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SEIDEL POLYNOMIAL AND ENERGY OF JOIN OF TWO GRAPHS

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ABSTRACT. For a graph G, Seidel matrix is defined as $S(G) = [s_{ij}]$ in which $s_{ij} = -1$ if the vertices v_i and v_j are adjacent, $s_{ij} = 1$ if the vertices v_i and v_j are not adjacent and $s_{ij} = 0$ for $i = j$. Seidel polynomial is the polynomial associated to the Seidel matrix. In the present work, we study Seidel polynomial for join of any two graphs by using the concept of coronal of a matrix and hence construct a pair of Seidel equienergetic graphs by joining a regular graph (which is among a pair of Seidel equienergetic graphs of same order and degree) with a non-regular graph.

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper we consider simple, finite, connected and undirected graphs. A graph G is a pair of sets $\big(V(G), E(G)\big)$, where $V(G)$ is a non empty set of elements called vertices, and $E(G)$ is a set of elements called edges. Two vertices that are joined by an edge are said to be adjacenct or neighbors, the degree of a vertex counts to the number of neighbors, G is regular if all of its vertices have same degree. The join of two disjoint graphs G_1 and G_2 denoted by $G_1 \bigtriangledown G_2$ is the graph obtained by taking a copy of G_1 and a copy of G_2, joining each vertex of G_1 to each vertex of G_2. If G is a graph on n vertices (of order n), adjacency matrix is defied as $A(G) = [a_{ij}]$ in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and 0 otherwise. The polynomial associated to $A(G)$ defined by, $\phi\big(A(G); x\big) = det\big(xI_n - A(G)\big)$ is called the characteristic polynomial, roots of the equation $det\big(xI_n - A(G)\big) = 0$ are eigenvalues (denote them by λ_i's), their collection is called the spectrum and sum of absolute values of these eigenvalues is the energy. Two graphs are said to be equienergetic, if they have same energy. Cospectral graphs are obviously equieneregtic, non cospectral equienergetic graphs are study of interest. For a graph G on n vertices, Seidel matrix is defied as $S(G) = [s_{ij}]$ in which $s_{ij} = -1$ if the vertices v_i and v_j are adjacent, $s_{ij} = 1$ if the vertices v_i and v_j are not adjacent and $s_{ij} = 0$ for $i = j$. The polynomial associated to $S(G)$ defined by, $\phi\big(S(G); x\big) = det\big(xI_n - S(G)\big)$ is called the Seidel

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polynomial, roots of the equation $det\Big(xI_n - S(G)\Big) = 0$ are Seidel eigenvalues (denote them by ξ_i's), their collection is called Seidel spectrum and sum of absolute values of these eigenvalues is the Seidel energy. Two graphs are said to be Seidel equienergetic, if they have same Seidel energy. Seidel cospectral graphs are obviously Seidel equieneregtic, non Seidel cospectral but Seidel equienergetic graphs are in scope. If G is an r-regular graph on n vertices then the Seidel eigenvalue corresponding to regularity r is, $n-2r-1$. It is noted that, for regular graphs row sum is constant (for all graph matrices) and hence whole regular graph can be taken as a single partition in studying join of a non-regular graph and a regular graph. J denotes the matrix with all etries equal to 1, I denotes the identity matrix, 1_n denotes the column vector of dimension n. For other graph theoretical definitions and notations, we follow [2, 3].

Definition 1.1. [6] Given a graph G on n vertices with the graph matrix M, where M is viewed as a matrix over the field of rational functions $\mathbb{C}(x)$ with $det(xI_n - M)$ non zero. The M-coronal $\Gamma_M(x) \in \mathbb{C}(x)$ of G is, $\Gamma_M(x) = 1_n^T\Big(xI_n - M\Big)^{-1}1_n$. If M has a constant row sum r, $\Gamma_M(x) = \dfrac{n}{x-r}$.

Proposition 1.2. (Schur Complement [2]) *Suppose that the order of all four matrices B_{11}, B_{12}, B_{21} and B_{22} satisfy the rules of operations on matrices. Then, we have*

$$det\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{cases} det(B_{22})\; det(B_{11} - B_{12}B_{22}^{-1}B_{21}), \\ \qquad\qquad\qquad\qquad \textit{if } B_{22} \textit{ is a non-singular matrix,} \\ det(B_{11})\, det(B_{22} - B_{21}B_{11}^{-1}B_{12}), \\ \qquad\qquad\qquad\qquad \textit{if } B_{11} \textit{ is a non-singular matrix.} \end{cases}$$

Seidel matrix is one of the adjacency based matrix associated to the graph structure. Seidel matrix is studied for its polyomial, eigenvalues and hence energy, which is due to the property of invariance of Seidel energy under the operation of Seidel switching which operates on the edge set of the graph structure. In literature Seidel matrix is studied for its eigenvalues [1, 13]; energy [4, 5, 7, 9, 11]; equienergetic graphs [10, 12] and borderenergetic graphs [8].

