

A NOTE ON LIN'S AND NUMERICAL RADII INEQUALITIES AND APPLICATION IN PORTFOLIO OPTIMIZATION

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ABSTRACT. We consider Lin's generalizations of Reid's and Halmos' inequalities via polar decomposition approach and establish several numerical radii inequalities for the product of two bounded linear operators on a Hilbert space. The established inequalities are a significant extension of Lin's inequalities. We then explore the applications of the inequalities in finance where when constructing an investment portfolio, investors often need to allocate capital across different assets to achieve a desired risk-return profile. Lin's inequalities can be used to ensure that the portfolio weights satisfy certain constraints, such as minimum and maximum exposure to particular asset classes or sectors.

1. INTRODUCTION AND PRELIMINARIES

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $B(H)$ denotes the C^* -algebra of all bounded linear operators on H . For $A \in B(H)$, the usual operator norm of an operator A is defined by $\|A\| = \sup_{\|x\|=1} \|Ax\|$, where $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$. The spectrum of an operator A , $\sigma(A)$, consists of those complex numbers λ for which $A - \lambda I$ is not invertible while the numerical range of $A \in B(H)$ is defined $W(A) = \{\langle Ax, x \rangle : x \in H, \|x\| = 1\}$. The numerical and spectral radii of A are given by $\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}$, and $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$ respectively. For a good account of the theory of spectra and numerical ranges, see [6, 8].

The study of numerical radii inequalities especially on Hilbert spaces still remains a fundamental area of interest in the recent times, see [1, 2, 3, 4, 5, 10, 11, 13, 14]. The extensive theory of both the classical and operator inequalities which are the core tools required in establishing better numerical radii inequalities, has attracted more interest in similar research. For instance, Halmos [8] generalized the Reid inequality; Kittaneh [10] provided the generalized form of mixed Schwarz inequality and also extended the Weyl's inequality. In [11], Lin presented generalizations of Reid's and Halmos inequalities via polar decomposition approach, while in [3], more generalization of the Reid's inequality was provided.

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In this paper, we consider Lin's inequalities established in [11] via polar decomposition approach and extend the inequalities to provide the numerical radii inequalities. For consistency, we shall inherit the notation used in [11] with the operators acting on a Hilbert space H .

We say that an operator A is positive (written $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. If S and K are hermitian, we write $S \geq K$ in case $S - K \geq 0$. $A = U|A|$ is the polar decomposition of A with U the partial isometry such that $N(U) = N(A)$, where $N(A)$ means the null space of A , and $|A|$ denotes the positive square root of the positive operator A^*A , that is, $|A| = (A^*A)^{\frac{1}{2}}$. Also, we have $A^* = |A|U^*$ and $|A^*| = (AA^*)^{\frac{1}{2}}$ with $N(U^*) = N(A^*)$.

Before stating the results of this paper, we give the following Lemmas from literature. The first Lemma is a basic result giving another interesting definition of the operator norm for an operator which is positive and self-adjoint.

Lemma 1.1. *If $A \in B(H)$ is self-adjoint and nonnegative in the sense that $\langle Ax, x \rangle \geq 0$ for all $x \in H$, then $\|A\| = \sup\{\langle Ax, x \rangle : \|x\| = 1\}$.*

The second Lemma, which is called Hölder-McCarthy inequality, is a well known result that follows from spectral theorem for positive operators and Jensen's inequality (see [10]).

Lemma 1.2. *Let $A \in B(H)$ be positive operator and let $x \in H$ be any unit vector. Then*

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle \quad \forall r \geq 1.$$

The next two Lemmas are due to C.-S. Lin [11]. They form the base for this paper and hence are of great importance.

Lemma 1.3. *Let $SK = V|SK|$ be the polar decomposition. Then the following inequalities hold for every $x, y \in H$ and $\alpha \in [0, 1]$:*

(1)

$$\begin{aligned} |\langle SKx, y \rangle|^2 &\leq \|K\|^{2(1-\alpha)} \langle |SK|^{2\alpha} x, x \rangle \langle |S^*|^{2(1-\alpha)} y, y \rangle \\ &\leq \|S\|^{2\alpha} \|K\|^{2(1-\alpha)} \langle |K|^{2\alpha} x, x \rangle \langle |S^*|^{2(1-\alpha)} y, y \rangle. \end{aligned}$$

