Annals of Mathematics and Computer Science ISSN: 2789-7206
Vol 25 (2024) 19-25 https://doi.org/10.56947/amcs.v25.364

SPECTRUM OF INNER PRODUCT TYPE INTEGRAL TRANSFORMERS ON HILBERT SPACES

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ABSTRACT. In this paper, we have computed the spectrum of the inner product type integral transformer on Hilbert spaces. We have also shown that the spectrum of the inner product type integral transformer is contained in the closure of its numerical range.

1. INTRODUCTION

Let $\mathcal{A}$ be a Banach algebra, $A, B \in \mathcal{A}$. Elementary operators, introduced by G. Lumer and M. Rosenblum in [7] are mappings from $\mathcal{A}$ to $\mathcal{A}$ of the form

$$X \longmapsto \sum_{i=1}^{n} A_i X B_i. \tag{1.1}$$

The finite sum in Equation (1.1) may be replaced by the infinite sum provided convergence conditions are placed. A similar mapping, called the inner product type integral transformer (i.p.t.i transformer), introduced by Danko in [3] is defined by

$$X \longmapsto \int_{\Omega} A_t X B_t d\mu(t), \tag{1.2}$$

where (Ω, μ) is a measure space, $t \longmapsto A_t, B_t$ are operator valued functions in $B(H)$ and $X \in B(H)$. Throughout this paper, we will denote the i.p.t.i. transformer in Equation (1.2) by

$$T_{A,B}(X) = \int_{\Omega} A_t X B_t d\mu(t) = \int_{\Omega} A_t \otimes B_t d\mu(t). \tag{1.3}$$

The spectrum of the operators of the form in Equation (1.1) has been widely investigated by various researchers [see [1], [8] and references there in]. For $A, B \in B(H)$, the spectrum $\sigma(M)$ of the operator defined by $MX = AXB$ was studied by G. Lumer and M. Rosenblum in [7]. They proved that the spectrum $\sigma(M)$ of M is given by

$$\sigma(M) = \sigma(A)\sigma(B) = \{\alpha\beta : \alpha \in \sigma(A), \beta \in \sigma(B)\}.$$

Date: Received: Jul 30, 2024; Accepted: Sep 22, 2024.
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2010 *Mathematics Subject Classification.* Primary 47A10; Secondary 47B48.
Key words and phrases. Inner product type integral transformer, Spectrum, Elementary operator.

The spectrum $\sigma(R)$ of the elementary operator $R : B(H) \longmapsto B(H)$ defined by

$$R(X) = \sum_{i=1}^{n} A_i X B_i,$$

was studied in [2] by Davis and Rosenthal. They showed that for two commuting families $\{A_i\}_{i=1}^{n}$ and $\{B_i\}_{i=1}^{n}$,

$$\sigma_\delta(R) \subset \{\sum_{t=1}^{n} \alpha_i\beta_i : \alpha_i \in \sigma_\delta(A_i), \beta_i \in \sigma_\pi(B_i)\},$$

where σ_π and σ_δ denote the approximate point spectrum and the defect spectrum respectively.

For two commuting families $A_1, ..., A_n$ and $B_1, ..., B_n$ of self-adjoint operators in $B(H)$, Khan and Kyle in [5] showed that

$$\sigma(R) = \{\lambda_i\mu_i : \lambda_i \in \sigma(A_i), \mu_i \in \sigma(B_i)\}.$$

The computation of the spectrum of a similar mapping of the form in Equation (1.3) has not been done. However, Danko in [3] using the Cauchy-Schwartz inequality showed that the operator $T_{A,B}$ is bounded.

In [4], Danko obtained results on the relationship between the spectra of the inner product type transformers and the unit disc. The spectral radius of $T_{A,B}$ has been determined in [4] and found to be

$$r\left(\int_\Omega A^* \otimes B d\mu\right) \leq \sqrt{r(\int_\Omega A^* \otimes A d\mu) r(\int_\Omega B^* \otimes B d\mu)}.$$

Therefore, the spectrum is contained in a disk if $r(T_{A,B}) < 1$. The numerical range of the operator $T_{A,B}$ has been determined in [9] and is defined as

$$V(T_{\{A,B\}}) = \left\{f(\int_\Omega A_t \otimes B_t d\mu(t)) : f \in P(\Omega)\right\}.$$

The properties of the determined numerical range have also been discussed in [9], Theorem 2.5. The aim of this paper is to determine the spectrum of the operator $T_{A,B}$ and determine the relationship between the spectrum and the numerical range of this operator.

