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TRAVELING WAVE SOLUTIONS OF A SUSCEPTIBLE-INFECTIOUS MODEL

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Abstract. This paper studies the traveling wave solutions of a susceptible and infectious (SI) mathematical model with and without recruitment rates. Our research provides numerical solutions for the proposed models, confirming the existence of traveling wave solutions. We meticulously calculate the minimal traveling wave speeds and analytically determine the spreading speed without turnover for the susceptible population. The paper also investigates the relationship between the spreading speeds and the model parameters. Additionally, we identify the threshold density of susceptible individuals, a crucial point below which the disease cannot persist. Our findings also confirm that the disease ceases to exist if the death rates surpass the rate of new cases of infections.

1. Introduction

Modeling infectious diseases has been widely used in mathematics. According to [6], Kermack and McKendrick consider the following model in 1927,

$$\frac{S(t)}{dt} = -\beta SI,$$
$$\frac{I(t)}{dt} = \beta SI - mI,$$
$$\frac{R(t)}{dt} = mI,$$

where the population divided into three states: susceptible individuals $S(t)$, infected individuals $I(t)$, and recovered individuals $R(t)$. Today, the literature is very rich in models that study infectious diseases; for instance, we refer to the following studies [1, 2, 3, 4, 5, 8, 9, 16, 20] and the references therein.

Propagation of traveling wave solutions is another issue widely discussed using infectious disease models [7, 10, 14, 15, 22]. The theory of the asymptotic speed of propagation can be used to study the behavior of an epidemic [11, 12, 13, 18, 19, 21]. The work by Alanazi et al. [3, 5] studies the theory of the asymptotic speed of propagation and applies it to the rabies epidemic that spread in Europe during the last century. Alanazi [1] discusses the spreading speeds of COVID-19 by applying the concepts of asymptotic propagation speed. See also Ruan [17]

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for an excellent survey about the theory of the asymptotic speed of propagation and infectious diseases.

This work studies the traveling wave solutions of a susceptible and infectious (SI) mathematical model with and without recruitment rates. The proposed models are solved numerically. We prove the existence of traveling wave solutions and calculate minimal traveling wave speeds. When we assume that no new people are recruited to the susceptible population, an equation for the spreading speed is analytically obtained. The relation between the spreading speeds and the model parameters is also discussed. We also find the threshold density of the susceptible individuals below which the disease cannot persist.

The plan of the paper is as follows. In section 2, we introduce a susceptible and infectious (SI) mathematical model with a recruitment rate. In section 4, we discuss the model without a recruitment rate and find analytically the asymptotic speeds of spread c^*. In sections 3 and 5, we show numerically the existence of traveling wave solutions of our proposed models. We discuss the results in section 6.

2. An SI model with recruitment rate

Assume $S(x,t)$ and $E(x,t)$ are the population densities of susceptible and infected individuals. Let Θ be the recruitment rate. The susceptible individuals transfer from the healthy state to the infected stage with rate α. m_S is the natural death rate of susceptible individuals, m_E is the infected individuals death rate, and m is the infected population recovery rate. D denotes the diffusion coefficient. Then, we assume the following system of partial differential equations describes the dynamics between the susceptible and infected individuals:

$$\begin{aligned} \partial_t S(x,t) &= \Theta - m_S S(x,t) - \alpha S(x,t)E(x,t), \\ \partial_t E(x,t) &= D\partial_x^2 E(x,t) + \alpha S(x,t)E(x,t) - mE(x,t) - m_E E(x,t), \end{aligned} \tag{2.1}$$