(2)

$$\begin{aligned} |\langle SKx, y \rangle|^2 &\leq \|S\|^{2\alpha} \langle |K|^{2\alpha} x, x \rangle \langle |(SK)^*|^{2(1-\alpha)} y, y \rangle \\ &\leq \|S\|^{2\alpha} \|K\|^{2(1-\alpha)} \langle |K|^{2\alpha} x, x \rangle \langle |S^*|^{2(1-\alpha)} y, y \rangle. \end{aligned}$$

(3) *If SK is Hermitian, then*

$$\begin{aligned} |\langle SKx, y \rangle|^2 &\leq \|K\|^{2\alpha} \langle |S^*|^{2\alpha} x, x \rangle \langle |SK|^{2(1-\alpha)} y, y \rangle \\ &\leq \|K\|^2 \langle |S^*|^{2\alpha} x, x \rangle \langle |S^*|^{2(1-\alpha)} y, y \rangle; \end{aligned}$$

and

$$\begin{aligned} |\langle SKx, y \rangle|^2 &\leq \|K\|^{2(1-\alpha)} \langle |SK|^{2\alpha} x, x \rangle \langle |S^*|^{2(1-\alpha)} y, y \rangle \\ &\leq \|K\|^2 \langle |S^*|^{2\alpha} x, x \rangle \langle |S^*|^{2(1-\alpha)} y, y \rangle. \end{aligned}$$

(4) If both S and SK are Hermitian, then

$$\begin{aligned} |\langle SKx, y \rangle|^2 &\leq \|K\|^{2\alpha} \langle |S|^{2\alpha} x, x \rangle \langle |SK|^{2(1-\alpha)} y, y \rangle \\ &\leq \|K\|^2 \langle |S|^{2\alpha} x, x \rangle \langle |S|^{2(1-\alpha)} y, y \rangle; \end{aligned}$$

and

$$\begin{aligned} |\langle SKx, y \rangle|^2 &\leq \|K\|^{2(1-\alpha)} \langle |SK|^{2\alpha} x, x \rangle \langle |S|^{2(1-\alpha)} y, y \rangle \\ &\leq \|K\|^2 \langle |S|^{2\alpha} x, x \rangle \langle |S|^{2(1-\alpha)} y, y \rangle. \end{aligned}$$

(5) If $S \geq 0$ and SK is Hermitian, then

$$\begin{aligned} |\langle SKx, y \rangle|^2 &\leq \|K\|^{2\alpha} \langle S^{2\alpha} x, x \rangle \langle |SK|^{2(1-\alpha)} y, y \rangle \\ &\leq \|K\|^2 \langle S^{2\alpha} x, x \rangle \langle S^{2(1-\alpha)} y, y \rangle; \end{aligned}$$

and

$$\begin{aligned} |\langle SKx, y \rangle|^2 &\leq \|K\|^{2(1-\alpha)} \langle |SK|^{2\alpha} x, x \rangle \langle S^{2(1-\alpha)} y, y \rangle \\ &\leq \|K\|^2 \langle S^{2\alpha} x, x \rangle \langle S^{2(1-\alpha)} y, y \rangle. \end{aligned}$$

The next Lemma considers the sharpening of the inequalities (3), (4) and (5) of Lemma 1.3 by the replacement of the norm of the operator by its spectral radius.

Lemma 1.4. *Let $SK = V|SK|$ be the polar decomposition. Then the following inequalities hold for every $x, y \in H$ and $\alpha \in [0, 1]$.*

(1) If $|S|^{2\alpha}K$ is Hermitian, then

$$|\langle SKx, y \rangle|^2 \leq (r(K))^2 \langle |S|^{2\alpha} x, x \rangle \langle |S^*|^{2(1-\alpha)} y, y \rangle.$$

(2) If both S and $|S|^{2\alpha}K$ are Hermitian, then

$$|\langle SKx, y \rangle|^2 \leq (r(K))^2 \langle |S|^{2\alpha} x, x \rangle \langle |S|^{2(1-\alpha)} y, y \rangle.$$

(3) If $S \geq 0$ and $S^{2\alpha}K$ is Hermitian, then

$$|\langle SKx, y \rangle|^2 \leq (r(K))^2 \langle S^{2\alpha} x, x \rangle \langle S^{2(1-\alpha)} y, y \rangle.$$

We give generalizations to Lemmas 1.3 and 1.4 in the next section.

In a financial market, the goal of the investor is to optimize on the returns. Portfolios can be constructed that minimizes risks and thereby optimize the investor's returns [9, 12]. Lin's inequality is a mathematical result used in finance and economics to analyse the feasibility of investment opportunities to determine optimal allocation of resources [7].

