}L_x \Leftrightarrow J_\rho H_xL_x^{-1} = J_\lambda R_{x^\rho}$

$$\Leftrightarrow J_\rho J_\lambda R_xM_xL_x^{-1} = J_\lambda R_{x^\rho} \Leftrightarrow R_xM_xL_x^{-1} = J_\lambda R_{x^\rho}$$

$$\Leftrightarrow W_xL_x^{-1} = J_\lambda R_{x^\rho} \Leftrightarrow W_x = J_\lambda R_{x^\rho}L_x,$$

since $H_x = J_\lambda R_xM_x$. By Theorem 1.4(4), $R_{x^\rho}L_x = R_zL_xR_{xz}^{-1}$. It follows that, $x \in Q'_* \Leftrightarrow W_x = J_\lambda R_{x^\rho}L_x \Leftrightarrow W_x = J_\lambda R_zL_xR_{xz}^{-1} \Leftrightarrow W_xR_{xz} = J_\lambda R_zL_x$.

(2) For every $x \in Q,\ x \in Q'_* \Leftrightarrow J_\lambda G_x = J_\rho H_x \Leftrightarrow G_x = J_\lambda^{-1}J_\rho H_x$

$$\Leftrightarrow G_x = J_\rho^2 H_x \Leftrightarrow J_\rho L_xM_x^{-1} = J_\rho J_\rho H_x \Leftrightarrow U'_x = J_\rho H_x,$$

since $G_x = J_\rho L_xM_x^{-1}$. Also, for every $x \in Q,\ x \in Q'_* \Leftrightarrow J_\lambda G_x = J_\rho H_x \Leftrightarrow$

$$J_\lambda G_x = J_\rho J_\lambda R_xM_x \Leftrightarrow J_\lambda G_x = R_xM_x \Leftrightarrow J_\lambda G_x = W_x.$$

(3) For every $x \in Q,\ x \in Q'_* \Leftrightarrow J_\rho H_x = J_\lambda L_xR_x^{-1} \Leftrightarrow J_\rho H_xR_x = J_\lambda L_x$

$$\Leftrightarrow J_\rho J_\lambda R_xM_xR_x = J_\lambda L_x \Leftrightarrow R_xM_xR_x = J_\lambda L_x$$

$\Leftrightarrow W_xR_x = J_\lambda L_x \Leftrightarrow W_x = J_\lambda L_xR_x^{-1} \Leftrightarrow W_x = J_\lambda T_x^{-1}$ since $H_x = J_\lambda R_xM_x$. Also, from the preceding part,

$$x \in Q'_* \Leftrightarrow R_xM_xR_x = J_\lambda L_x \Leftrightarrow M_xR_x = R_x^{-1}J_\lambda L_x$$

$$\Leftrightarrow M_x = R_x^{-1}J_\lambda L_xR_x^{-1} \Leftrightarrow M_x = R_x^{-1}J_\lambda R_{x^\lambda}L_x \Leftrightarrow M_x = R_x^{-1}J_\lambda T_x^{-1}.$$

(4) For every $x \in Q,\ x \in Q'_*$

$$\Leftrightarrow J_\rho L_{x^\lambda}R_x = J_\lambda G_x \Leftrightarrow L_{x^\lambda}R_x = J_\rho^{-1}J_\lambda G_x \Leftrightarrow L_{x^\lambda}R_x = J_\lambda^2 G_x.$$

By Theorem 1.4(4), $L_{x^\lambda}R_x = J_\lambda^2G_x \Leftrightarrow L_yR_xL_{yx}^{-1} = J_\lambda^2G_x$

$$\Leftrightarrow R_xL_{yx}^{-1} = L_y^{-1}J_\lambda^2G_x \Leftrightarrow L_yR_x = J_\lambda^2G_xL_{yx} \Leftrightarrow J_\rho L_yR_x = J_\lambda G_xL_{yx}$$

$$\Leftrightarrow J_\rho L_yR_x = J_\lambda J_\rho L_xM_x^{-1}L_{yx} \Leftrightarrow J_\rho L_yR_x = L_xM_x^{-1}L_{yx}$$

$$\Leftrightarrow J_\rho L_yR_x = U'_xL_{yx} \Leftrightarrow J_\rho L_yR_xL_{yx}^{-1} = U'_x.$$

(5) For every $x \in Q$,

$$x \in Q'_* \Leftrightarrow J_\lambda R_{x^\rho}L_x = J_\rho L_{x^\lambda}R_x.$$

