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APPROXIMATION OF COMMON FIXED POINT OF TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE FOR A NEW ITERATION PROCESS IN CAT(0) SPACES

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Abstract. In this paper, we establish strong and $\Delta-$convergence for a new iteration process containing two asymptotically nonexpansive mappings in the intermediate sense which is broader than the class of asymptotically nonexpansive mappings in the context of $CAT(0)$ spaces. Our results extend, generalize and improve many well-known results in the literature.

1. Introduction and preliminaries

Consider (X, d) as a metric space. A geodesic path between $\theta \in X$ and $\eta \in X$ (or, to put it another way, a geodesic from θ to η) is a map r from $[0, l]$ to X with $r(0) = \theta$, $r(l) = \eta$ and $(d(r(t), r(t_0)) = |t - t_0|$, for any $t, t_0 \in [0, l]$. Thus r is an isometry and $d(\theta, \eta) = l$. The image of r is a geodesic (or metric) segment that joins θ and η. When geodesic is unique, it is denoted by $[\theta, \eta]$.

The space (X, d) is said to be a geodesic space if any two points in X are connected by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic between θ and η for any θ, η in X. If D contains every geodesic segment connecting any two points, the subset $D \subseteq X$ is convex.

A geodesic triangle $\triangle(\theta_1, \theta_2, \theta_3)$ in a geodesic metric space (X, d) is made up of three points $\theta_1, \theta_2, \theta_3$ in X (the vertices of $\triangle$), with a geodesic segment connecting each pair of vertices (the edge of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(\theta_1, \theta_2, \theta_3)$ in (X, d) is a triangle $\bar{\triangle}(\theta_1, \theta_2, \theta_3) = \triangle(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$ in Euclidean space $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{\theta}_i, \bar{\theta}_j) = d(\theta_i, \theta_j)$ for $i, j \in \{1, 2, 3\}$ [16].

Let $\bar{\triangle}$ be a comparison triangle for a geodesic triangle $\triangle$ in X. The $\triangle$ is satisfy the $CAT(0)$ inequality if $\forall\ \theta, \eta \in \triangle$ and all comparison points $\bar{\theta}, \bar{\eta} \in \bar{\triangle}$ such that

$$d(\theta, \eta) \le d_{\mathbb{R}^2}(\bar{\theta}, \bar{\eta})$$

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A complete $CAT(0)$ space is often called *Hadamard space* [13]. Let θ, η, ζ are points of X and η_0 be the midpoint of segment $[\eta, \zeta]$, denoted by $\frac{\eta \oplus \zeta}{2}$, then the $CAT(0)$ inequality gives

$$d^2(\theta, \eta_0) \leq \frac{1}{2}d^2(\theta, \eta) + \frac{1}{2}d^2(\theta, \zeta) - \frac{1}{4}d^2(\eta, \zeta).$$

This is called (CN) inequality of Bruhat and Tits [3]. A geodesic space is said to be $CAT(0)$ space if and only if it satisfies the (CN) inequality [16]

Fixed point theory in $CAT(0)$ spaces was first studied by Kirk [9]. He proved that every nonexpansive mapping defined on a closed bounded convex subset of a complete $CAT(0)$ space always had a fixed point.

Let D be a non-empty subset of a $CAT(0)$ space X and let $T : D \to D$ be a mapping. The **Mann iteration** [14] process is defined by the sequence $\{\theta_n\}$,

$$\begin{cases} \theta_1 \quad \in D \\ \theta_{n+1} = (1 - a_n)\theta_n + a_n T\theta_n, \ n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{a_n\}$ is a sequence in $(0, 1)$.

The **Ishikawa iteration** [8] process is defined by the sequence $\{\theta_n\}$,

$$\begin{cases} \theta_1 \quad \in D \\ \theta_{n+1} = (1 - a_n)\theta_n + a_n T\eta_n, \\ \eta_n = (1 - b_n)\theta_n + b_n T\theta_n, \ n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences in $(0, 1)$. This iteration process reduces to the **Mann iteration** process when $b_n = 0$ for all $n \in \mathbb{N}$.

In 2007 Agarwal, O' Regan and Sahu [1] introduced the **S-iteration** process in Banach space,

$$\begin{cases} \theta_1 \quad \in D \\ \theta_{n+1} = (1 - a_n)T\theta_n + a_n T\eta_n, \\ \eta_n = (1 - b_n)\theta_n + b_n T\theta_n, \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences in $(0, 1)$. Note that (1.3) is independent of (1.2) (and hence of (1.1)).