2. LIN'S GENERALIZATIONS AND THE NUMERICAL RADII INEQUALITIES

In this section, we consider the operators $S, K \in B(H)$ to be self-adjoint. But it's important to note that this does not imply that SK is self-adjoint. In most cases, self-adjoint operators are also referred to as hermitian even though not all self-adjoint operators are hermitian. A hermitian operator is a bounded self-adjoint operator, that is, all hermitian operators are self-adjoint. On the other hand, if SK is self-adjoint with S and K being self-adjoint separately, then KS is also self-adjoint. Our first result is an extension of Lemma 1.3 where we provide several numerical radii inequalities.

Proposition 2.1. *Let $SK = V|SK|$ be the polar decomposition. Then the following inequalities hold for every $\alpha \in [0, 1]$:*

(1) *If $|SK|^{2\alpha}|S^*|^{2(1-\alpha)}$ is hermitian, then*

$$\begin{aligned}\omega(SK) &\leq \|K\|^{1-\alpha} \| |SK|^{2\alpha} |S^*|^{2(1-\alpha)} \|_{\frac{1}{2}} \\ &\leq \|S\|^\alpha \|K\|^{1-\alpha} \| |K|^{2\alpha} |S^*|^{2(1-\alpha)} \|_{\frac{1}{2}}.\end{aligned}$$

(2) *If $|K|^{2\alpha} |(SK)^*|^{2(1-\alpha)}$ is hermitian, then*

$$\begin{aligned}\omega(SK) &\leq \|S\|^\alpha \| |K|^{2\alpha} |(SK)^*|^{2(1-\alpha)} \|_{\frac{1}{2}} \\ &\leq \|S\|^\alpha \|K\|^{1-\alpha} \| |K|^{2\alpha} |S^*|^{2(1-\alpha)} \|_{\frac{1}{2}}.\end{aligned}$$

(3) *If both SK and $|S^*||SK|$ are Hermitian, then*

$$\begin{aligned}\omega(SK) &\leq \|K\|^\alpha \| |S^*|^{2\alpha} |SK|^{2(1-\alpha)} \|_{\frac{1}{2}} \\ &\leq \|K\| \| |S^*|^2 \|_{\frac{1}{2}};\end{aligned}$$

and

$$\begin{aligned}\omega(SK) &\leq \|K\|^{1-\alpha} \| |SK|^{2\alpha} |S^*|^{2(1-\alpha)} \|_{\frac{1}{2}} \\ &\leq \|K\| \| |S^*|^2 \|_{\frac{1}{2}}.\end{aligned}$$

(4) *If both SK and $|S||SK|$ are Hermitian, then*

$$\begin{aligned}\omega(SK) &\leq \|K\|^\alpha \| |S|^{2\alpha} |SK|^{2(1-\alpha)} \|_{\frac{1}{2}} \\ &\leq \|K\| \| |S|^2 \|_{\frac{1}{2}};\end{aligned}$$

and

$$\begin{aligned}\omega(SK) &\leq \|K\|^{1-\alpha} \| |SK|^{2\alpha} |S|^{2(1-\alpha)} \|_{\frac{1}{2}} \\ &\leq \|K\| \| |S|^2 \|_{\frac{1}{2}}.\end{aligned}$$

(5) *If $S \geq 0$ and $SK, S|SK|$ are Hermitian, then*

$$\begin{aligned}\omega(SK) &\leq \|K\|^\alpha \| S^{2\alpha} |SK|^{2(1-\alpha)} \|_{\frac{1}{2}} \\ &\leq \|K\| \| S^2 \|_{\frac{1}{2}};\end{aligned}$$

and

$$\begin{aligned}\omega(SK) &\leq \|K\|^{1-\alpha} \| |SK|^{2\alpha} S^{2(1-\alpha)} \|_{\frac{1}{2}} \\ &\leq \|K\| \| S^2 \|_{\frac{1}{2}}.\end{aligned}$$

Proof. Consider inequality (1) of Lemma 1.3, then we have

$$\begin{aligned}|\langle SKx, x \rangle|^2 &\leq \|K\|^{2(1-\alpha)} \langle |SK|^{2\alpha} x, x \rangle \langle |S^*|^{2(1-\alpha)} x, x \rangle \\ &\leq \|K\|^{2(1-\alpha)} \langle |SK|^{2\alpha} |S^*|^{2(1-\alpha)} x, x \rangle, \quad (\text{by Lemma 1.2}).\end{aligned}$$

By taking supremum on both sides over all unit vectors $x \in H$, and from Lemma 1.1, we obtain

$$\omega(SK) \leq \|K\|^{1-\alpha} \| |SK|^{2\alpha} |S^*|^{2(1-\alpha)} \|^{1/2}.$$

The second inequality of (1) then follows at once. Similarly, each other inequality of Lemma 1.3 generates the corresponding inequalities (2), (3), (4), and (5) of this theorem. We omit the details. $\square$

The next result is also an extension of Lemma 1.3.