Using Lemma 2.4, $R_{x^\rho} = J_\rho L_xM_x^{-1}L_x^{-1}$. Then $J_\lambda R_{x^\rho}L_x = J_\rho L_{x^\lambda}R_x$

$$\Leftrightarrow J_\lambda J_\rho L_xM_x^{-1}L_x^{-1}L_x = J_\rho L_{x^\lambda}R_x \Leftrightarrow L_xM_x^{-1} = J_\rho L_{x^\lambda}R_x \Leftrightarrow U'_x = J_\rho L_{x^\lambda}R_x.$$

Next, for every $x \in Q,\ x \in Q'_* \Leftrightarrow J_\rho L_{x^\lambda}R_x = J_\rho L_xR_x^{-1}$

$$\Leftrightarrow L_{x^\lambda} = J_\rho^{-1}J_\lambda L_xR_x^{-1}R_x^{-1} \Leftrightarrow L_{x^\lambda} = J_\lambda^2L_xR_x^{-2}.$$

By Lemma 2.4,

$$L_{x^\lambda} = J_\lambda R_x M_x R_x^{-1},\ L_{x^\lambda} = J_\lambda^2 L_x R_x^{-2} \Leftrightarrow J_\lambda R_x M_x R_x^{-1} = J_\lambda^2 L_x R_x^{-2}$$
$$\Leftrightarrow R_x M_x R_x^{-1} = J_\lambda^{-1} L_x R_x^{-2} \Leftrightarrow R_x M_x = J_\lambda^{-1} L_x R_x^{-1} \Leftrightarrow W_x = J_\lambda^{-1} L_x R_x^{-1}$$

(6) For every $x \in Q$,

$$x \in Q'_* \Leftrightarrow J_\lambda G_x = J_\rho R_x L_x^{-1} \Leftrightarrow J_\lambda G_x L_x = J_\rho R_x$$
$$\Leftrightarrow J_\lambda J_\rho M_x^{-1} L_x = J_\rho R_x \Leftrightarrow M_x^{-1} L_x = L_x^{-1} J_\rho R_x \Leftrightarrow M_x^{-1} = L_x^{-1} J_\rho R_x L_x^{-1}$$
$$\Leftrightarrow M_x^{-1} = L_x^{-1} J_\rho L_{x^\lambda} \Leftrightarrow M_x^{-1} = L_x^{-1} J_\rho T_x.$$

(7) For every $x \in Q$,

$$x \in Q'_* \Leftrightarrow J_\lambda R_{x^\rho} L_x = J_\rho L_{x^\lambda} R_x \Leftrightarrow J_\lambda R_{x^\rho} = J_\rho L_{x^\lambda} R_x L_x^{-1}$$
$$\Leftrightarrow J_\lambda R_{x^\rho} = J_\rho L_{x^\lambda} L_{x^\lambda} R_x \Leftrightarrow J_\lambda R_{x^\rho} = J_\rho L_{x^\lambda} T_x \Leftrightarrow R_{x^\rho} = L_{x^\rho} T_x$$

(8) For every $x \in Q$, $x \in Q'_* \Leftrightarrow J_\rho L_{x^\lambda} R_x = J_\lambda L_x R_x^{-1}$

$$\Leftrightarrow J_\rho L_{x^\lambda} = J_\lambda L_x R_x^{-2} \Leftrightarrow J_\rho = J_\lambda L_x R_x^{-2} L_{x^\lambda}^{-1}.$$

By [11], in a Basarab loop, $L_{x^\lambda}^{-1} = R_x L_{yx} R_x^{-1} L_y^{-1}$. It follows that,

$$J_\rho = J_\lambda L_x R_x^{-2} L_{x^\lambda}^{-1} \Leftrightarrow J_\rho = J_\lambda L_x R_x^{-2} R_x L_{yx} R_x^{-1} L_y^{-1}$$
$$\Leftrightarrow J_\rho = J_\lambda L_x R_x^{-1} L_{yx} R_x^{-1} L_y^{-1} \Leftrightarrow J_\rho L_y R_x = J_\lambda L_x R_x^{-1} L_{yx} \Leftrightarrow L_y R_x = L_x R_x^{-1} L_{yx}$$
$$\Leftrightarrow L_y R_x = T_x^{-1} L_{yx} \Leftrightarrow T_x^{-1} = L_y R_x L_{yx}^{-1}.$$

