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PROPERTY (Baw_1) AND WEYL TYPE THEOREMS

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Abstract. In this paper, we study property (Baw_1) for bounded linear operators defined by Zariouh and Lombarkia in [12] as a variant of Weyl type theorem. We discuss property (Baw_1) for operators satisfying single valued extension property (SVEP). We explore conditions related to the direct summands to ensure the extension of the property (Baw_1) from $\mathfrak{I}$ and $\mathfrak{S}$ to their direct sum $\mathfrak{I} \oplus \mathfrak{S}$.

1. Introduction and preliminaries

Let $\mathfrak{B}(\mathfrak{X})$ stand for the Banach algebra of all bounded linear operators on an infinite-dimensional complex Banach space $\mathfrak{X}$. For $\mathfrak{I} \in \mathfrak{B}(\mathfrak{X})$, let $N(\mathfrak{I})$, $R(\mathfrak{I})$, $\sigma(\mathfrak{I})$ and $A_\sigma(\mathfrak{I})$ represent, in order, null space, range space, spectrum and approximate spectrum. Let $\alpha(\mathfrak{I}) = dim\, N(\mathfrak{I})$ be the nullity and $\beta(\mathfrak{I}) = codim\, R(\mathfrak{I})$ be the deficiency of $\mathfrak{I}$. If the range $R(\mathfrak{I})$ of $\mathfrak{I}$ is closed and $\alpha(\mathfrak{I}) < \infty$ (resp. $\beta(\mathfrak{I}) < \infty$), then $\mathfrak{I}$ is referred to as an upper (resp., a lower) semi-Fredholm operator. If $\mathfrak{I} \in B(\mathfrak{X})$ is either an upper or a lower semi-Fredholm, then $\mathfrak{I}$ is referred to as a semi-Fredholm operator and the index of $\mathfrak{I}$ is defined by $ind(\mathfrak{I}) = \alpha(\mathfrak{I}) - \beta(\mathfrak{I})$. An operator $\mathfrak{I} \in B(\mathfrak{X})$ is referred to as a Fredholm operator if $\mathfrak{I}$ is upper as well as lower semi-Fredholm. If $\mathfrak{I} \in B(\mathfrak{X})$ is a Fredholm operator of index zero, it is referred to as a Weyl operator. $W_\sigma(\mathfrak{I}) = \{\lambda \in \mathbb{C} : \mathfrak{I} - \lambda\mathfrak{T} \text{ is not Weyl}\}$ defines the Weyl spectrum of $\mathfrak{I}$.

For a nonnegative integer n, let $\mathfrak{I}_n$ be the restriction of a bounded linear operator $\mathfrak{I} \in B(\mathfrak{X})$ to $R(\mathfrak{I}^n)$ regarded as a map from $R(\mathfrak{I}^n)$ into itself (more specifically $(\mathfrak{I}_0 = \mathfrak{I})$. When $\mathfrak{I}_n$ is an upper (or lower) semi-Fredholm operator and the range space $R(\mathfrak{I}_n)$ is closed for some integer n, $\mathfrak{I}$ is referred to as an upper (or lower) semi-B-Fredholm operator. An upper or lower semi-B-Fredholm operator is a semi-B-Fredholm operator. In [6, Proposition 2.1], Berkani and Sarih mentioned that if $\mathfrak{I}_n$ is a semi-Fredholm operator then $\mathfrak{I}_m$ is also a semi-Fredholm operator for each $m \geq n$ and $ind(\mathfrak{I}_m) = ind(\mathfrak{I}_n)$ (see [5, 6]). Thus, index of the semi-Fredholm operator $\mathfrak{I}_n$ is referred as the index of a semi-B-Fredholm operator $\mathfrak{I}$. If an operator $\mathfrak{I} \in B(\mathfrak{X})$ is a B-Fredholm operator with

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index 0, it is referred to as B-Weyl operator. The B-Weyl spectrum $BW_\sigma(\mathfrak{I})$ of $\mathfrak{I}$ is defined as $BW_\sigma(\mathfrak{I}) = \{\lambda \in \mathbb{C} : \mathfrak{I} - \lambda\mathfrak{T}$ is not B-Weyl operator$\}$.