$x \in \mathbb{R}^n$ and $t > 0$. Let the initial conditions given by

$$S(x,0) = \hat{S}(x), \;\; E(x,0) = \hat{E}(x). \tag{2.2}$$

The boundary conditions are defined to be

$$E(x,t) = 0 \,\text{on}\, \partial\Omega \times (0,\infty). \tag{2.3}$$

3. Traveling wave solutions of the model (2.1)

The proposed model (2.1) is now in the form of ordinary differential equations as the following:

$$\begin{aligned} P'(x_i,t) &= \Theta - m_S S(x_i,t) - \alpha S(x_i,t)E(x_i,t), \\ E'(x_i,t) &= D\Big[\frac{E(x_{i-1},t) - 2E(x_i,t) + E(x_{i+1},t)}{\Delta x^2}\Big] \\ &+ \alpha S(x_i,t)E(x_i,t) - mE(x_i,t) - m_E E(x_i,t), \end{aligned} \tag{3.1}$$

where the index $i = 1, \ldots, N$. The initial conditions are assumed to be

$$S(x,0) = 1, \;\; E(x,0) = 0.1. \tag{3.2}$$

The boundary conditions are defined in (2.3). The approximated solutions of $S(x,t)$ and $E(x,t)$ are presented in Fig. 1–2. Fig. 1–2 provide that the approximated wave speed is $c = 0.41$.

Also, we simulate the model (2.1) when the natural death rates of the susceptible and infected individuals are set to be zeros, i.e., $m_S = m_E = 0$. Therefore, the model takes the form,

$$\begin{aligned} P'(x_i,t) &= \Theta - \alpha S(x_i,t)E(x_i,t), \\ E'(x_i,t) &= D\Big[\frac{E(x_{i-1},t) - 2E(x_i,t) + E(x_{i+1},t)}{\Delta x^2}\Big] \\ &+ \alpha S(x_i,t)E(x_i,t) - mE(x_i,t), \end{aligned} \tag{3.3}$$

The approximated solutions of the model (3.3) are given in Fig. 3–4. By Fig. 3–4, the wave speed is about $c = 0.5$.

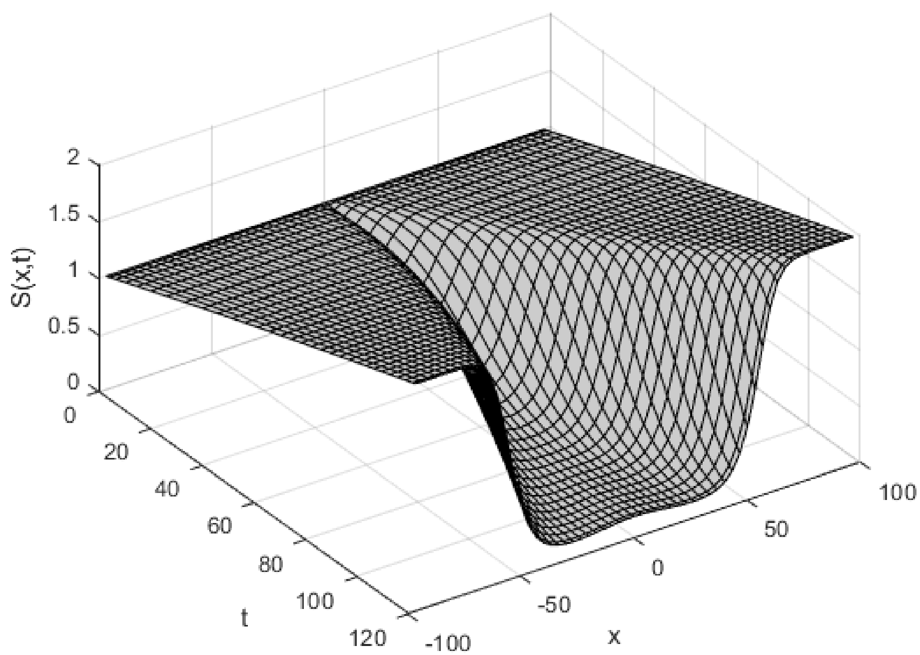

FIGURE 1. The numerical solution of $S(x,t)$ of the model (3.1). Here, $\Theta = 0.01, m_S = 0.001, \alpha = 0.1, D = 0.7, m = 0.01$, and $m_E = 0.02$.