In 2016, Ramane et al. studied the Seidel polynomial for join of two graphs $G_1 \bigtriangledown G_2$ and given an explicit expression for the associated Seidel polynomial interms of that of underlying graphs G_1 and G_2. Hence, given a method for the costruction of Seidel equienergetic graphs [10] usig join of two regular graphs (where one regular graph is among a pair of Seidel equienergetic graphs of same order and degree).

In the present work, we study polynomial associated to the Seidel matrix for join of two graphs using the concept of coronal of a matrix which reduces the calculations which were seen to be more as in [10] (which involves row, column opeartions and expansion of determinant) and hence provide a method for the construction of Seidel equienergetic graphs by joining a regular graph (which is among a pair of Seidel equienergetic graphs of same order and degree) with a non-regular graph.

2. Seidel polynomial for join of two graphs

The current section focuses on the polynomial associated to the Seidel matrix for join of any two graphs. Further, study the Seidel eigenvalues and hence Seidel energy for this structure, thereby provide a method for the construction of Seidel equienergetic graphs.

Theorem 2.1. *For two graphs G_1 and G_2 on n_1 and n_2 vertices, the Seidel polynomial of $G_1 \bigtriangledown G_2$ is*

$$\phi\Big(S\big(G_1 \bigtriangledown G_2\big);\, x\Big) \;=\; \phi\big(S(G_2)\ ;\ x\big)\ det\Big(xI_{n_1} - S(G_1) - \Gamma_{S(G_2)}(x)\ J_{n_1}\Big).$$

Proof. The Seidel matrix due to the structure of $G_1 \bigtriangledown G_2$ is

$$S(G_1 \bigtriangledown G_2) = \begin{pmatrix} S(G_1) & -J_{n_1\times n_2} \\ -J_{n_2\times n_1} & S(G_2) \end{pmatrix}.$$

The Seidel polynomial is

$$\begin{aligned}\phi\Big(S(G_1 \bigtriangledown G_2);\, x\Big) &= det\Big(x\, I_{n_1+n_2} - S(G_1 \bigtriangledown G_2)\Big)\\ &= det\left(\begin{array}{c|c} xI_{n_1} - S(G_1) & J_{n_1\times n_2} \\ \hline J_{n_2\times n_1} & xI_{n_2} - S(G_2)\end{array}\right).\end{aligned}$$

From Proposition 1.2, we have

$$\begin{aligned}\phi\Big(S(G_1 \bigtriangledown G_2);\, x\Big) &= det\Big(x\, I_{n_2} - S(G_2)\Big)\\ &\quad\ det\Big(xI_{n_1} - S(G_1) - J_{n_1\times n_2}\Big[xI_{n_2} - S(G_2)\Big]^{-1} J_{n_2\times n_1}\Big)\\ &= \phi\big(S(G_2)\ ;\ x\big)\\ &\quad\ det\Big(xI_{n_1} - S(G_1) - J_{n_1\times n_2}\Big[xI_{n_2} - S(G_2)\Big]^{-1} J_{n_2\times n_1}\Big).\end{aligned} \tag{2.1}$$

Since,

$$\begin{aligned}J_{n_1\times n_2}\Big[xI_{n_2} - S(G_2)\Big]^{-1} J_{n_2\times n_1} &= 1_{n_1}\ 1_{n_2}^T\ \Big[xI_{n_2} - S(G_2)\Big]^{-1}\ 1_{n_2}\ 1_{n_1}^T\\ &= 1_{n_1}\ \Gamma_{S(G_2)}(x)\ 1_{n_1}^T\\ &= \Gamma_{S(G_2)}(x)\ J_{n_1}.\end{aligned} \tag{2.2}$$

From (2.1) and (2.2), result follows. ☐

Corollary 2.2. *If G_i are r_i-regular graphs on n_i vertices for $i = 1, 2$ then Seidel polyomial of $G_1 \bigtriangledown G_2$ is*

$$\begin{aligned}\phi\Big(S(G_1 \bigtriangledown G_2);\, x\Big) &= \prod_{i=2}^{n_1}\Big(x - \xi_i(G_1)\Big)\ \prod_{i=2}^{n_2}\Big(x - \xi_i(G_2)\Big)\\ &\quad\ \Big(\big(x - (n_1 - 2r_1 - 1)\big)\ \big(x - (n_2 - 2r_2 - 1)\big) - n_1 n_2\Big).\end{aligned}$$