2. Main results

In this section, we prove that the spectrum $\sigma(T_{A,B})$ of the i.p.t.i. transformer $T_{A,B}$ is given by

$$\sigma(T_{A,B}) = \{\int_\Omega \alpha_t\beta_t d\mu(t) : \alpha_t \in \sigma(A_t, B(H)), \beta_t \in \sigma(B_t, B(H))\}.$$

To begin with, we give a condition for self-adjointness of the operator valued function A_t.

Lemma 2.1. *Let $A_t : H \mapsto H$ be a bounded operator valued linear operator on a complex Hilbert space H. If $\langle A_t f, f\rangle$ is real $\forall f \in H$ and $t \in \mathbb{R}$ then the operator A_t is self-adjoint.*

Proof. If $\langle A_t f, f\rangle$ is real $\forall f \in H$ and $t \in \mathbb{R}$, then $\langle A_t f, f\rangle = \overline{\langle A_t f, f\rangle} = \overline{\langle f, A_t^* f\rangle} = \langle A_t^* f, f\rangle$

Then $0 = \langle A_t f, f\rangle - \langle A_t^* f, f\rangle = \langle (A_t - A_t^*) f, f\rangle$ and since H is a complex Hilbert space, we have that $A_t - A_t^* = 0$, hence A_t is self-adjoint. □

Remark 2.2. The above Lemma implies that the operator $T_{A,B}$ is self-adjoint provided A_t and B_t are self-adjoint.

We define mappings $^+$ and $^-$ from $B(H)$ to $B(B(H))$ that associates with any element $A_t \in B(H)$, the operators $A_t^+, A_t^- \in B(B(H))$ defined by

$$A_t^+ = A_tX \text{ and } A_t^- = XA_t \;\; \forall \;\; X \in B(H)$$

That is, $^+$ and $^-$ are multiplication on the right and left respectively by a bounded operator X.

Remark 2.3. For $A_t, B_t \in B(H)$, we have

$$A_t^+B_t^-(X) = A_t^+(XB_t) = A_tXB_t$$

and

$$B_t^-A_t^+(X) = B_t^-(A_tX) = A_tXB_t$$

This implies that $A_t^+B_t^- = B_t^-A_t^+$

In Lemma 2.7, we have verified that the mappings $^+$ and $^-$ are one-to-one and onto mappings from the Banach algebra $B(H)$ onto $B(B(H))$.
The relationship between the spectra in $B(H)$ and $B(B(H))$ can be obtained using the result in [[7], Lemma 2].

Lemma 2.4. *Let A_t be an operator valued function on the Banach algebra $B(H)$ and $A_t^+, A_t^- \in B(B(H))$. Then*

$$\sigma(A_t, B(H)) = \sigma(A_t^+, B(B(H)))$$

and

$$\sigma(A_t, B(H)) = \sigma(A_t^-, B(B(H)))$$

Proof. The proof follows closely that of Lemma 2 in [7] and will be omitted in this paper. □

In this paper, we study the operator $T_{A,B} \in B(B(H))$ defined by

$$\begin{aligned} T_{A,B}(X) &= \int_\Omega A_tXB_t d\mu(t) \\ &= \int_\Omega A_t^+B_t^- d\mu(t), \end{aligned} \tag{2.1}$$

where A_t is an operator-valued function in $B(H)$, the operators $A_t^+, A_t^- \in B(B(H))$ for some spectral measure μ on the compact space Ω and $X \in B(H)$. We shall determine the spectrum of $T_{A,B}$.
The remark below gives some of the abelian properties of sets which we shall need in the proof of our main result.

Remark 2.5. For any set $Z \in B(H)$, there is a smallest Von Neumann algebra W containing Z. We call W the Von Neumann algebra generated by Z. If Z consists of commuting elements, then W is abelian. In this case $Z \subseteq Z'$ where Z' is the commutant of Z. If Z is self adjoint and contains identity, then $W = Z''$, where Z'' is the double commutant of Z. Always $Z \subseteq Z'' \subseteq W$ [See [10]]. Thus if Z is a subalgebra generated by commuting elements $z_1, z_2, ..., z_n$ and identity I, then there exists an abelian Banach algebra Z'' such that

$$Z \subseteq Z'' \subset B(H).$$

We now prove that the spectrum $\sigma(T_{\{A,B\}})$ of the i.p.t.i. transformer $T_{\{A,B\}}$ using the abelian properties of sets and the concept of commutators as in the Von Neumann algebras is given by

$$\sigma(T_{\{A,B\}}) = \{\int_\Omega \alpha_t\beta_t d\mu(t) : \alpha_t \in \sigma(A_t, B(H)), \beta_t \in \sigma(B_t, B(H))\}.$$