4. AN SI MODEL WITHOUT RECRUITMENT RATE

In this part, we assume that the recruitment rate and the natural death rate for the susceptible are zero. Therefore, the model (2.1) is now of the form:

$$\begin{aligned} \partial_t S(x,t) &= -\alpha S(x,t)E(x,t), \\ \partial_t E(x,t) &= D\partial_x^2 E(x,t) + \alpha S(x,t)E(x,t) - mE(x,t) - m_E E(x,t), \end{aligned} \tag{4.1}$$

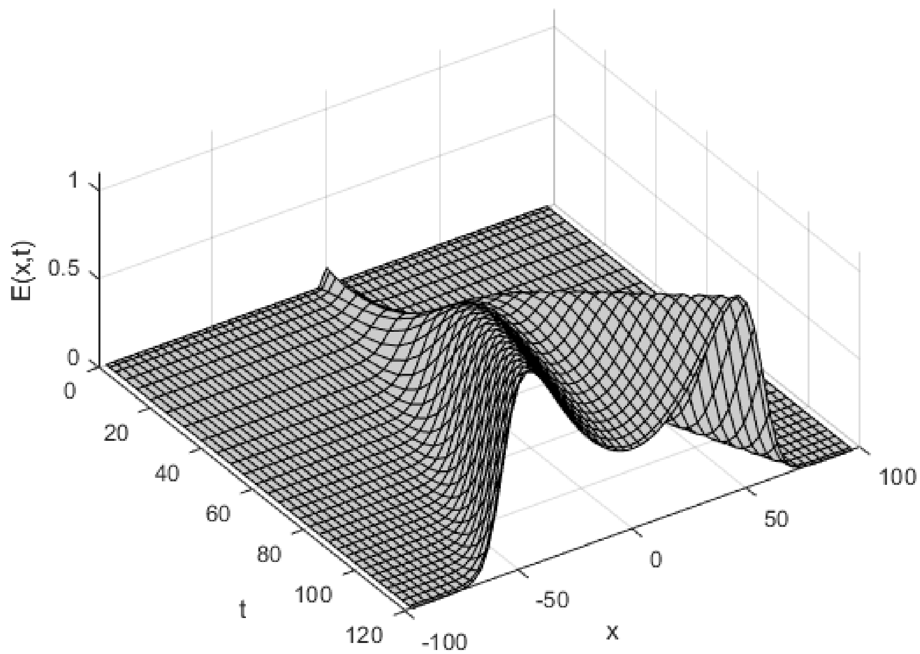

FIGURE 2. The numerical solution of $E(x,t)$ of the model (3.1). Here, $\Theta = 0.01, m_S = 0.001, \alpha = 0.1, D = 0.7, m = 0.01$, and $m_E = 0.02$.

$x \in \mathbb{R}^n$ and $t > 0$. For $m_E = 0$, the model discussed by many authors, see [10, 16]. The initial and boundary conditions are given by Let the initial conditions given by

$$\begin{aligned} &S(x,0) = \hat{S},\ \ E(x,0) = \hat{E}, \\ &E(x,t) = 0 \,\text{on}\ \partial\Omega \times (0,\infty), \end{aligned} \tag{4.2}$$

where $\hat{S}$ and $\hat{E}$ are constants. However, we want here to find the asymptotic speeds of spread c^* of the system (4.1). The model can be rewritten as

$$\begin{aligned} \partial_t S(x,t) &= -\alpha S(x,t)E(x,t), \\ \partial_t E(x,t) &= D\partial_x^2 E(x,t) - \partial_t S(x,t) - \rho E(x,t), \end{aligned} \tag{4.3}$$

where $\rho = m + m_E$. The second equation provides the following solution,

$$\begin{aligned} \alpha E(x,t) = &- \alpha \int_0^t \int_{\mathbb{R}^n} \Gamma(Ds, x-y)\partial_t S(y,t-s)e^{-\rho s} dyds \\ &+ \alpha \int_{\mathbb{R}^n} \Gamma(Dt, x-y)\hat{E}e^{-\rho t} dy. \end{aligned} \tag{4.4}$$