Let $p(\mathfrak{I}) := asc(\mathfrak{I})$ be the ascent of an operator $\mathfrak{I}$ i.e., the smallest nonnegative integer n such that $N(\mathfrak{I}^n) = N(\mathfrak{I}^{n+1})$. We set $asc(\mathfrak{I}) = \infty$, if such an integer does not exist. Analogously, let $q(\mathfrak{I}) := dsc(\mathfrak{I})$ be the descent of an operator $\mathfrak{I}$ i.e. the smallest nonnegative integer such that $R(\mathfrak{I}^n) = R(\mathfrak{I}^{n+1})$. We set $dsc(\mathfrak{I}) = \infty$, if such an integer does not exist. If $p(\mathfrak{I})$ and $q(\mathfrak{I})$ are both finite then $p(\mathfrak{I}) = q(\mathfrak{I})$. If an operator $\mathfrak{I}$ has finite ascent and descent, it is called Drazin invertible. The Drazin spectrum of $\mathfrak{I}$ is defined by $D_\sigma(\mathfrak{I}) = \{\lambda \in \mathbb{C} : \mathfrak{I} - \lambda\mathfrak{T}$ is not Drazin invertible$\}$. We note that $D_\sigma(\mathfrak{I}) = \sigma(\mathfrak{I}) \setminus \pi(\mathfrak{I})$, where $\pi(\mathfrak{I})$ is the set of poles of $\mathfrak{I}$. We denote by $E(\mathfrak{I})$, $E_a(\mathfrak{I})$ the eigenvalues of $\mathfrak{I}$ that are isolated in the spectrum $\sigma(\mathfrak{I})$ and approximate point spectrum $A_\sigma(\mathfrak{I})$ respectively and by $E^0(\mathfrak{I})$, $E^0_a(\mathfrak{I})$ the eigenvalues of finite multiplicity of $\mathfrak{I}$ that are isolated in the spectrum $\sigma(\mathfrak{I})$ and approximate point spectrum $A_\sigma(\mathfrak{I})$ respectively. We recall the definition of some required theorems and properties. These have been studied in ([2, 3, 4, 5, 6, 12]) and the references therein.

Weyl's theorem	$\sigma(\mathfrak{I}) \setminus W_\sigma(\mathfrak{I}) = E^0(\mathfrak{I})$	Browder's theorem	$\sigma(\mathfrak{I}) \setminus W_\sigma(\mathfrak{I}) = \pi^0(\mathfrak{I})$
Generalized Weyl's theorem	$\sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I}) = E(\mathfrak{I})$	Generalized Browder's theorem	$\sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I}) = \pi(\mathfrak{I})$
Property (Baw)	$\sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I}) = E^0_a(\mathfrak{I})$	Property (Baw_1)	$\sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I}) \subset E^0_a(\mathfrak{I})$

In this paper, necessary and sufficient conditions are provided for a bounded linear operator defined on a Banach space for which the property (Baw_1) holds. If $\mathfrak{I}$ and $\mathfrak{S}$ satisfy property (Baw_1), then it is not necessary that their direct sum $\mathfrak{I} \oplus \mathfrak{S}$ satisfies property (Baw_1). We explore conditions on $\mathfrak{I}$ and S with which the property (Baw_1) transfers from direct summands to the direct sum. We can summarize the findings as follows: In the second section, we prove that $\mathfrak{I}$ satisfies property (Baw_1) if and only if generalized Browder's theorem holds for $\mathfrak{I}$ and $\pi(\mathfrak{I}) \subset E^0_a(\mathfrak{I})$. In the third section, we discuss the preservation of property (Baw_1) under direct sum, utilizing the union of B-Weyl spectra of its summands and under the assumption of equality of their point spectra.

2. Property (Baw_1)

The operator $\mathfrak{I} \in B(\mathfrak{X})$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0 \in \mathbb{C}$), if for every open disc U of λ_0 the only analytic function $f : U \to \mathfrak{X}$ which satisfies the equation $(\mathfrak{I} - \lambda\mathfrak{T})f(\lambda) = 0$ for all $\lambda \in U$, is the function $f = 0$. An operator $\mathfrak{I} \in B(\mathfrak{X})$ is said to have SVEP if $\mathfrak{I}$ has SVEP at every point $\lambda \in \mathbb{C}$. The single valued extension property was introduced by Dunford ([9, 10]) and it plays an important role in local spectral theory and Fredholm theory ([1], [11]). Duggal [8] gave the following important results:

Theorem 2.1 ([8, Proposition 3.9(a)]). *The following statements are equivalent.*

(i) $\mathfrak{I}$ *satisfies generalized Browder's theorem.*
(ii) $\mathfrak{I}$ *has SVEP at points* $\lambda \notin BW_{\sigma}(\mathfrak{I})$.