Proposition 2.2. *Let $SK = V|SK|$ be the polar decomposition. Then the following inequalities hold for $\alpha \in [0, 1]$;*

(1) *If SK is hermitian, then*

$$\begin{aligned} \omega(SK) &\leq \frac{1}{2} \|K\|^{1-\alpha} \| |SK|^{2\alpha} + |S^*|^{2(1-\alpha)} \| \\ &\leq \frac{1}{2} \|S\|^\alpha \|K\|^{1-\alpha} \| |K|^{2\alpha} + |S^*|^{2(1-\alpha)} \|. \end{aligned}$$

(2) *If $(SK)^*$ is hermitian, then*

$$\begin{aligned} \omega(SK) &\leq \frac{1}{2} \|S\|^\alpha \| |K|^{2\alpha} + |(SK)^*|^{2(1-\alpha)} \| \\ &\leq \frac{1}{2} \|S\|^\alpha \|K\|^{1-\alpha} \| |K|^{2\alpha} + |S^*|^{2(1-\alpha)} \|. \end{aligned}$$

(3) *If SK is Hermitian, then*

$$\begin{aligned} \omega(SK) &\leq \frac{1}{2} \|K\|^\alpha \| |S^*|^{2\alpha} + |SK|^{2(1-\alpha)} \| \\ &\leq \frac{1}{2} \|K\| \| |S^*|^{2\alpha} + |S^*|^{2(1-\alpha)} \|; \end{aligned}$$

and

$$\begin{aligned} \omega(SK) &\leq |S^*|^{2\alpha} \|K\|^{1-\alpha} \| |SK|^{2\alpha} + |S^*|^{2(1-\alpha)} \| \\ &\leq \frac{1}{2} \|K\| \| |S^*|^{2\alpha} + |S^*|^{2(1-\alpha)} \|. \end{aligned}$$

(4) *If SK is Hermitian, then*

$$\begin{aligned} \omega(SK) &\leq \frac{1}{2} \|K\|^\alpha \| |S|^{2\alpha} + |SK|^{2(1-\alpha)} \| \\ &\leq \frac{1}{2} \|K\| \| |S|^{2\alpha} + |S|^{2(1-\alpha)} \|; \end{aligned}$$

and

$$\begin{aligned} \omega(SK) &\leq \frac{1}{2} \|K\|^{1-\alpha} \| |SK|^{2\alpha} + |S|^{2(1-\alpha)} \| \\ &\leq \frac{1}{2} \|K\| \| |S|^{2\alpha} + |S|^{2(1-\alpha)} \|. \end{aligned}$$

(5) *If $S \geq 0$ and SK is Hermitian, then*

$$\begin{aligned} \omega(SK) &\leq \frac{1}{2} \|K\|^\alpha \| S^{2\alpha} + |SK|^{2(1-\alpha)} \| \\ &\leq \frac{1}{2} \|K\| \| S^{2\alpha} + S^{2(1-\alpha)} \|; \end{aligned}$$

and

$$\begin{aligned} \omega(SK) &\leq \frac{1}{2} \|K\|^{1-\alpha} \| |SK|^{2\alpha} + S^{2(1-\alpha)} \| \\ &\leq \frac{1}{2} \|K\| \| S^{2\alpha} + S^{2(1-\alpha)} \|. \end{aligned}$$

Proof. Consider the first inequality of Lemma 1.3. Then

$$\begin{aligned} |\langle SKx, x \rangle|^2 &\leq \|K\|^{2(1-\alpha)} \langle |SK|^{2\alpha} x, x \rangle \langle |S^*|^{2(1-\alpha)} x, x \rangle \\ &\leq \frac{1}{4} \|K\|^{2(1-\alpha)} \left(\langle |SK|^{2\alpha} x, x \rangle + \langle |S^*|^{2(1-\alpha)} x, x \rangle \right)^2, \\ &\quad (\text{by the arithmetic - geometric mean inequality}). \end{aligned}$$