In 1991 Schu [23], The modified **Mann iteration** process which is a **generalization of the Mann iteration** process,

$$\begin{cases} \theta_1 \quad \in D \\ \theta_{n+1} = (1 - a_n)\theta_n + a_n T^n\theta_n, n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{a_n\}$ is a sequence in $(0, 1)$.

In 1994,Tan and Xu [26], studied the **modified Ishikawa iteration** process which is a generalization of the **Ishikawa iteration** process,

$$\begin{cases} \theta_1 \quad \in D \\ \theta_{n+1} = (1 - a_n)\theta_n + a_n T\eta_n, \\ \eta_n = (1 - b_n)\theta_n + b_n T\theta_n, \ n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where the sequences $\{a_n\}$ and $\{b_n\}$ are in $(0,1)$. This iteration process reduces to the **modified Mann iteration** process when $b_n = 0$ for all $n \in \mathbb{N}$. Recently, Agarwal, O'Regan and Sahu [1] introduced the **modified S-iteration** process in a Banach space,

$$\begin{cases} \theta_1 \quad \in D \\ \theta_{n+1} = (1-a_n)T^n\theta_n + a_n T^n \eta_n, \\ \eta_n = (1-b_n)\theta_n + b_n T^n \theta_n, \ n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where the sequences $\{a_n\}$ and $\{b_n\}$ are in $(0,1)$. Note that (1.6) is independent of (1.5)(and hence of (1.4)).

Sahin and Basarir[21] modified iteration process (1.6) in a $CAT(0)$ space as follows. Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : D \to D$ be an asymptotically quasi-nonexpansive mapping with $S_f(T) = \{\theta \in K : T\theta = \theta\} \neq \emptyset$. Suppose that $\{\theta_n\}$ is a sequence generated iteratively by

$$\begin{cases} \theta_1 \quad \in D \\ \theta_{n+1} = (1-a_n)T^n\theta_n \oplus a_n T^n \eta_n, \\ \eta_n = (1-b_n)\theta_n \oplus b_n T^n \theta_n, \ n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences such that $0 \leq a_n, b_n \leq 1$, for all $n \in \mathbb{N}$.

Consider D to be a nonempty closed, convex subset of a complete $CAT(0)$ space X and $T_1, T_2 : D \to D$ be two asymptotically quasi-nonexpansive mappings in the intermediate sense with $S_f(T_1, T_2) = S_f(T_1) \cap S_f(T_2) \neq \emptyset$. Suppose that $\{\theta_n\}$ is a sequence generated iteratively by

$$\begin{cases} \theta_1 \quad \in D \\ \theta_{n+1} = (1-a_n)T_1^n\theta_n \oplus a_n T_2^n \eta_n, \\ \eta_n = (1-b_n)T_2^n\theta_n \oplus b_n T_1^n \theta_n, \ n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where the sequences $\{a_n\}$ and $\{b_n\}$, throughout the paper, are such that $0 \leq a_n, b_n \leq 1$ for all $n \geq 1$.

$$\begin{cases} \theta_1 \quad \in D \\ \theta_{n+1} = (1-a_n)T_1^n\theta_n \oplus a_n T_2^n \eta_n, \\ \eta_n = (\frac{b_n}{1-a_n})T_2^n\theta_n \oplus \frac{c_n}{1-a_n} T_1^n \theta_n, \ n \in \mathbb{N}, \end{cases} \tag{1.9}$$

where $a_n, b_n, c_n \in (0,1)$ and $a_n + b_n + c_n = 1$.

2. Preliminaries

Let us recall some definitions and known results in the existing literature on this concept. Goebel and Kirk [6] proposed the concept of an asymptotically nonexpansive mapping in 1972. Many authors have studied the iterative approximation problems for asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings in a Banach space and a CAT(0) space. Many authors studied the iterative approximation problem for asymptotically quasi-nonexpansive mappings in Banach space and a CAT(0) space[11, 20, 19, 22, 24].