Here, we want to relate the spectra of operator $T_{\{A,B\}}$ to the spectra of its left regular and right regular representations. A simplification of this is achieved by defining mappings $^+$ and $^-$ from $B(H)$ to $B(B(H))$ that associates with any element $A_t \in B(H)$, the operators $A_t^+, A_t^- \in B(B(H))$ defined by

$$A_t^+ = A_tX \text{ and } A_t^- = XA_t \ \forall \ X \in B(H)$$

That is, $^+$ and $^-$ are multiplication on the right and left respectively by a bounded operator X.

Remark 2.6. For $A_t, B_t \in B(H)$, we have

$$A_t^+B_t^-(X) = A_t^+(XB_t) = A_tXB_t$$

and

$$B_t^-A_t^+(X) = B_t^-(A_tX) = A_tXB_t$$

This implies that $A_t^+B_t^- = B_t^-A_t^+$.
Furthermore, we can put

$$B(H)^+ = \{A_t^+ : A_t \in B(H)\}$$

and

$$B(H)^- = \{A_t^- : A_t \in B(H)\}$$

Lemma 2.7. *The mappings $^+$ and $^-$ are one-to-one and onto mappings from Banach algebra $B(H)$ onto $B(B(H))$.*

Proof. Let $A_t \in B(H)$ .The mapping $^+$ is defined as

$$^+ : A_t \longmapsto A_tX \in B(B(H)), \forall X \in B(H)$$

Let $x_1, x_2 \in X$ such that

$$\begin{aligned} A_t(X) &= A_tx_1 \text{ and} \\ A_t(X) &= A_tx_2, \end{aligned}$$

then,

$$\begin{aligned} A_tx_1 - A_tx_2 &= 0 \\ A_t(x_1 - x_2) &= 0. \end{aligned}$$

Since A_t is not equal to zero, then $x_1 - x_2 = 0$, which implies that $x_1 = x_2$. Hence the mapping $^+$ is one-to-one.
Now, the mapping $^+$ maps an element $A_t \in B(H)$ to an element $A_tX \in B(B(H))$. Therefore, the mapping $^+$ is onto.
Similarly, the mapping $^-$ is a one-to-one and onto mapping from Banach algebra $B(H)$ onto $B(B(H))$. □

Theorem 2.8. *Let $\{A_t\}_{t\in\Omega}$ and $\{B_t\}_{t\in\Omega}$ be two commuting families of bounded self-adjoint operators on $B(H)$. If $T_{A,B}$ is as defined in Equation (2.1), then*

$$\sigma(T_{A,B}) = \{\int_\Omega \alpha_t\beta_t d\mu(t) : \alpha_t \in \sigma(A_t), \beta_t \in \sigma(B_t)\}.$$

Proof. Let $Z \subset B(B(H))$ and let Z contain the elements $\{A_t^+\}_{t\in\Omega}$ and $\{B_t^-\}_{t\in\Omega}$ which are pairwise commutative. Then, Z'', the double commutant of Z, is the smallest abelian Von Neumann algebra containing Z and I_t (the identity operator). Thus $Z \subset Z'' \subset B(B(H))$ [see [[11], P. 72] for more details on commutators. Let $\sigma(A_t^+, Z'') = \sigma(A_t^+, B(B(H)))$ and $\sigma(B_t^-, Z'') = \sigma(B_t^-, B(B(H)))$ for any two elements $A_t^+, B_t^- \in Z$.
Now suppose ϕ is a multiplicative linear functional on Z''. Then

$$\begin{aligned} \phi(T_{A,B}(X)) &= \phi\left(\int_\Omega A_t^+ B_t^- d\mu(t)\right) \\ &= \int_\Omega \phi\left(A_t^+ B_t^- d\mu(t)\right) \\ &= \int_\Omega \phi(A_t^+)\phi(B_t^-) d\mu(t) \end{aligned}$$