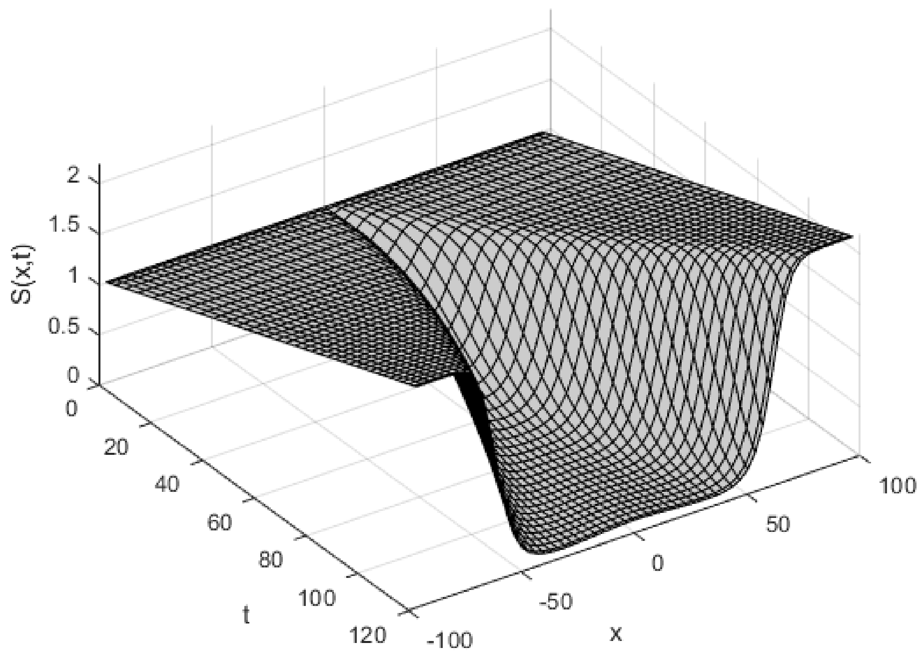

FIGURE 3. The numerical solution of $S(x,t)$ of the system (3.3). Here, $\Theta = 0.01, \alpha = 0.1, D = 0.7$, and $m = 0.01$.

Integrating both sides gives us the following equation

$$\begin{aligned} \alpha \int_0^t E(x,\nu)d\nu = & -\alpha \int_0^t \int_0^t \int_{\mathbb{R}^n} \Gamma(Ds, x-y)\partial_\nu S(y,\nu - s)e^{-\rho s} dyds d\nu \\ & + \alpha \int_0^t \int_{\mathbb{R}^n} \Gamma(Dt, x-y)\hat{E}e^{-\rho t} dy d\nu. \end{aligned} \tag{4.5}$$

We could interchange the order of integration and solve the integral $\left(-\int_0^t \partial_\nu S(y,\nu)d\nu\right)$, then we will get

$$\begin{aligned} \alpha \int_0^t E(x,\nu)d\nu = & \alpha \int_0^t \int_{\mathbb{R}^n} \Gamma(Ds, x-y)\Big(\hat{S} - S(y,t-s)\Big)e^{-\rho s} dyds \\ & + \alpha \int_0^t \int_{\mathbb{R}^n} \Gamma(Dt, x-y)\hat{E}e^{-\rho t} dy d\nu. \end{aligned} \tag{4.6}$$

To find c^*, we have to use the integral kernel, which we could find from the nonlinear equation (4.6) as discussed by Alanazi et al. [5, 3] and by Alanazi [1]. By (4.6), the integral kernel is