(9) For every $x \in Q$,

$$x \in Q'_* \Leftrightarrow J_\rho H_x = J_\lambda R_{x^\rho} L_x \Leftrightarrow H_x = J_\rho^{-1} J_\lambda R_{x^\rho} L_x \Leftrightarrow H_x = J_\lambda J_\lambda R_{x^\rho} L_x$$
$$\Leftrightarrow J_\lambda R_x M_x = J_\lambda^2 R_{x^\rho} L_x \Leftrightarrow R_x M_x = J_\lambda R_{x^\rho} L_x \Leftrightarrow R_x M_x L_x^{-1} = J_\lambda R_{x^\rho}$$
$$\Leftrightarrow W_x = J_\lambda R_{x^\rho} L_x \Leftrightarrow W_x = J_\lambda T_x^{-1} \Leftrightarrow W_x T_x = J_\lambda.$$

□

2.3. Characterization of a Subloop of a Basarab loop by Inverse Translation Mappings.

Let $(Q, \cdot)$ be a Basarab loop and let $Q^{ii} = \{x \in Q : xJ_\lambda = xJ_\rho\}$. From Lemma 2.4,

$$G_x M_x L_x^{-1} = J_\rho = R_{x^\rho} L_x M_x L_x^{-1} = L_x R_x^{-1} M_x L_x^{-1} \tag{2.5}$$

$$\text{and}\quad H_x M_x^{-1} R_x^{-1} = J_\lambda = L_{x^\lambda} R_x M_x^{-1} R_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1}. \tag{2.6}$$

Corollary 2.5. Let Q be a Basarab loop. If $Q^{ii}, Q'_* \neq \emptyset$, then $Q^{ii} = Q'_*$.

Proof. This follows from Theorem 2.5. □

Theorem 2.7. Let $(Q, \cdot)$ be a Basarab loop. Then Q^{ii} is a subloop of Q, if Q has AIP or AAIP.

Proof. $Q^{ii} \neq \emptyset \Leftrightarrow e^\lambda = e^\rho$. Let $x, y \in Q^{ii}$, then $x^\rho = x^\lambda$ and $y^\rho = y^\lambda$. By AIP, $(xy)^\rho = x^\rho y^\rho = x^\lambda y^\lambda = (xy)^\lambda \Leftrightarrow (xy)^\rho = (xy)^\lambda$ for all $x, y \in Q^{ii}$. By AAIP, $(xy)^\rho = y^\rho x^\rho = y^\lambda x^\lambda = (xy)^\lambda \Leftrightarrow (xy)^\rho = (xy)^\lambda$ for all $x, y \in Q^{ii}$. □

Remark 2.2. In Jaiyéọlá and Effiong [13], it was shown that for Basarab loop $(Q, \cdot)$, any two implies the third among the following statements:

(1) $(Q,\cdot)$ is an AIPL (AAIPL);
(2) $T_x \in Q_\rho^\lambda \leq SYM(Q)$;
(3) $yx^\rho = x^\lambda y$ ($x^\rho = x^\lambda$) for all $x, y \in Q$;

where $Q_\rho^\lambda = \{A \in SYM(Q) | J_\lambda A J_\rho = A\}$. By Theorem 2.7, Q^{ii} is a subloop of $(Q,\cdot)$ while by [Theorem 2.5,[13]], Q_ρ^λ is a subgroup of $SYM(Q)$.

Theorem 2.8. Let $(Q,\cdot)$ be a Basarab loop. Then for every $x \in Q,$ x is contained in Q^{ii} if and only if any of the following conditions hold :

(1) $W_x G_x^2 = U_x'$ (2) $G_x = J_\lambda U_x'$ (3) $H_x = J_\rho W_x$.

Proof. By equations (2.5) and (2.6), it follows that:

(1) For every $x \in Q$,

$$x \in Q^{ii} \Leftrightarrow G_x M_x L_x^{-1} = H_x M_x^{-1} R_x^{-1} \Leftrightarrow G_x M_x = H_x M_x^{-1} R_x^{-1} L_x$$
$$\Leftrightarrow H_x^{-1} G_x M_x = M_x^{-1} R_x^{-1} L_x \Leftrightarrow G_x G_x M_x = M_x^{-1} R_x^{-1} L_x$$
$$\Leftrightarrow G_x^2 M_x = M_x^{-1} R_x^{-1} L_x \Leftrightarrow G_x^2 = (R_x M_x)^{-1} L_x M_x^{-1} \Leftrightarrow G_x^2 = W_x^{-1} U_x' \Leftrightarrow W_x G_x^2 = U_x'.$$