Remark 2.2. Duggal [8] proved that $\mathfrak{I}^*$ satisfies generalized Browder's theorem if and only if $\mathfrak{I}$ satisfies Browder's theorem as $\sigma(\mathfrak{I}) = \sigma(\mathfrak{I}^*)$, $BW_{\sigma}(\mathfrak{I}) = BW_{\sigma}(\mathfrak{I}^*)$ and $\pi(\mathfrak{I}) = \pi(\mathfrak{I}^*)$.

In this section, We investigate the relationships of the property (Baw_1) with other spectral properties and Weyl type theorems as property (Baw) and generalized Browder's theorem.

Theorem 2.3. *Property* (Baw) *holds for* $\mathfrak{I}$ *if and only if* $\mathfrak{I}$ *satisfies property* (Baw_1) *and* $BW_{\sigma}(\mathfrak{I}) \cap E_a^0(\mathfrak{I}) = \emptyset$.

Proof. Suppose that $\mathfrak{I}$ satisfies property (Baw), then property (Baw_1) holds for $\mathfrak{I}$ and $BW_{\sigma}(\mathfrak{I}) \cap E_a^0(\mathfrak{I}) = \emptyset$. For the converse, if $\lambda \in E_a^0(\mathfrak{I})$, $\lambda \notin BW_{\sigma}(\mathfrak{I})$ since $BW_{\sigma}(\mathfrak{I}) \cap E_a^0(\mathfrak{I}) = \emptyset$. Thus $\lambda \in \sigma(\mathfrak{I}) \setminus BW_{\sigma}(\mathfrak{I})$. Hence, it implies $E_a^0(\mathfrak{I}) \subseteq \sigma(\mathfrak{I}) \setminus BW_{\sigma}(\mathfrak{I})$. □

Thus property (Baw) implies property (Baw_1) but the converse need not be true. The following example shows that property (Baw_1) does not imply property (Baw) in general.

Example 2.4. Let R $\in L(l^2(\mathbb{N}))$ be the unilateral right shift and U be defined by $U(x_1, x_2, \dots) = (0, x_2, x_3, \dots)$, $(x_n) \in l^2(\mathbb{N})$. Let $\mathfrak{I} = R \oplus U$, then $\sigma(\mathfrak{I}) = D(0,1)$, $BW_{\sigma}(\mathfrak{I}) = D(0,1)$, $E_a^0(\mathfrak{I}) = \{0\}$. Thus property (Baw_1) is satisfied.

Theorem 2.5. *If* $\mathfrak{I} \in B(\mathfrak{X})$ *satisfies property* (Baw_1). *Then generalized Browder's theorem holds for* $\mathfrak{I}$ *and* $\sigma(\mathfrak{I}) = BW_{\sigma}(\mathfrak{I}) \cup E_a^0(T)$.

Proof. By Theorem 2.3, it is sufficient to prove that $\mathfrak{I}$ has SVEP at every $\lambda \notin BW_{\sigma}(\mathfrak{I})$. Let us assume that $\lambda \notin BW_{\sigma}(\mathfrak{I})$.

Case (i): If $\lambda \notin \sigma(\mathfrak{I})$ then T has SVEP at λ.

Case (ii): If $\lambda \in \sigma(\mathfrak{I})$ and suppose that $\mathfrak{I}$ satisfies property (Baw_1) then $\lambda \in \sigma(\mathfrak{I}) \setminus BW_{\sigma}(\mathfrak{I}) \subset E_a^0(\mathfrak{I})$ and hence $\lambda \in iso\, \sigma_a(\mathfrak{I})$, so also in this case $\mathfrak{I}$ has SVEP at λ.