Definition 2.1. *Let (X, d) be a metric space and D, its nonempty subset. Let $T : D \to D$ be a mapping. A point $\theta \in D$ is called a fixed point of T if $T\theta = \theta$. We will also denote by $S_f(T)$ the set of fixed points of T, that is , $S_f(T) = \{\theta \in K : T\theta = \theta\}$.*

Definition 2.2. *Let (X, d) be a metric space and D its nonempty subset. Let $T_1, T_2 : D \to D$ be mappings. A point $\theta \in D$ is called a common fixed point of T_1 and T_2 if $T_1\theta = T_2\theta = \theta$, and $S_f(T_1, T_2) = \{\theta \in K : T_1\theta = T_2\theta = \theta\}$ is the set of common fixed points of T_1 and T_2 .*

Definition 2.3. *Let (X, d) be a $CAT(0)$ space and D be its nonempty subset of X in $CAT(0)$ space . Then $T : D \to D$ is said to be*

(1) *nonexpansive, if $d(T\theta, T\eta) \leq d(\theta, \eta)$, for all $\theta, \eta \in D$;*
(2) *uniformly L-Lipschitzian, if there exists a $L \in (0, \infty)$ such that $d(T^n\theta, T^n\eta) \leq Ld(\theta, \eta)$ for all $\theta, \eta \in D$ and $n \geq 1$;*
(3) *asymptotically nonexpansive, if there exists a sequence $u_n \in [0, \infty)$ with the property $\lim_{n\to\infty} u_n = 0$ and such that $d(T^n(\theta), T^n\eta) \leq (1 + u_n)d(\theta, \eta)$, for all $\theta, \eta \in D$;*
(4) *semi-compact, if for a sequence $\{\theta_n\}$ in D with $\lim_{n\to\infty} d(\theta_n, T\theta_n) = 0$, there exists a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\theta_{n_k} \to p \in D$;*
(5) *asymptotically quasi-nonexpansive type, if $S_f(T) \neq \emptyset$ and $\lim_{n\to\infty} \sup \{\sup (d(T^n\theta, p) - d(\theta, p)), \theta \in D, p \in S_f(T)\} \leq 0$.*

Definition 2.4. [10] *A sequence $\{\theta_n\}$ in $CAT(0)$ space X is said to be $\Delta-$converge to $\theta \in X$, if x is unique asymptotic center of $\{\theta_n\}$ for every subsequence $\{u_n\}$ of $\{\theta_n\}$.*

In this case we write $\Delta - \lim_{n\to\infty} \theta_n = \theta$ and call θ the Δ limit of $\{\theta_n\}$.

Definition 2.5. [7] *Let D be closed, convex subset of a $CAT(0)$ space X. A bounded sequence $\{\theta_n\}$ in D is said to converge weakly to $q \in D$ iff $\phi(q) = \inf_{\theta\in D} \phi(\theta)$, $\phi(q) = \limsup_{n\to\infty} d(\theta_n, \theta)$.*

Note that $\{\theta_n\} \rightharpoonup q$ iff $A_D\{\theta_n\} = \{q\}$.

In 1993, Bruck, Kuczumow, and Reich[2] introduced a notion of asymptotically nonexpansive mapping in the intermediate sense. A mapping $T : D \to D$ is said to be asymptotically nonexpansive in the intermediate sense provided that T is uniformly continuous and

$\limsup_{n\to\infty} \sup_{\theta,\eta\in D} \{d(T^n\theta, T^n\eta) - d(\theta, \eta)\} \leq 0$

From the above definition, it follows that an asymptotically nonexpansive mapping must be asymptotically nonexpansive mapping in the intermediate sense.

Lemma 2.1. [18] *Let X be a $CAT(0)$ space.*

(i) Let $\theta, \eta \in X$, For each $t \in [0,1]$, there exists a unique point $\zeta \in [\theta, \eta]$ such that

$$d(\theta, \zeta) = td(\theta, \eta), \quad d(\eta, \zeta) = (1-t)d(\theta, \eta). \tag{2.1}$$

We use the notation $(1-t)\theta \oplus t\eta$ for unique point ζ satisfying (2.1)

(ii) For all $t \in [0,1]$ and $\theta, \eta, \zeta \in X$

$$d(((1-t)\theta \oplus)t\eta, \zeta) \leq (1-t)d(\theta, \zeta) + td(\eta, \zeta). \tag{2.2}$$

(iii) For any $t \in [0,1]$ and $\theta, \eta, \zeta \in X$

$$d^2(\zeta, t\theta \oplus (1-t)\eta) \leq td^2(\zeta, \theta) + (1-t)d^2(\zeta, \eta) - t(1-t)d^2(\theta, \eta). \tag{2.3}$$

Let $\{\theta_n\}$ be a bounded sequence in closed, convex subset D of a $CAT(0)$-space X. For θ in X, set $r(\theta, \{\theta_n\}) = \lim_{n\to\infty} \sup d(\theta, \theta_n)$. The asymptotic radius $r(\{\theta_n\})$ of $\{\theta_n\}$ is given by $r(\{\theta_n\}) = \inf\{r(\theta, \{\theta_n\})\} : \theta \in X$ and the asymptotic centre $A(\{\theta_n\})$ of $\{\theta_n\}$ is the set $A(\{\theta_n\}) = \{\theta \in X : r(\{\theta_n\}) = r(\theta, \{\theta_n\})\}$, to be known that, $A(\{\theta_n\})$ consists of exactly one point in $CAT(0)$ space. In this paper, the symbol " $\rightharpoonup$ " for weak convergence.