(2) For every $x \in Q$,

$$x \in Q^{ii} \Leftrightarrow L_x R_x^{-1} M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1} \Leftrightarrow R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} R_x L_x^{-1} M_x^{-1}$$
$$\Leftrightarrow R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} H_x M_x^{-1} \Leftrightarrow R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} J_\lambda R_x M_x M_x^{-1}$$
$$\Leftrightarrow R_x^{-1} M_x L_x^{-1} R_x = L_x^{-1} J_\lambda R_x \Leftrightarrow R_x^{-1} M_x L_x^{-1} = L_x^{-1} J_\lambda \Leftrightarrow R_x^{-1} M_x = L_x^{-1} J_\lambda L_x$$
$$\Leftrightarrow I = H_x J_\lambda U_x' \Leftrightarrow G_x = J_\lambda U_x'.$$

(3) For every $x \in Q$,

$$x \in Q^{ii} \Leftrightarrow L_x R_x^{-1} M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1}$$
$$\Leftrightarrow G_x M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1} \Leftrightarrow J_\rho L_x M_x^{-1} M_x L_x^{-1} = R_x L_x^{-1} M_x^{-1} R_x^{-1}$$
$$\Leftrightarrow J_\rho = R_x L_x^{-1} M_x^{-1} R_x^{-1} \Leftrightarrow R_x^{-1} J_\rho R_x = L_x^{-1} M_x^{-1} \Leftrightarrow H_x = J_\rho W_x.$$

□

Theorem 2.9. Let $(Q,\cdot)$ be a Basarab loop and let $G_x = J_\rho L_x M_x^{-1}$ and $H_x = J_\lambda R_x M_x$ be defined on Q. Then for every $x \in Q,$ x is contained in Q^{ii} if and only if any of the following conditions hold :

(1) $J_\lambda = U_x' J_\rho W_x$
(2) $J_\rho = U_x' J_\lambda W_x$
(3) $J_\rho = T_x^{W_x} U_x'$
(4) $W_x J_\rho = T_x^{U_x'}$
(5) $J_\rho = T_x U_x'$
(6) $U_x' T_x = J_\rho$.

Proof. From Lemma 2.4 and Theorem 2.8,

(1)

$$U_x' = W_x G_x^2 \Leftrightarrow G_x^2 = W_x^{-1} U_x' \Leftrightarrow G_x^2 = (R_x M_x)^{-1} L_x M_x^{-1} \Leftrightarrow G_x^2 M_x = M_x^{-1} R_x^{-1} L_x$$

Put $G_x = J_\rho L_x M_x^{-1}$. Then

$$G_x G_x M_x = M_x^{-1} R_x^{-1} L_x \Leftrightarrow J_\rho L_x M_x^{-1} J_\rho L_x M_x^{-1} M_x = M_x^{-1} R_x^{-1} L_x$$
$$\Leftrightarrow J_\rho L_x M_x^{-1} J_\rho L_x = M_x^{-1} R_x^{-1} L_x \Leftrightarrow J_\rho L_x M_x^{-1} J_\rho = M_x^{-1} R_x^{-1}$$
$$\Leftrightarrow L_x M_x^{-1} J_\rho = J_\lambda M_x^{-1} R_x^{-1} \Leftrightarrow U_x' J_\rho = J_\lambda W_x^{-1} \Leftrightarrow J_\lambda = U_x' J_\rho W_x.$$

(2)

$$W_x = U'_x H_x^2 \Leftrightarrow U_x'^{-1} = H_x^2 W_x^{-1} \Leftrightarrow M_x L_x^{-1} R_x = H_x^2 M_x^{-1} \Leftrightarrow M_x L_x^{-1} R_x = H_x H_x M_x^{-1}$$

Put $H_x = J_\lambda R_x M_x$. Then

$$M_x L_x^{-1} R_x = H_x^2 M_x^{-1} \Leftrightarrow M_x L_x^{-1} R_x = J_\lambda R_x M_x J_\lambda R_x M_x M_x^{-1}$$

$$\Leftrightarrow M_x L_x^{-1} R_x = J_\lambda R_x M_x J_\lambda R_x \Leftrightarrow M_x L_x^{-1} = J_\lambda R_x M_x J_\lambda$$

$$\Leftrightarrow M_x L_x^{-1} J_\rho = J_\lambda R_x M_x \Leftrightarrow U_x'^{-1} J_\rho = J_\lambda W_x \Leftrightarrow J_\rho = U'_x J_\lambda W_x.$$