Proof. From Theorem 2.1, we have

$$\begin{aligned}\phi\Big(S\big(G_1 \bigtriangledown G_2\big);\ x\Big) &= \phi\big(S(G_2)\ ;\ x\big)\ det\Big(xI_{n_1} - S(G_1)\\ &\qquad -J_{n_1\times n_2}\Big[xI_{n_2} - S(G_2)\Big]^{-1} J_{n_2\times n_1}\Big).\end{aligned}$$

Since both G_1 and G_2 are regular, we have

$$\begin{aligned}\phi\big(S(G_2)\ ;\ x\big) &= det\Big(xI_{n_2} - S(G_2)\Big)\\ &= \Big(x-(n_2-2r_2-1)\Big)\prod_{i=2}^{n_2}\Big(x-\xi_i(G_2)\Big) \qquad (2.3)\end{aligned}$$

and

$$\begin{aligned}& det\Big(xI_{n_1} - S(G_1) - J_{n_1\times n_2}\Big[xI_{n_2} - S(G_2)\Big]^{-1} J_{n_2\times n_1}\Big)\\ &= \prod_{i=2}^{n_1}\Big(x-\xi_i(G_1)\Big)\left(x-(n_1-2r_1-1)-\frac{n_1n_2}{\big(x-(n_2-2r_2-1)\big)}\right)\\ &= \prod_{i=2}^{n_1}\Big(x-\xi_i(G_1)\Big)\frac{\Big(\big(x-(n_1-2r_1-1)\big)\ \big(x-(n_2-2r_2-1)\big)-n_1n_2\Big)}{\Big(x-(n_2-2r_2-1)\Big)}. \qquad (2.4)\end{aligned}$$

Since,

$$\begin{aligned}J_{n_1\times n_2}\Big[xI_{n_2} - S(G_2)\Big]^{-1} J_{n_2\times n_1} &= 1_{n_1}\ 1_{n_2}^T\ \Big[xI_{n_2} - S(G_2)\Big]^{-1}\ 1_{n_2}\ 1_{n_1}^T\\ &= 1_{n_1}\ \Gamma_{S(G_2)}\ 1_{n_2}^T\\ &= \Gamma_{S(G_2)}\ J_{n_1}\\ &= \frac{n_2\ J_{n_1}}{x-(n_2-2r_2-1)}.\end{aligned}$$

Here, $\Gamma_{S(G_2)} = \dfrac{n_2}{x-(n_2-2r_2-1)}$. Because G_2 is an r_2 regular graph on n_2 vertices with a constant row sum of n_2-2r_2-1.

Thus, $J_{n_1\times n_2}\Big[xI_{n_2} - S(G_2)\Big]^{-1} J_{n_2\times n_1} = \dfrac{n_1\ n_2}{x-(n_2-2r_2-1)}$ corresponding to Seidel eigenvalue n_1-2r_1-1 of graph G_1 and zero otherwise. Hence, from (2.3) and (2.4), result follows. $\square$

Corollary 2.3. *If G_i are r_i-regular graphs on n_i vertices for $i=1,2$ then Seidel energy of $G_1 \bigtriangledown G_2$ is*

$$\begin{aligned}\mathcal{E}_S(G_1 \bigtriangledown G_2) &= \mathcal{E}_S(G_1) - (n_1-2r_1-1) + \mathcal{E}_S(G_2) - (n_2-2r_2-1)\\ &\quad + \sqrt{\Big[(n_1-2r_1-1)-(n_2-2r_2-1)\Big]^2 + 4n_1n_2}.\end{aligned}$$

Proof. Result follows from Corollary 2.2 by finding roots of the polynomial equation $\phi\Big(S\big(G_1 \bigtriangledown G_2\big);\, x\Big) = 0$ for Seidel eigenvalues and hence applying definition of Seidel energy. □

Theorem 2.4. *For a non-regular graph G_1 on n_1 vertices and an r_2-regular graph G_2 on n_2 vertices, Seidel polyomial of $G_1 \bigtriangledown G_2$ is*

$$\phi\Big(S\big(G_1 \bigtriangledown G_2\big);\, x\Big) \;=\; \prod_{i=2}^{n_2}\Big(x-\xi_i(G_2)\Big)\;\phi\big(Q;\, x\big)$$

where, $Q = \begin{pmatrix} S(G_1) & -n_2 J_{n_1\times 1} \\ -J_{1\times n_1} & n_2-2r_2-1 \end{pmatrix}$.