$$A(x,t) = \alpha \Gamma(Dt, x)\hat{S}e^{-\rho t}.$$

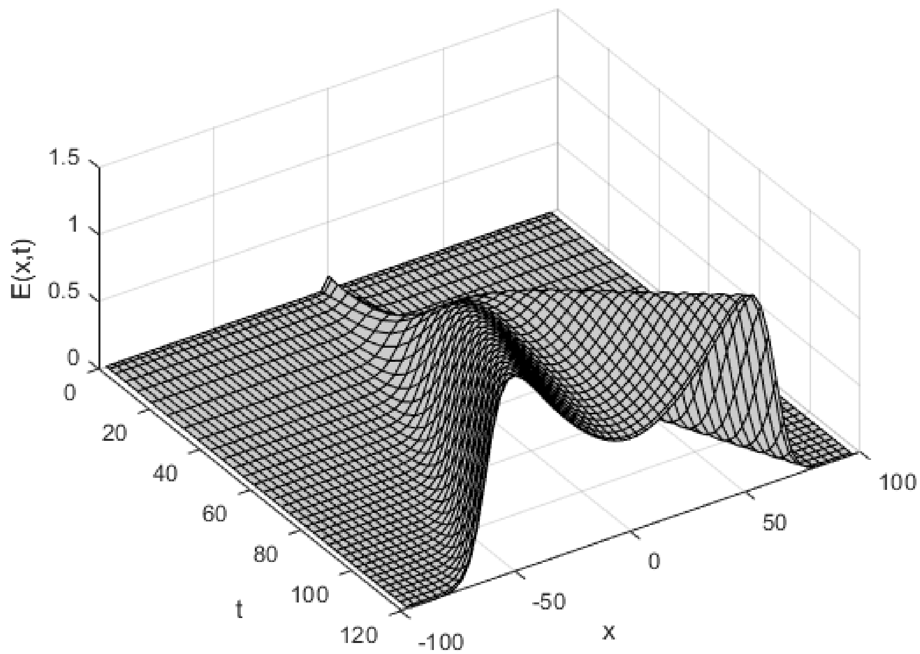

FIGURE 4. The numerical solution of $E(x,t)$ of the system (3.3). Here, $\Theta = 0.01, \alpha = 0.1, D = 0.7$, and $m = 0.01$.

It is well known by [19, 21] that the Laplace transform is given by

$$T(c,\lambda) = \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cz+y_1)} A(z,y) dy dz. \tag{4.7}$$

Therefore,

$$T(c,\lambda) = \alpha \hat{S} \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cz+y_1)} \Gamma(Dz,y) e^{-\rho z} dy dz. \tag{4.8}$$

Since $\rho = m + m_E$, we have

$$T(c,\lambda) = \alpha \hat{S} \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cz+y_1)} \Gamma(Dz,y) e^{-(m+m_E)z} dy dz. \tag{4.9}$$

By [21, Proposition 4.2], $\int_{\mathbb{R}^n} e^{-\lambda y_1} \Gamma(Dz,y) dy = e^{\lambda^2 Dz}$. Therefore,

$$\begin{aligned} T(c,\lambda) &= \alpha \hat{S} \int_0^\infty e^{-\lambda cz} e^{-(m+m_E)z} e^{\lambda^2 Dz} dz, \\ &= \frac{\alpha \hat{S}}{m_E + c\lambda + m - D\lambda^2}. \end{aligned} \tag{4.10}$$

According to [19, 21], we could find c^* by solving the following system,

$$T(c^*,\lambda) = 1, \qquad \frac{\partial}{\partial \lambda} T(c^*,\lambda) = 0. \tag{4.11}$$

Therefore, we have the following system of equations,

$$\begin{aligned} \frac{\alpha \hat{S}}{m_E + c^*\lambda + m - D\lambda^2} &= 1, \\ \alpha \hat{S}(2D\lambda - c^*) &= 0. \end{aligned} \tag{4.12}$$

From (4.10), $\lambda = \frac{c^*}{2D}$. Hence, the asymptotic speed of spread c^* is given by

$$c^* = \sqrt{4D(\alpha \hat{S} - m - m_E)}. \tag{4.13}$$

Let $\hat{S} = 1, \alpha = 0.1, D = 0.7, m = 0.01$, $m_E = 0.02$, then the asymptotic speed of spread is $c^* = 0.5$.