To prove that $\sigma(\mathfrak{I}) = BW_{\sigma}(\mathfrak{I}) \cup E_a^0(\mathfrak{I})$, we observe that $BW_{\sigma}(\mathfrak{I}) \cup E_a^0(\mathfrak{I}) \subseteq \sigma(\mathfrak{I})$ for every $\mathfrak{I} \in B(X)$. For the reverse inclusion, consider $\lambda \in \sigma(\mathfrak{I})$. If $\lambda \notin BW_{\sigma}(\mathfrak{I})$ then $\lambda \in \sigma(\mathfrak{I}) \setminus BW_{\sigma}(\mathfrak{I})$. As T satisfies property (Baw_1), $\lambda \in E_a^0(\mathfrak{I})$. It implies that $\sigma(\mathfrak{I}) \subseteq BW_{\sigma}(\mathfrak{I}) \cup E_a^0(\mathfrak{I})$. Thus $\sigma(\mathfrak{I}) = BW_{\sigma}(\mathfrak{I}) \cup E_a^0(\mathfrak{I})$. □

Now we give a characterization of property (Baw_1):

Theorem 2.6. *If* $\mathfrak{I} \in B(\mathfrak{X})$, *then the following statements are equivalent:*

(i) $\mathfrak{I}$ *satisfies property* (Baw_1),
(ii) *generalized Browder's theorem holds for* $\mathfrak{I}$ *and* $\pi(\mathfrak{I}) \subset E_a^0(\mathfrak{I})$.

Proof. (i)⇒(ii): Assume that $\mathfrak{I}$ satisfies property (Baw_1). By Theorem 2.5 it is sufficient to prove the inequality $\pi(\mathfrak{I}) \subset E_a^0(\mathfrak{I})$. Let $\lambda \in \pi(\mathfrak{I}) = \sigma(\mathfrak{I}) \setminus BW_{\sigma}(\mathfrak{I}) \subset E_a^0(\mathfrak{I})$.

(ii)$\Rightarrow$(i): If $\lambda \in \sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I})$. Then generalized Browder's theorem implies that $\lambda \in \pi(\mathfrak{I}) \subset E_a^0(\mathfrak{I})$. Thus $\sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I}) \subset E_a^0(\mathfrak{I})$. Thus property (Baw_1) is satisfied. □

Theorem 2.7. *Let $\mathfrak{I} \in B(\mathfrak{X})$. If $\mathfrak{I}$ or $\mathfrak{I}^*$ has SVEP at $\sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I})$, then $\mathfrak{I}$ satisfies property (Baw_1) if and only if $\pi(\mathfrak{I}) \subset E_a^0(\mathfrak{I})$.*

Proof. The hypothesis $\mathfrak{I}$ or $\mathfrak{I}^*$ has SVEP at $\sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I}) = \sigma(\mathfrak{I}^*) \setminus BW_\sigma(\mathfrak{I}^*)$ implies that $\mathfrak{I}$ satisfies generalized Browder's theorem (see Theorem 2.3 and Remark 2.2).
Hence if $\pi(\mathfrak{I}) \subset E_a^0(\mathfrak{I})$ then $\sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I}) = \pi(\mathfrak{I}) \subset E_a^0(\mathfrak{I})$. □

Definition 2.8. Operators $\mathfrak{S}, \mathfrak{I} \in B(\mathfrak{X})$ are said to be injectively interwined, denoted, $\mathfrak{S} \prec_i \mathfrak{I}$, if there exists an injection $U \in B(\mathfrak{X})$ such that $\mathfrak{I}U = U\mathfrak{S}$.
If $\mathfrak{S} \prec_i \mathfrak{I}$, then $\mathfrak{I}$ has SVEP at a point λ implies $\mathfrak{S}$ has SVEP at λ. To prove this, let $\mathfrak{I}$ has SVEP at λ, let U be an open neighbourhood of λ and let $f : U \to X$ be an analytic function such that $(\mathfrak{S} - \mu)f(\mu) = 0$ for every $\mu \in U$.
Then $U(\mathfrak{S} - \mu)f(\mu) = (\mathfrak{I} - \mu)Uf(\mu) = 0 \Rightarrow Uf(\mu) = 0$. As U is injective, $f(\mu) = 0$, i.e. $\mathfrak{S}$ has SVEP at λ.

Theorem 2.9. *Let $\mathfrak{S}, \mathfrak{I} \in B(\mathfrak{X})$. If $\mathfrak{I}$ has SVEP and $\mathfrak{S} \prec_i \mathfrak{I}$, then property (Baw_1) holds for $\mathfrak{S}$ if and only if $\pi(\mathfrak{S}) \subset E_a^0(T)$.*

Proof. Suppose that $\mathfrak{I}$ has SVEP. Since $\mathfrak{S} \prec_i \mathfrak{I}$, therefore $\mathfrak{S}$ has SVEP. Hence the result follows from Theorem 2.7. □

In the following results, we study the stability of property (Baw) under commutative perturbations.