(3)

$$W_x = U'_x H_x^2 \Leftrightarrow R_x M_x L_x = L_x M_x^{-1} H_x R_x$$

Put $H_x = J_\lambda R_x M_x$. Then

$$R_x M_x L_x = L_x M_x^{-1} J_\lambda R_x M_x R_x \Leftrightarrow M_x L_x^{-1} R_x M_x L_x = J_\lambda R_x M_x R_x$$

$$\Leftrightarrow M_x L_x^{-1} R_x = J_\lambda R_x M_x R_x L_x^{-1} M_x^{-1} \Leftrightarrow R_x^{-1} J_\rho M_x L_x^{-1} R_x = M_x R_x L_x^{-1} M_x^{-1}$$

$$\Leftrightarrow J_\rho M_x L_x^{-1} = R_x M_x R_x L_x^{-1} M_x^{-1} R_x^{-1} \Leftrightarrow J_\rho U_x'^{-1} = W_x T_x W_x^{-1} \Leftrightarrow J_\rho = T_x^{W_x} U'_x.$$

(4)

$$W_x G_x^2 = U'_x \Leftrightarrow W_x G_x L_x = U'_x R_x \Leftrightarrow R_x M_x G_x L_x = L_x M_x^{-1} R_x$$

Put $G_x = J_\rho L_x M_x^{-1}$. Then

$$R_x M_x G_x L_x = L_x M_x^{-1} R_x \Leftrightarrow R_x M_x J_\rho L_x M_x^{-1} L_x = L_x M_x^{-1} R_x$$

$$\Leftrightarrow R_x M_x J_\rho L_x = L_x M_x^{-1} R_x L_x^{-1} M_x \Leftrightarrow L_x^{-1} R_x M_x J_\rho L_x = M_x^{-1} R_x L_x^{-1} M_x$$

$$\Leftrightarrow R_x M_x J_\rho = L_x M_x^{-1} R_x L_x^{-1} M_x L_x^{-1} \Leftrightarrow W_x J_\rho = T_x^{U'_x}.$$

(5)

$$W_x G_x^2 = U'_x \Leftrightarrow W_x G_x R_{x^\rho} L_x = U'_x \Leftrightarrow W_x G_x R_{x^\rho} = U'_x L_x^{-1}$$

$$\Leftrightarrow R_{x^\rho}^{-1} H_x W_x^{-1} = L_x U_x'^{-1} \Leftrightarrow R_{x^\rho}^{-1} H_x M_x^{-1} = L_x M_x L_x^{-1} R_x.$$

Put $H_x = J_\lambda R_x M_x$. Then

$$R_{x^\rho}^{-1} H_x M_x^{-1} = L_x M_x L_x^{-1} R_x \Leftrightarrow R_{x^\rho}^{-1} J_\lambda R_x M_x M_x^{-1} = L_x M_x L_x^{-1} R_x$$

$$\Leftrightarrow R_{x^\rho}^{-1} J_\lambda R_x = L_x M_x L_x^{-1} R_x \Leftrightarrow R_{x^\rho}^{-1} J_\lambda = L_x M_x L_x^{-1}$$

$$\Leftrightarrow J_\lambda = R_{x^\rho} L_x M_x L_x^{-1} \Leftrightarrow J_\lambda = T_x^{-1} U_x'^{-1} \Leftrightarrow J_\lambda = (U'_x T_x)^{-1} \Leftrightarrow J_\rho = T_x U'_x.$$

(6)

$$W_x^{-1} U'_x = G_x^2 \Leftrightarrow W_x^{-1} L_x = H_x^{-1} G_x M_x \Leftrightarrow M_x^{-1} R_x^{-1} L_x = H_x^{-1} R_{x^\rho} L_x M_x$$

Put $H_x = J_\lambda R_x M_x$. Then,

$$M_x^{-1} R_x^{-1} L_x = H_x^{-1} R_{x^\rho} L_x M_x \Leftrightarrow M_x^{-1} R_x^{-1} L_x = (J_\lambda R_x M_x)^{-1} R_{x^\rho} L_x M_x$$

$$\Leftrightarrow M_x^{-1} R_x^{-1} L_x = M_x^{-1} R_x^{-1} J_\lambda^{-1} R_{x^\rho} L_x M_x \Leftrightarrow M_x^{-1} R_x^{-1} L_x M_x^{-1} = M_x^{-1} R_x^{-1} J_\lambda^{-1} R_{x^\rho} L_x$$

$$\Leftrightarrow L_x M_x^{-1} = J_\lambda^{-1} R_{x^\rho} L_x \Leftrightarrow L_x M_x^{-1} L_x^{-1} = J_\lambda^{-1} R_{x^\rho}$$

$$\Leftrightarrow L_x M_x^{-1} = J_\rho R_{x^\rho} L_x \Leftrightarrow U'_x = J_\rho T_x^{-1} \Leftrightarrow U'_x T_x = J_\rho.$$