5. Traveling wave solutions of the model (4.1)

This section seeks numerical solutions for the model defined in (4.1). Therefore, the ordinary differential equation form of the model (4.1) is

$$\begin{aligned} P'(x_i, t) &= -\alpha S(x_i, t) E(x_i, t), \\ E'(x_i, t) &= D\Big[\frac{E(x_{i-1}, t) - 2E(x_i, t) + E(x_{i+1}, t)}{\Delta x^2}\Big] \\ &+ \alpha S(x_i, t) E(x_i, t) - mE(x_i, t) - m_E E(x_i, t), \end{aligned} \tag{5.1}$$

where $i = 1, \ldots, N$. The initial conditions are

$$S(x, 0) = 1, \ E(x, 0) = 0.1. \tag{5.2}$$

The boundary conditions are defined in the previous section. The numerical solutions of $S(x,t)$ and $E(x,t)$ are demonstrated in Fig. 5–6. By Fig. 5–6, the wave speed is approximately $c = 0.37$.

6. Discussion

In this work, we discuss a susceptible and infectious mathematical model, known as an SI model, in two different ways. First, we assume the model incorporates a recruitment rate. In the second case, we assume no new susceptible are recruited to the population. We solve the model numerically for both cases and find the minimal traveling wave speeds. We can find the spreading speed analytically when there is no turnover for the susceptible population. The recruitment rate increases the spreading speeds of traveling wave solutions.

For the model (2.1), The numerical solutions of the proposed model are depicted in Fig. 1–2. The recruitment rate that was assigned to the system (2.1) helped the density of susceptible to grow above the initial number of the susceptible individuals, as shown in Fig. 1. Also, we study the model when the natural death rates of the susceptible and infected individuals are set to be zeros, i.e., $m_S = m_E = 0$. The results are given in Fig. 3–4. The results also clearly show the existence of traveling wave solutions as demonstrated in Fig. 1–4. These waves propagate with speed $c = 0.41$ for the model (2.1), and they spread with speed $c = 0.5$ for the model (3.3). When there is no disease in the population, i.e., $E(x,t) = 0$, the steady state solution is $S(x,t) = \frac{\Theta}{m_S}$. The equation $S(x,t) = \frac{\Theta}{m_S}$

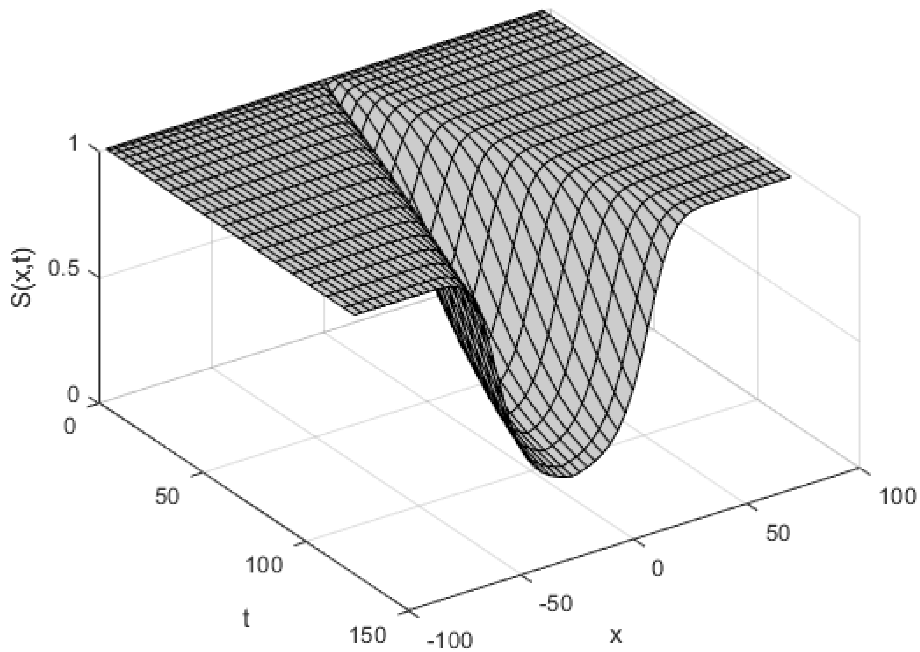

FIGURE 5. The numerical solution of $S(x,t)$. Here, $\alpha = 0.1, D = 0.7, m = 0.01$, and $m_E = 0.02$.

considers as a threshold density below which the disease cannot survive in the population.