Theorem 2.10. *Let $\mathfrak{I} \in B(\mathfrak{X})$. If $\mathfrak{I}$ has property (Baw) and F is a finite rank operator in $B(\mathfrak{X})$ that commutes with $\mathfrak{I}$, then $\mathfrak{I} + F$ has property (Baw) if and only if $\pi(\mathfrak{I} + F) = E_a^0(\mathfrak{I} + F)$.*

Proof. If $\mathfrak{I}+F$ has property (Baw) then $\pi(\mathfrak{I}+F) = E_a^0(\mathfrak{I}+F)$. Conversely, if $\pi(\mathfrak{I}+F) = E_a^0(\mathfrak{I}+F)$. As F is a finite rank operator, we have $BW_\sigma(\mathfrak{I}) = BW_\sigma(\mathfrak{I}+F)$ [4, Theorem 4.3]. Since F commutes with $\mathfrak{I}$, therefore $D_\sigma(\mathfrak{I}) = D_\sigma(\mathfrak{I} + F)$ [3, Theorem 2.7]. As $\mathfrak{I}$ satisfies generalized Browder's theorem, therefore $BW_\sigma(\mathfrak{I}) = D_\sigma(\mathfrak{I})$. Now $\sigma(\mathfrak{I}+F) \setminus BW_\sigma(\mathfrak{I}+F) = \sigma(\mathfrak{I}+F) \setminus D_\sigma(\mathfrak{I}+F) = \pi(\mathfrak{I}+F) = E_a^0(\mathfrak{I}+F)$. Therefore, $\mathfrak{I} + F$ satisfies property (Baw). □

Theorem 2.11. *Let $\mathfrak{I} \in B(\mathfrak{X})$ and let N be a nilpotent operator commuting with $\mathfrak{I}$. If $\mathfrak{I}$ satisfies property (Baw), then $\mathfrak{I} + N$ satisfies property (Baw) if and only if $BW_\sigma(\mathfrak{I} + N) = BW_\sigma(\mathfrak{I})$.*

Proof. Assume that $\mathfrak{I} + N$ satisfies property (Baw), then $\sigma(\mathfrak{I} + N) \setminus BW_\sigma(\mathfrak{I} + N) = E_a^0(\mathfrak{I} + N)$. As $A_\sigma(\mathfrak{I} + N) = A_\sigma(T)$ and $E_a^0(\mathfrak{I} + N) = E_a^0(\mathfrak{I})$ [7, Lemma 3.1], then, $\sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I} + N) = E_a^0(\mathfrak{I})$. Since $\mathfrak{I}$ satisfies property (Baw), then $\sigma(\mathfrak{I}) \setminus BW_\sigma(\mathfrak{I}) = E_a^0(\mathfrak{I})$. Therefore $BW_\sigma(\mathfrak{I} + N) = BW_\sigma(\mathfrak{I})$. Conversely assume that $BW_\sigma(\mathfrak{I} + N) = BW_\sigma(\mathfrak{I})$, then as $\mathfrak{I}$ satisfies property (Baw), it follows that $\mathfrak{I} + N$ also satisfies property (Baw). □

3. Property (Baw_1) for Direct Sum

In this section, we discuss the stability of property (Baw_1) under direct sum of operators. For the purposes of this discussion and future references, the sets of all isolated and accumulation points of $A_\sigma(\mathfrak{I})$ will be denoted by $iso\, A_\sigma(\mathfrak{I})$ and $acc\, A_\sigma(\mathfrak{I})$, respectively.

Theorem 3.1. *Let $\mathfrak{I} \in B(\mathfrak{X})$ and $\mathfrak{S} \in B(\mathfrak{Y})$ are such that $\sigma^0_P(\mathfrak{I}) = \sigma^0_P(\mathfrak{S})$. If $\mathfrak{I}$ and $\mathfrak{S}$ both possess property (Baw_1), then $\mathfrak{I} \oplus \mathfrak{S}$ possesses property (Baw_1) if and only if*

$$BW_\sigma(\mathfrak{I} \oplus \mathfrak{S}) = BW_\sigma(\mathfrak{I}) \cup BW_\sigma(\mathfrak{S}).$$