□

2.4. **Characterization of a Subloop of a Basarab loop by Middle Translation Mapping.** Let $(Q,\cdot)$ be a Basarab loop and let $Q^{iii} := \{x \in Q | M_x = M_x^{-1}\}$. From Lemma 2.4,

$$L_x^{-1}J_\lambda G_x = M_x^{-1} = L_x^{-1}J_\lambda R_{x^\rho}L_x = L_x^{-1}J_\lambda L_x R_x^{-1} \tag{2.7}$$

$$\text{and } \ R_x^{-1}J_\rho H_x = M_x = R_x^{-1}J_\rho L_{x^\lambda}R_x = R_x^{-1}J_\rho R_x L_x^{-1}. \tag{2.8}$$

Theorem 2.10. Let $(Q,\cdot)$ be a Basarab loop. Then Q^{iii} is a subloop of Q, if

(1) $J_\rho = J_\lambda,\ M_{xy}^2 = (M_x M_y)^2$ for all $x, y \in Q^{iii}$; and
(2) $[M_x, M_y] = I$ for all $x, y \in Q^{iii}$.

Proof. Obviously, $Q^{iii} \neq \emptyset \Leftrightarrow M_e^2 = I \Leftrightarrow J_\rho^2 = I \Leftrightarrow |J_\rho| = 2$

$$\Leftrightarrow J_\rho J_\rho = I \Leftrightarrow J_\rho J_\rho = J_\rho J_\lambda \Leftrightarrow J_\rho = J_\lambda.$$

Next, $x, y \in Q^{iii} \Leftrightarrow M_x^2 = I = M_y^2$. Thus, $M_{xy}^2 = (M_x M_y)^2 = M_x M_y M_x M_y = M_x^2 M_y^2 = II = I$. □

Theorem 2.11. Let $(Q,\cdot)$ be a Basarab loop. Then for every $x \in Q,\ x$ is contained in Q^{iii} if and only if any of the following conditions hold :

(1) $J_\lambda G_x^2 = G_x J_\rho$ (2) $H_x J_\lambda = J_\rho H_x^2$.

Proof. By equations (2.7) and (2.8), it follows that:

(1) For every $x \in Q$,

$$x \in Q^{iii} \Leftrightarrow L_x^{-1}J_\lambda G_x = R_x^{-1}J_\rho H_x \Leftrightarrow L_x^{-1}J_\lambda G_x H_x^{-1} = R_x^{-1}J_\rho$$

$$\Leftrightarrow L_x^{-1}J_\lambda G_x G_x = R_x^{-1}J_\rho \Leftrightarrow L_x^{-1}J_\lambda G_x^2 = R_x^{-1}J_\rho$$

$$\Leftrightarrow J_\lambda G_x^2 = L_x R_x^{-1}J_\rho \Leftrightarrow J_\lambda G_x^2 = G_x J_\rho.$$

(2) For every $x \in Q$,

$$x \in Q^{iii} \Leftrightarrow L_x^{-1}J_\lambda G_x = R_x^{-1}J_\rho H_x \Leftrightarrow L_x^{-1}J_\lambda = R_x^{-1}J_\rho H_x G_x^{-1}$$

$$\Leftrightarrow L_x^{-1}J_\lambda = R_x^{-1}J_\rho H_x H_x \Leftrightarrow L_x^{-1}J_\lambda = R_x^{-1}J_\rho H_x^2$$

$$\Leftrightarrow R_x L_x^{-1}J_\lambda = J_\rho H_x^2 \Leftrightarrow H_x J_\lambda = J_\rho H_x^2.$$

□

2.5. **Characterization of a Subloop of a Basarab loop by Inverse elements.** Let $(Q,\cdot)$ be a Basarab loop and $Q^{iv} := \{x \in Q : R_{x^\rho} = L_{x^\lambda}\}$. From Lemma 2.4,

$$G_x L_x^{-1} = J_\rho L_x M_x^{-1} L_x^{-1} = R_{x^\rho} = L_x R_x^{-1} L_x^{-1} \tag{2.9}$$

$$\text{and } \ H_x R_x^{-1} = J_\lambda R_x M_x R_x^{-1} = L_{x^\lambda} = R_x L_x^{-1} R_x^{-1}. \tag{2.10}$$

Theorem 2.12. Let $(Q,\cdot)$ be a Basarab loop. Then Q^{iv} is a subloop of Q, if Q is an abelian group.