For the second case, we discuss the model (4.1). For this case, we assume no more susceptible are recruited into the population. These assumptions helped us to find the analytic formula for the asymptotic speed of spread c^*. We found that

$$c^* = \sqrt{4D(\alpha\hat{S} - m - m_E)}.$$

According to this formula, if the death rates are higher than the rate of new cases of infectives, then the disease dies out. If $m + m_E > \alpha\hat{S}$, there will be no epidemic wave. On the other hand, the disease persists as long as $m+m_E < \alpha\hat{S}$. We also use this formula to calculate the asymptotic speed of spread c^*. The result shows that $c^* = 0.5$ when $\hat{S} = 1, \alpha = 0.1, D = 0.7, m = 0.01$, $m_E = 0.02$. Furthermore, this formula indicates that c^* is an increasing function with the diffusion coefficient D, the transformation rate from susceptible to infected stage α, and the initial number of susceptible $\hat{S}$. In addition, the formula for c^* provides that the spreading speed is a decreasing function with m and m_E. The numerical solutions of the model (4.1) are demonstrated in Fig. 5–6. Since we have not allowed more susceptible to come into the system (4.1), the density of susceptible decays over time, as we see in Fig. 5. Furthermore, Fig. 5–6 prove the existence of traveling wave solutions numerically. We found that the wave speed equals

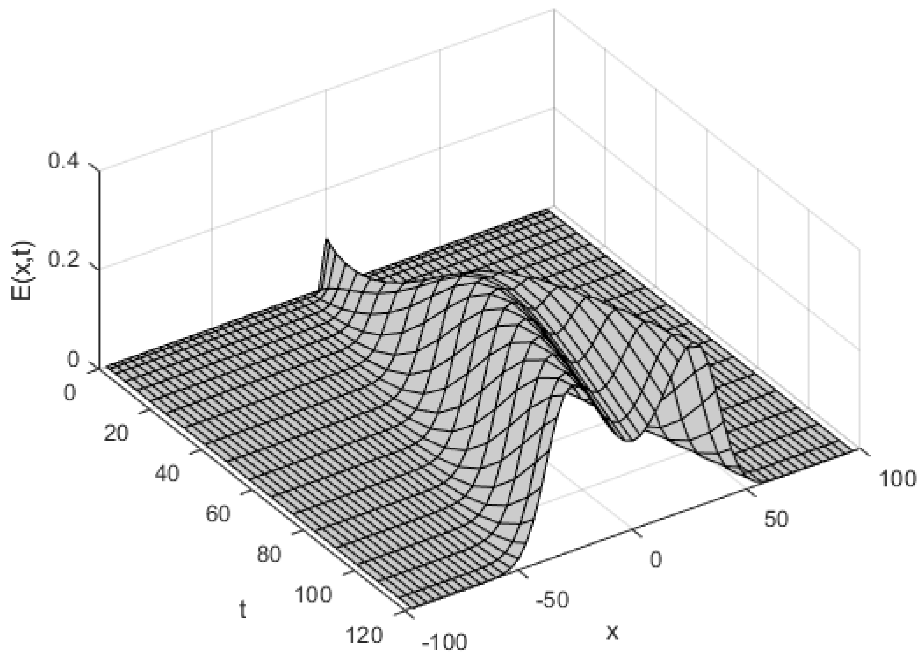

FIGURE 6. The numerical solution of $E(x,t)$. Here, $\alpha = 0.1, D = 0.7, m = 0.01$, and $m_E = 0.02$.

$c = 0.37$. It is not surprising that $c^* > c$ since we calculate c^* on $\mathbb{R}^n$, while we calculate c on a sub-domain of $\mathbb{R}$.

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