Proof. Suppose that $BW_\sigma(\mathfrak{I}\oplus\mathfrak{S}) = BW_\sigma(\mathfrak{I})\cup BW_\sigma(\mathfrak{S})$. As $\mathfrak{I}$ and $\mathfrak{S}$ both satisfy property (Baw_1), we have

$$\begin{aligned}\sigma(\mathfrak{I} \oplus \mathfrak{S}) \setminus BW_\sigma(\mathfrak{I} \oplus \mathfrak{S}) &= [\sigma(\mathfrak{I}) \cup \sigma(\mathfrak{S})] \setminus [BW_\sigma(\mathfrak{I}) \cup BW_\sigma(\mathfrak{S})] \\ &\subset [E^0_a(\mathfrak{I}) \cap \rho(\mathfrak{S})] \cup [E^0_a(\mathfrak{S}) \cap \rho(\mathfrak{I})] \cup [E^0_a(\mathfrak{I}) \cap E^0_a(\mathfrak{S})].\end{aligned}$$

Since $\sigma^0_P(\mathfrak{I}) = \sigma^0_P(\mathfrak{S})$, we have $E^0_a(\mathfrak{I}) \cap \rho\,(\mathfrak{S}) = \emptyset$ and $E^0_a(\mathfrak{S}) \cap \rho(\mathfrak{I}) = \emptyset$. Thus $\sigma(\mathfrak{I} \oplus \mathfrak{S}) \setminus BW_\sigma(\mathfrak{I} \oplus \mathfrak{S}) \subset\ E^0_a(\mathfrak{I}) \cap E^0_a(\mathfrak{S})$.
We know that

$$\begin{aligned}E^0_a(\mathfrak{I} \oplus \mathfrak{S}) &= iso\, A_\sigma(\mathfrak{I} \oplus \mathfrak{S}) \cap \sigma^0_P(\mathfrak{I} \oplus \mathfrak{S}) \\ &= iso[A_\sigma(\mathfrak{I}) \cup A_\sigma(\mathfrak{S})] \cap [\sigma^0_P(\mathfrak{I}) \cup \sigma^0_P(\mathfrak{S})] \\ &= [E^0_a(\mathfrak{I}) \cap A_\rho(\mathfrak{S})] \cup [E^0_a(\mathfrak{S}) \cap A_\rho(\mathfrak{I})] \cup [E^0_a(\mathfrak{I}) \cap iso\, \sigma_a(\mathfrak{S})] \\ &\quad \cup [E^0_a(\mathfrak{S}) \cap iso\, \sigma_a(\mathfrak{I})] \\ &= E^0_a(\mathfrak{I}) \cap E^0_a(\mathfrak{S}),\end{aligned}$$

as $E^0_a(\mathfrak{I}) \cap A_\rho(\mathfrak{S}) = \emptyset$ and $E^0_a(\mathfrak{S}) \cap A_\rho(\mathfrak{I}) = \emptyset$.
Hence, $\sigma(\mathfrak{I} \oplus \mathfrak{S}) \setminus BW_\sigma(\mathfrak{I} \oplus \mathfrak{S}) \subset E^0_a(\mathfrak{I} \oplus \mathfrak{S})$.
Conversely, if $\mathfrak{I}$, $\mathfrak{S}$ and $\mathfrak{I} \oplus \mathfrak{S}$ satisfy property (Baw_1), then $BW_\sigma(\mathfrak{I} \oplus \mathfrak{S}) = BW_\sigma(\mathfrak{I}) \cup BW_\sigma(\mathfrak{S})$. To prove this, we have from Theorem 2.5 that $\mathfrak{I}$, $\mathfrak{S}$ and $\mathfrak{I} \oplus \mathfrak{S}$ satisfy generalized Browder's theorem. Thus $BW_\sigma(\mathfrak{I} \oplus \mathfrak{S}) = D_\sigma(\mathfrak{I} \oplus \mathfrak{S})$, $BW_\sigma(\mathfrak{I}) = D_\sigma(\mathfrak{I})$, $BW_\sigma(\mathfrak{S}) = D_\sigma(\mathfrak{S})$. As $D_\sigma(\mathfrak{I} \oplus \mathfrak{S}) = D_\sigma(\mathfrak{I}) \cup D_\sigma(\mathfrak{S})$ is always true, $BW_\sigma(\mathfrak{I} \oplus \mathfrak{S}) = D_\sigma(\mathfrak{I} \oplus \mathfrak{S}) = D_\sigma(\mathfrak{I}) \cup D_\sigma(\mathfrak{S}) = BW_\sigma(\mathfrak{I}) \cup BW_\sigma(\mathfrak{S})$. □

We have that $BW_\sigma(\mathfrak{I} \oplus \mathfrak{S}) = BW_\sigma(\mathfrak{I}) \cup BW_\sigma(\mathfrak{S})$.