The ranges of $\phi(T_{A,B}), \phi(A_t^+)$ and $\phi(B_t^-)$ as ϕ ranges over all multiplicative linear functionals on Z'' are $\sigma(T_{A,B}, B(B(H))), \sigma(A_t^+, Z''), \sigma(B_t^-, Z'')$
Hence

$$\begin{aligned} \sigma(T_{A,B}, B(B(H))) &= \int_\Omega \sigma(A_t^+, Z'')\sigma(B_t^-, Z'') d\mu(t) \\ &= \int_\Omega \sigma(A_t^+, B(B(H)))\sigma(B_t^-, B(B(H))) d\mu(t) \end{aligned}$$

and by Lemma 2.4, we have

$$\begin{aligned} \sigma(T_{A,B}, B(B(H))) &= \int_\Omega \sigma(A_t, B(H))\sigma(B_t, B(H)) d\mu(t) \\ &= \{\int_\Omega \alpha_t \beta_t d\mu(t) : \alpha_t \in \sigma(A_t, B(H)), \beta_t \in \sigma(B_t, B(H))\} \end{aligned}$$

□

Next we give a relationship between the spectrum $\sigma(T_{A,B})$ of $T_{A,B}$ and its numerical range. The numerical range of the operator $T_{A,B}$ has been computed in [9].

Theorem 2.9. *The spectrum of the inner product type integral transformer is contained in the closure of its numerical range.*

Proof. To prove this, we use the fact that the boundary of the spectrum is contained in the approximate point spectrum $\sigma_{app}(T_{A,B})$ of $T_{A,B}$. Let $V(T_{A,B}) = \int_\Omega \langle A_t x, x\rangle\langle B_t y, y\rangle d\mu(t) = \int_\Omega \lambda_t\beta_t d\mu(t) = \alpha$, where $V(T_{A,B})$ is the algebraic numerical range of $T_{A,B}$, we first show that $\int_\Omega \lambda_t\beta_t d\mu(t) \in \sigma_{app}(T_{A,B})$, and it will be enough therefore to show that if $\int_\Omega \lambda_t\beta_t d\mu(t) = \alpha \in V(T_{A,B})$, then $\alpha \in \overline{W(T_{A,B})}$, where $W(T_{A,B})$ is the usual numerical range. Suppose that $\exists$ a sequence $\{f_n : n \in \mathbb{N}\}$ of unit vectors in H such that $\|f_n\| = 1$ and $\phi(T_{A,B}(f_n)) \longrightarrow \alpha$.

Now,

$$\begin{aligned}\|\phi(T_{A,B}-\alpha I)(f_n)\|^2 &= \|\phi(T_{A,B}(f_n)-\alpha(f_n))\|^2\\ &= \langle\phi(T_{A,B}(f_n)-\alpha(f_n)),\phi(T_{A,B}(f_n)-\alpha(f_n))\rangle\\ &= \langle\phi(T_{A,B}(f_n)),\phi(T_{A,B}(f_n))\rangle-\langle\phi(T_{A,B}(f_n)),\alpha(f_n)\rangle\\ &- \langle\alpha(f_n),\phi(T_{A,B}(f_n))\rangle+\langle\alpha(f_n),\alpha(f_n)\rangle\\ &= \|\phi(T_{A,B}(f_n))\|^2-2\|\alpha\|^2+\|\alpha(f_n)\|^2\\ &= |\alpha|^2-2|\alpha|^2+|\alpha|^2\longrightarrow 0\end{aligned}$$

where the convergence is caused by the convergence of the components and the equality is caused by the linearity of ϕ. Therefore, the convergence above implies that $\alpha=\int_\Omega\lambda_t\beta_t d\mu(t)\in\sigma_{app}(T_{A,B})$. By the definition of the approximate point spectrum, we can deduce that $\sigma_{app}(T_{A,B})=\int_\Omega\lambda_t\beta_t d\mu(t)$.
Since $\|\phi(T_{A,B}-\alpha I)(f_n)\|^2\longrightarrow 0$, with $\{f_n\}\neq 0$, it implies that $\phi(T_{A,B}-\alpha)\longrightarrow 0$, therefore, $\alpha=\int_\Omega\langle A_tx,x\rangle\langle B_ty,y\rangle d\mu(t)\in\sigma(T_{A,B})$. Since each $\langle A_tx,x\rangle,\langle B_ty,y\rangle$ lies in the $W(A_t)$ and $W(B_t)$ respectively, it follows that $\int_\Omega\langle A_tx,x\rangle\langle B_ty,y\rangle d\mu(t)\in\overline{W(T_{A,B})}$. Hence, $\sigma_{app}(T_{A,B})\subseteq\overline{W(T_{A,B})}$ and so $\sigma(T_{A,B})=\sigma_{app}(T_{A,B})\subseteq\overline{W(T_{A,B})}$. □

Acknowledgement. The first author is grateful to DAAD for their financial support through their In-Country Scholarship.

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