Proof. As matrix associated with the structure of $G_1 \bigtriangledown G_2$ is

$$S(G_1 \bigtriangledown G_2) = \begin{pmatrix} S(G_1) & -J_{n_1\times n_2} \\ -J_{n_2\times n_1} & S(G_2) \end{pmatrix}.$$

Since G_2 is an r_2-regular graph on n_2 vertices, the block $S(G_2)$ has the constant row sum: (n_2-2r_2-1), taking all n_2 vertices in a single partition, above matrix is reduced to

$$Q = \begin{pmatrix} S(G_1) & -n_2 J_{n_1\times 1} \\ -J_{1\times n_1} & n_2-2r_2-1 \end{pmatrix},$$

which is called as quotient matrix. The eigenvalues that donot belong to the quotient due to the block $S(G_2)$ coincide with eigenvalues associated to it except n_2-2r_2-1, which is obtained due to equation (2.3) in the form of product of (n_2-1) factors as,

$$\prod_{i=2}^{n_2}\Big(x-\xi_i(G_2)\Big) \tag{2.5}$$

Hence, result follows from (2.5) and polynomial due to Q. □

Corollary 2.5. *For a non-regular graph G_1 on n_1 vertices and an r_2-regular graph G_2 on n_2 vertices, Seidel energy of $G_1 \bigtriangledown G_2$ is*

$$\mathcal{E}_S(G_1 \bigtriangledown G_2) = \mathcal{E}_S(G_2) - (n_2-2r_2-1) + \sum_{i=1}^{n_1+1}\big|\xi_i(Q)\big|$$

where $\xi_i(Q)$ are the eigenvalues associated to matrix Q.

Proof. Result follows from Theorem 2.4 by finding roots of the polynomial equation $\phi\Big(S\big(G_1 \bigtriangledown G_2\big);\, x\Big) = 0$ for Seidel eigenvalues and hence applying definition of Seidel energy. □

Corollary 2.6. *For a non-regular graph G_1 on n_1 vertices, if G_a and G_b are a pair of Seidel equienergetic graphs on n_2 vertices and both are of r_2-regular then*

$$G_1 \bigtriangledown G_a \quad \text{and} \quad G_1 \bigtriangledown G_b$$

are also Seidel equienergetic graphs.

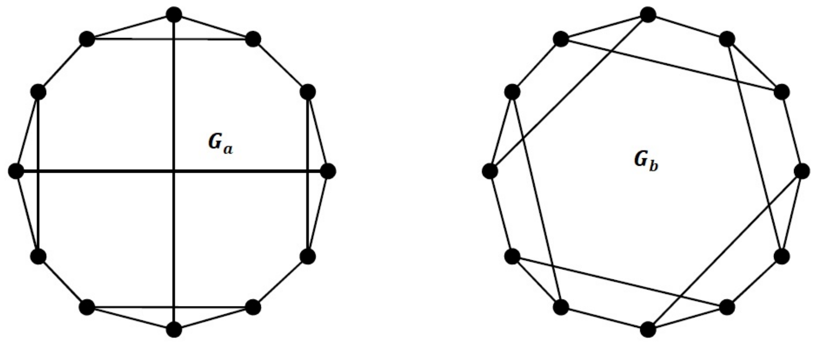

FIGURE 1. Seidel equienergetic graphs

Proof. As G_a and G_b are both r_2-regular graphs on n_2 vertices, thus quotient matrices associated to both $G_1 \bigtriangledown G_a$ and $G_1 \bigtriangledown G_b$ are same. Also, G_a and G_b are Seidel equienergetic graphs. Result follows from Corollary 2.5. □

Example 2.7. Consider path on 3 vertices (P_3), a pair of Seidel equienergetic graphs G_a and G_b as in Figure 1 with $\mathcal{E}_S(G_1) = \mathcal{E}_S(G_2) = 34$. Then graphs $P_3 \bigtriangledown G_a$ and $P_3 \bigtriangledown G_b$ are Seidel equienergetic.
Since, $\mathcal{E}_S\Big(P_3 \bigtriangledown G_a\Big) = \mathcal{E}_S\Big(P_3 \bigtriangledown G_b\Big) = 45.14133968535387.$

Remark 2.8. For the structure $G_1 \bigtriangledown G_2$,

i. Corollary 2.2 signifies the importance of coronal in studying the Seidel polynomial of join of two graphs. Since, it reduces calculations which were seen to be more as in [10].
ii. Corollary 2.6 gives the construction of Seidel equienergetic graphs by joining a regular graph (which is among a pair of Seidel equienergetic graphs of same order and degree) with a non-regular graph.
iii. Corollary 2.6 generalizes the results due to Ramane et al. [10], where construction of Seidel equienergetic graphs is done by joining two regular graphs (where one regular graph is among a pair of Seidel equienergetic graphs of same order and degree).

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