Theorem 3.2. *Suppose $\mathfrak{I} \in B(\mathfrak{X})$ has no isolated point in its approximate spectrum and $\mathfrak{S} \in B(\mathfrak{Y})$ satisfies property (Baw_1). If $\sigma_{BW}(\mathfrak{I} \oplus \mathfrak{S}) = \sigma(\mathfrak{I}) \cup \sigma_{BW}(\mathfrak{S})$, then property ($Baw_1$) holds for $\mathfrak{I} \oplus \mathfrak{S}$.*

Proof. As $\sigma(\mathfrak{I} \oplus \mathfrak{S}) = \sigma(\mathfrak{I}) \cup \sigma(\mathfrak{S})$ for any pair of operators.

Thus

$$\begin{aligned}\sigma(\mathfrak{I}\oplus\mathfrak{S})\setminus BW_\sigma(\mathfrak{I}\oplus\mathfrak{S}) &= [\sigma(\mathfrak{I})\cup\sigma(\mathfrak{S})]\setminus[\sigma(\mathfrak{I})\cup BW_\sigma(\mathfrak{S})]\\ &= \sigma(\mathfrak{S})\setminus[\sigma(\mathfrak{I})\cup BW_\sigma(\mathfrak{S})]\\ &= [\sigma(\mathfrak{S})\setminus BW_\sigma(\mathfrak{S})]\setminus\sigma(\mathfrak{I})\\ &\subset E_a^0(\mathfrak{S})\cap\rho(\mathfrak{I})\\ &\subseteq E_a^0(\mathfrak{S})\cap A_\rho(\mathfrak{I})\end{aligned}$$

where $\rho(\cdot)=\mathbb{C}\setminus\sigma(\cdot)$ and $A_\rho(\cdot)=\mathbb{C}\setminus A_\sigma(\cdot)$.
Now $iso\, A_\sigma(\mathfrak{I}\oplus\mathfrak{S})$ is the set of isolated points of $A_\sigma(\mathfrak{I}\oplus\mathfrak{S})=A_\sigma(\mathfrak{I})\cup A_\sigma(\mathfrak{S})$. If $iso\, A_\sigma(\mathfrak{I})=\phi$, it implies that $A_\sigma(\mathfrak{I})=acc\, A_\sigma(\mathfrak{I})$, where $acc\, A_\sigma(\mathfrak{I})=A_\sigma(\mathfrak{I})\setminus iso\, A_\sigma(\mathfrak{I})$ is the set of all accumulation points of $A_\sigma(\mathfrak{I})$. Thus, we have

$$\begin{aligned}iso\, A_\sigma\,(\mathfrak{I}\oplus\mathfrak{S}) &= [iso\, A_\sigma(\mathfrak{I})\cup iso\, A_\sigma(\mathfrak{S})]\setminus[(iso\, A_\sigma(\mathfrak{I})\cap acc\, A_\sigma(\mathfrak{S}))\\ &\qquad\cup(acc\, A_\sigma(\mathfrak{I})\cap iso\, A_\sigma(\mathfrak{S}))]\\ &= (iso\, A_\sigma(\mathfrak{I})\setminus acc\, A_\sigma(\mathfrak{S}))\cup(iso\, A_\sigma(\mathfrak{S})\setminus acc\, A_\sigma(\mathfrak{I}))\\ &= iso\, A_\sigma(\mathfrak{S})\setminus A_\sigma(\mathfrak{I})\\ &= iso\, A_\sigma(\mathfrak{S})\cap A_\rho(\mathfrak{I}).\end{aligned}$$