Thus

$$|\langle SKx, x \rangle| \leq \frac{1}{2} \|K\|^{1-\alpha} \left(\langle |SK|^{2\alpha} + |S^*|^{2(1-\alpha)} x, x \rangle \right)^{\frac{1}{2}}.$$

By taking supremum on both sides over all unit vectors $x \in H$, and by Lemma 1.1, we obtain

$$\omega(SK) \leq \frac{1}{2} \|K\|^{1-\alpha} \| |SK|^{2\alpha} + |S^*|^{2(1-\alpha)} \|.$$

Now the second inequality is obvious. Every other inequality follows similarly from the corresponding inequality of Lemma 1.3. We omit the details. $\square$

We note that the inequalities in Lemma 1.4 are of Halmos type. In our next results, we shall extend these inequalities to include the numerical radii of product of two operators on a Hilbert space. Our first extension is the following;

Proposition 2.3. *Let $SK = V|SK|$ be the polar decomposition. Then the following inequalities hold for $\alpha \in [0, 1]$;*

(1) *If $|S|^{2\alpha}K$ is Hermitian, then*

$$\omega(SK) \leq r(K) \| |S|^{2\alpha} \|^{1/2}$$

(2) *If $S \geq 0$ and $S^{2\alpha}K$ is Hermitian, then*

$$\omega(SK) \leq r(K) \| S^2 \|^{1/2}$$

Proof. From Lemma 1.4, we have

$$\begin{aligned} |\langle SKx, x \rangle|^2 &\leq (r(K))^2 \langle |S^*|^{2\alpha} x, x \rangle \langle |S^*|^{2(1-\alpha)} x, x \rangle \\ &\leq (r(K))^2 \langle |S^*|^{2\alpha} |S^*|^{2(1-\alpha)} x, x \rangle, \text{ by Lemma 1.2, and} \\ &\quad \text{taking note that } S \text{ is self-adjoint,} \end{aligned}$$

and so

$$\omega(SK) \leq r(K) \| |S^*|^{2\alpha} |S^*|^{2(1-\alpha)} \|^{1/2}$$

which gives the desired result. The inequalities (1) and (2) of Lemma 1.4 yield the same result. Similarly, it is easy to prove inequality (2). $\square$

We conclude this paper with the following result which gives another extension of Lemma 1.4 but incorporates the arithmetic - geometric mean inequality in its proof.

Proposition 2.4. *Let $SK = V|SK|$ be the polar decomposition. Then the following inequalities hold for $\alpha \in [0, 1]$;*

(1) *If $|S|^{2\alpha}K$ is Hermitian, then*

$$\omega(SK) \leq \frac{1}{2} r(K) \| |S|^{2\alpha} + |S|^{2(1-\alpha)} \|$$

(2) *If $S \geq 0$ and $S^{2\alpha}K$ is Hermitian, then*

$$\omega(SK) \leq \frac{1}{2} r(K) \| |S|^{2\alpha} + |S|^{2(1-\alpha)} \|$$

Proof. Follows by a simple application of arithmetic - geometric mean inequality to Lemma 1.4. We omit the details. $\square$

3. APPLICATION IN PORTFOLIO OPTIMIZATION

3.1. Constructing a minimum variance portfolio. We can derive the equation of the efficient frontier in the return-risk space in case of two securities S and K . We have the equations

$$1 = x_S + x_K$$

$$E_p = x_S E_S + x_K E_K$$

$$V_p = x_S^2 V_S + x_K^2 V_K + 2x_S x_K C_{SK}$$

where x_S and x_K are portfolio weights, E_p is the return on the portfolio and V_p is the variance of portfolio returns.

V_p is minimized when

$$x_S = \frac{V_K - C_{SK}}{V_S + V_K - 2C_{SK}}$$

Indeed, consider the above case of two securities. The portfolio variance is given by

$$V_p = x_S^2 V_S + x_K^2 V_K + 2x_S x_K C_{SK}$$

Rewriting in terms of the proportion x_S ,

$$V_p = x_S^2 V_S + (1 - x_S)^2 V_K + 2x_S(1 - x_S)C_{SK}.$$

Differentiating with respect to x_S , we get

$$\frac{dV_p}{dx_S} = 2x_S V_S - 2V_K + 2x_S V_K + 2C_{SK} - 4x_S C_{SK}.$$

Equating to zero gives

$$x_S V_S - V_K + x_S V_K + C_{SK} - 2x_S C_{SK} = 0,$$

which can be written as $x_S(V_S + V_K - 2C_{SK}) = V_K - C_{SK}$, thereby yielding $x_S = \frac{V_K - C_{SK}}{(V_S + V_K - 2C_{SK})}$ as desired.