Proof. $Q^{iv} \neq \emptyset$ because $R_{e^\rho} = L_{e^\lambda} \Leftrightarrow R_e = L_e \Leftrightarrow e \in Q$. If Q is an abelian group, then for all $x, y, z \in Q,\ z \cdot x^\rho y^\rho = zx^\rho \cdot y^\rho = z \cdot x^\lambda y^\lambda = zx^\lambda \cdot y^\lambda$

$$\Leftrightarrow z \cdot x^\rho y^\rho = zx^\rho \cdot y^\rho = x^\lambda y^\lambda \cdot z = y^\lambda \cdot x^\lambda z$$

$$\Leftrightarrow zR_{(x^\rho y^\rho)} = zR_{x^\rho}R_{y^\rho} = zL_{(x^\lambda y^\lambda)} = zL_{x^\lambda}L_{y^\lambda}$$

$$\Leftrightarrow zR_{(xy)^\rho} = zR_{x^\rho}R_{y^\rho} = zL_{(xy)^\lambda} = zL_{x^\lambda}L_{y^\lambda} \text{ (by AIP)}$$

$$\Leftrightarrow zR_{(xy)^\rho} = zL_{(xy)^\lambda} \Leftrightarrow R_{(xy)^\rho} = L_{(xy)^\lambda}$$

$$\Leftrightarrow R_{u^\rho} = L_{u^\lambda} \text{ for all } u = xy \Leftrightarrow u \in Q^{iv}.$$

□

Theorem 2.13. Let $(Q, \cdot)$ be a Basarab loop. Then for every $x \in Q,\ x$ is contained in Q^{iv} if and only if any of the following conditions hold :

(1) $R_x = L_xT_x^2$
(2) $J_\lambda T_x = U'_xL_x^{-1}R_x$
(3) $J_\rho = W_xR_x^{-1}L_xT_x$
(4) $R_x = L_xT_xJ_\lambda U'_x$
(5) $J_\rho^2U'_x = W_xR_x^{-1}L_x$.

Proof. By equations 2.9 and 2.10, it follows that:

(1) For every $x \in Q,\ x \in Q^{iv} \Leftrightarrow G_xL_x^{-1} = H_xR_x^{-1}$

$$\Leftrightarrow L_xR_x^{-1}L_x^{-1} = R_xL_x^{-1}R_x^{-1} \Leftrightarrow R_x^{-1\ L_x} = L_x^{-1\ R_x} \Leftrightarrow T_x^{-1}L_x^{-1} = T_xR_x^{-1}$$

$$\Leftrightarrow (L_xT_x)^{-1}T_xR_x^{-1} \Leftrightarrow R_x = L_xT_x^2.$$

(2) For every $x \in Q,\ x \in Q^{iv} \Leftrightarrow H_xR_x^{-1} = J_\rho L_xM_x^{-1}L_x^{-1}$

$$\Leftrightarrow R_xL_x^{-1}R_x^{-1} = J_\rho L_xM_x^{-1}L_x^{-1} \Leftrightarrow L_x^{-1}J_\lambda R_xL_x^{-1}R_x^{-1} = M_x^{-1}L_x^{-1}$$

$$\Leftrightarrow L_x^{-1}J_\lambda R_xL_x^{-1} = M_x^{-1}L_x^{-1}R_x \Leftrightarrow$$

$$M_xL_x^{-1}J_\lambda T_x = L_x^{-1}R_x \Leftrightarrow U_x'^{-1}J_\lambda T_x = L_x^{-1}R_x \Leftrightarrow J_\lambda T_x = U'_xL_x^{-1}R_x.$$

(3) For every $x \in Q,\ x \in Q^{iv} \Leftrightarrow G_xL_x^{-1} = J_\lambda R_xM_xR_x^{-1}$

$$\Leftrightarrow L_xR_x^{-1}L_x^{-1} = J_\lambda R_xM_xR_x^{-1} \Leftrightarrow R_x^{-1}J_\rho L_xR_x^{-1} = M_xR_x^{-1}L_x$$

$$\Leftrightarrow R_x^{-1}J_\rho L_x = M_xR_x^{-1}L_xR_x \Leftrightarrow J_\rho = R_xM_xR_x^{-1}L_xR_xL_x^{-1}$$

$$\Leftrightarrow J_\rho = W_xR_x^{-1}L_xT_x.$$

(4) For every $x \in Q,\ x \in Q^{iv} \Leftrightarrow G_xL_x^{-1} = J_\lambda R_xM_xR_x^{-1} \Leftrightarrow G_xL_x^{-1}R_xM_x^{-1} = J_\lambda R_x$

$$\Leftrightarrow L_x^{-1}R_xM_x^{-1} = G_x^{-1}J_\lambda R_x \Leftrightarrow L_x^{-1}R_xM_x^{-1} = H_xJ_\lambda R_x$$

$$\Leftrightarrow L_x^{-1}R_x = H_xJ_\lambda U'_x \Leftrightarrow L_x^{-1}R_xU_x'^{-1} = H_xJ_\lambda \Leftrightarrow T_xJ_\lambda = L_x^{-1}R_xU_x'^{-1} \Leftrightarrow L_xT_xJ_\lambda U'_x = R_x.$$