Let $\sigma_P(\mathfrak{I})$ denote the point spectrum of $\mathfrak{I}$ and $\sigma_P^0(\mathfrak{I})$ denote the set of all eigen values of $\mathfrak{I}$ of finite multiplicity.
We have that $\sigma_P(\mathfrak{I}\oplus\mathfrak{S})=\sigma_P(\mathfrak{I})\cup\sigma_P(\mathfrak{S})$ and $dim\, N(\mathfrak{I}\oplus\mathfrak{S})=dim\, N(\mathfrak{I})+dim\, N(\mathfrak{S})$ for every pair of operators, so that

$$\sigma_P^0(\mathfrak{I}\oplus\mathfrak{S})=\{\lambda\in\sigma_P^0(\mathfrak{I})\cup\sigma_P^0(\mathfrak{S}):dim\, N(\lambda I-\mathfrak{I})+dim\, N(\lambda I-\mathfrak{S})<\infty\}.$$

We have that $\sigma_P^0(\mathfrak{I}\oplus\mathfrak{S})=\sigma_P^0(\mathfrak{I})\cup\sigma_P^0(\mathfrak{S})$.
Therefore

$$\begin{aligned}E_a^0(\mathfrak{I}\oplus\mathfrak{S}) &= iso\, A_\sigma(\mathfrak{I}\oplus\mathfrak{S})\cap\sigma_P^0(\mathfrak{I}\oplus\mathfrak{S})\\ &= iso\, A_\sigma(\mathfrak{S})\cap A_\rho(\mathfrak{I})\cap\sigma_P^0(\mathfrak{S})\\ &= E_a^0(\mathfrak{S})\cap A_\rho(\mathfrak{I}).\end{aligned}$$

Thus $\sigma(\mathfrak{I}\oplus\mathfrak{S})\setminus BW_\sigma(\mathfrak{I}\oplus\mathfrak{S})\subset E_a^0(\mathfrak{I}\oplus V)$. Hence $\mathfrak{I}\oplus\mathfrak{S}$ satisfies property (Baw_1). $\square$

Let $\sigma_1(\mathfrak{I})$ denote the compliment of $BW_\sigma(\mathfrak{I})$ in $\sigma(\mathfrak{I})$ i.e., $\sigma_1(\mathfrak{I})=\sigma(\mathfrak{I})\setminus BW_\sigma(\mathfrak{I})$. Following corollaries can be obtained by application of Theorem 3.2.

Corollary 3.3. *Suppose $\mathfrak{I}\in B(\mathfrak{X})$ is such that $iso\, A_\sigma(\mathfrak{I})=\phi$ and $\mathfrak{S}\in B(\mathfrak{Y})$ satisfies property (Baw_1) with $iso\, A_\sigma(\mathfrak{S})\cap\sigma_P^0(\mathfrak{S})=\phi$ and $\sigma_1(\mathfrak{I}\oplus\mathfrak{S})=\varphi$, then $\mathfrak{I}\oplus\mathfrak{S}$ satisfies property (Baw_1).*

Proof. As $\mathfrak{S}$ satisfies property (Baw_1), therefore given condition $iso\, A_\sigma(\mathfrak{S})\cap\sigma_P^0(\mathfrak{S})=\phi$ implies that $\sigma(\mathfrak{S})=BW_\sigma(\mathfrak{S})$. The condition $\sigma_1(\mathfrak{I}\oplus\mathfrak{S})=\phi$ gives that $\sigma(\mathfrak{I}\oplus\mathfrak{S})=BW_\sigma(\mathfrak{I}\oplus\mathfrak{S})=\sigma(\mathfrak{I})\cup BW_\sigma(\mathfrak{S})$. Using Theorem 3.2, we have that $\mathfrak{I}\oplus\mathfrak{S}$ satisfies property (Baw_1). $\square$

Corollary 3.4. *Suppose* $\mathfrak{I} \in B(\mathfrak{X})$ *is such that* $\sigma_1(\mathfrak{I}) \cup iso\, A_\sigma(\mathfrak{I}) = \phi$ *and* $\mathfrak{S} \in B(\mathfrak{Y})$ *satisfies property* (Baw_1). *If* $BW_\sigma(\mathfrak{I} \oplus \mathfrak{S}) = BW_\sigma(\mathfrak{I}) \cup BW_\sigma(\mathfrak{S})$, *then* $\mathfrak{I} \oplus \mathfrak{S}$ *satisfies property* (Baw_1).

4. Conclusion

The transfer of property (Baw_1) from individual operators to their orthogonal direct sum is not automatic. However, under specific conditions discussed in the paper, property (Baw_1) transfers from the direct summands to the direct sum as well.

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