3.2. Derivation of the efficient frontier in case of N securities. Here we use the Lagrangian function

$$Z = \sum_i \sum_j C_{ij} x_i x_j - \lambda \left(\sum x_i E_i - E - \mu \left(\sum_i x_i - 1 \right) \right)$$

This is differentiated with respect to the x'_i , λ and μ to give

$$\begin{aligned} \frac{\partial Z}{\partial x_i} &= 2 \sum_j C_{ij} x_j - \lambda E_i - \mu \\ \frac{\partial Z}{\partial \lambda} &= E - \sum x_i E_i \\ \frac{\partial Z}{\partial \mu} &= 1 - \sum x_i \end{aligned}$$

setting the derivatives to zero and solving $N + 2$ equations in $N + 2$ unknowns gives the equation of the efficient frontier in the expected return-variance space. One way of solving the minimization problem is the method of Langragian multipliers. We set up to optimize a suitable Langragian function and use it to solve constrained optimization problem as follows. They are static and the constraints are all strict inequalities.

- maximize or minimize some function $f(x_1, x_2, \dots, x_N)$
- subject to a set of M constraints

$$g_j(x_1, x_2, \dots, x_N) = c_j, \quad j = 1, 2, \dots, M$$

- By choice of N variables

We wish to minimize the portfolio variance V subject to the two constraints $\sum_i x_i$ and $E = E_p$ by choice of the securities x_i , $i = 1, 2, \dots, N$.

We set the Langragian function of the form

$$Z = f(x_1, x_2, \dots, x_N) - \sum_{j=1}^M \lambda_j (g_j(x_1, x_2, \dots, x_N))$$

The Langragian function is

$$Z = V - \lambda(E - E_p) - \mu\left(\sum_i x_i - 1\right)$$

or

$$Z = \sum_i \sum_j C_{ij} x_i x_j - \lambda \left(\sum_i E_i x_i - E_p \right) - \mu \left(\sum_i x_i - 1 \right),$$

where

- V, E and x_i are as defined above
- c_{ij} the covariance between security i and j returns
- E_p and 1 are the constraining constraints and
- λ and μ are known as Langragian multipliers

This is to minimize the variance V to the expected return and all the constraints. The partial derivative of Z with respect to x_i and equated to zero as given above. The resultant $N + 2$ equations in $N + 2$ unknowns can be solved to find optimal values of the security proportions of the portfolio expectation E_p . These can then be substituted in the variance equation to give the expression of the portfolio variance hence risk measure as a function of expectation giving the minimum variance.

3.3. Matrix notation of the $N + 2$ equations. The matrix notation of the $N + 2$ equations is given by

$$Ay = d,$$

where

$$\begin{aligned} A &= \begin{bmatrix} 2C & -E & -I \\ E & 0 & 0 \\ I^T & 0 & 0 \end{bmatrix} \\ y^T &= \{x^T \quad \lambda \quad \mu\} \end{aligned}$$

$$d^T = \{0 \quad 0 \quad \dots \quad 0 \quad E_p \quad 1\}$$

(N zeros)

$$x^T = \{x_1 \quad x_2 \quad \dots \quad x_N\}$$

$$E^T = \{E_1 \quad E_2 \quad \dots \quad E_N\}$$

$$I^T = \{1 \quad 1 \quad \dots \quad 1\}$$

(N ones)

the solution is then

$$y = A^{-1}d$$

x_i is linear in E_p

By substituting in the variance equation

$$V = \{x^T \quad C \quad x\}$$

The corresponding V_p is quadratic in E_p (portfolio variance is a quadratic function of portfolio return).

- A is $(N + 2) \times (N + 2)$
- The sub matrix C is $N \times N$ i.e

$$C = \begin{bmatrix} C_{11} & \dots & C_{1N} \\ \dots & & \\ C_{N1} & \dots & C_{NN} \end{bmatrix}$$

- y is $(N + 2) \times 1$
- d is $(N + 2) \times 1$.

CONCLUSION

It has been demonstrated that investors can apply the Lin's inequality in Finance when constructing an optimal investment portfolio based on a risk- return profile. Lin's inequality thus enables investors to ensure that the portfolio weights satisfy certain constraints, such as minimum and maximum exposure to particular assets or sectors. This paper demonstrates the case of two assets and a generalization for N assets.

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