(5) For every $x \in Q,\ x \in Q^{iv} \Leftrightarrow J_\rho L_xM_x^{-1}L_x^{-1} = J_\lambda R_xM_xR_x^{-1}$

$$\Leftrightarrow J_\rho J_\rho L_xM_x^{-1}L_x^{-1} = R_xM_xR_x^{-1} \Leftrightarrow J_\rho^2L_xM_x^{-1} = R_xM_xR_x^{-1}L_x$$

$$\Leftrightarrow J_\rho^2U'_x = W_xR_x^{-1}L_x \Leftrightarrow J_\rho^2U'_x = W_xR_x^{-1}L_x.$$

□

2.6. **Characterization of a Subloop of a Basarab loop by Commutativity.** Let $(Q,\cdot)$ be a Basarab loop and let $Q^v := \{x \in Q : L_x = R_x\} = \{x \in G : xz = zx,\ \forall\ z \in Q\} = C(Q)$. From Lemma 2.4,

$$J_\lambda G_x M_x = L_x = J_\lambda R_{x^\rho} L_x M_x = J_\lambda L_x R_x^{-1} M_x \tag{2.11}$$

$$\text{and } J_\rho H_x M_x^{-1} = R_x = J_\rho L_{x^\lambda} R_x M_x^{-1} = J_\rho R_x L_x^{-1} M_x^{-1}. \tag{2.12}$$

Theorem 2.14. Let $(Q,\cdot)$ be a Basarab loop. Then for every $x \in Q$, x is contained in Q^v if and only if any of the following conditions hold :

(1) $T_x J_\rho^2 = M_x^2$ (2) $T_x J_\rho^2 W_x = M_x^2 L_x M_x$ (3) $T_x = J_\rho^2 U_x' M_x R_x^{-1} M_x^2$.

Proof. By equations 2.11 and 2.12, it follows that:

(1) For every $x \in Q,\ x \in Q^v \Leftrightarrow J_\rho H_x M_x^{-1} = J_\lambda G_x M_x$

$$\Leftrightarrow J_\lambda^{-1} J_\rho H_x = G_x M_x M_x \Leftrightarrow J_\rho^2 H_x = G_x M_x^2$$

$$J_\rho^2 T_x = T_x^{-1} M_x^2 \Leftrightarrow T_x J_\rho^2 = M_x^2.$$

(2) For every $x \in Q,\ x \in Q^v \Leftrightarrow J_\rho R_x L_x^{-1} M_x^{-1} = J_\lambda L_x R_x^{-1} M_x$

$$\Leftrightarrow J_\rho R_x M_x M_x^{-1} L_x M_x^{-1} = J_\lambda T_x^{-1} M_x \Leftrightarrow J_\rho^2 W_x = T_x^{-1} M_x M_x L_x M_x$$

$$\Leftrightarrow T_x J_\rho^2 W_x = M_x^2 L_x M_x \Leftrightarrow T_x J_\rho^2 W_x = M_x^2 L_x M_x.$$

(3) For every $x \in Q,\ x \in Q^v \Leftrightarrow J_\lambda G_x M_x = J_\rho R_x L_x^{-1} M_x^{-1}$

$$\Leftrightarrow J_\lambda L_x R_x^{-1} M_x = J_\rho R_x L_x^{-1} M_x^{-1} \Leftrightarrow J_\lambda L_x M_x^{-1} M_x R_x^{-1} M_x = J_\rho T_x M_x^{-1}$$

$$\Leftrightarrow J_\rho^2 U_x' M_x R_x^{-1} M_x^2 = T_x.$$

□

Remark 2.3. By Theorem 2.14, we have new characterizations of the centrum of a Basarab loop. It should however be noted that in Effiong et al. [11], it was shown that $Z(Q) = C(Q)$ for any Basarab loop Q.

3. Conclusion

. The algebraic properties of some bi-variate functions defined on a Basarab loop Q have been established and these were used to explore the properties of some important mono-variate functions defined on Q. Also, some subloops of a Basarab loop have been characterized by some important permutations and new characterizations of the centrum of a Basarab loop were obtained.

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