Lemma 2.2. [15] *Given $\{\theta_n\} \subset X$ such that $\{\theta_n\}$ $\triangle$converges to θ and given $\eta \in X$ with $\eta \neq \theta$, then*
$\lim_{n\to\infty} \sup d(\theta_n, \theta) < \lim_{n\to\infty} \sup d(\theta_n, \eta)$

The above condition is known as Opial property in Banach space.

Lemma 2.3. [17] *Let $\{\theta_n\}$ be a bounded sequence in $CAT(0)$ space X, and let D be a closed convex subset of X which contains $\{\theta_n\}$. Then*
(i) $\Delta \lim \theta_n = \theta$ implies $\theta_n \rightharpoonup \theta$.
(ii) The convergence of (i) is true if $\{\theta_n\}$ is regular.

Lemma 2.4. [4] *If $\{\theta_n\}$ be a bounded sequence in $CAT(0)$ space X, with $A(\{\theta_n\}) = \{\theta\}$, and $\{u_n\}$ is a subsequence of $\{\theta_n\}$, with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(\theta_n, u)\}$ converges, then $\theta = u$*

Lemma 2.5. [5] *If D is a closed convex subset of a $CAtheT(0)$ space X and if $\{\theta_n\}$ is bounded sequence in D, then the asymptotically center of $\{\theta_n\}$ is in D*

Lemma 2.6. [25] *Suppose $\{a_n\}$ and $\{b_n\}$ are two non-negative sequences of real numbers such that $a_{n+1} \leq a_n + b_n$, for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges then $\lim\limits_{n\to\infty} a_n$ exists.*

Lemma 2.7. [10] *Let X be a complete $CAT(0)$ space, D be a nonempty closed convex subset of X. If $T : D \to D$ is an asymptotically nonexpansive mapping in the intermediate sense, then T has a fixed point.*

Lemma 2.8. [10] *Let X be a complete $CAT(0)$ space, D be a nonempty closed convex subset of X. If $T : D \to D$ is an asymptotically nonexpansive mapping in the intermediate sense, then $S_f(T)$ is closed and convex.*

Lemma 2.9. [10] *(Demiclosed principle) Let D be a closed convex subset of a complete $CAT(0)$ space X and $T : D \to D$ is an asymptotically nonexpansive mapping in the intermediate sense. If $\{\theta_n\}$ is a bounded sequence in D such that $\lim_{n\to\infty} d(\theta_n, T\theta_n) = 0$ and $\{\theta_n\} \rightharpoonup w$, then $Tw = w$*

Lemma 2.10. [10] *Let D be a closed convex subset of a complete $CAT(0)$ space X and $T : D \to D$ is an asymptotically nonexpansive mapping in the intermediate sense. If $\{\theta_n\}$ is bounded sequence in D $\Delta-$converging to θ and $\lim_{n\to\infty} d(\theta_n, T\theta_n) = 0$, then $\theta \in D$ and $T\theta = \theta$.*

3. Main results

In this section, we prove the following lemmas using a new scheme (1.9) and also prove convergence results using these lemmas.

Lemma 3.1. *Let D be a closed and convex subset of complete $CAT(0)$ space X and $T_1, T_2 :\to D$ be two asymptotically nonexpansive mappings in the intermediate sense with $S_f(T_1, T_2) \neq \emptyset$. Suppose that $\{\theta_n\}$ is defined by the interation (1.9). Put*

$$\alpha_n = \max\left\{0, \sup_{\theta,\eta\in D, n\geq 1} (d(T_1^n\theta, T_1^n\eta) - d(\theta, \eta))\right\} \tag{3.1}$$

and

$$\beta_n = \max\left\{0, \sup_{\theta,\eta\in D, n\geq 1} (d(T_2^n\theta, T_2^n\eta) - d(\theta, \eta))\right\} \tag{3.2}$$

such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$. Suppose that $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0, 1)$. Then :
(i) $\lim_{n\to\infty} d(\theta_n, p)$ exists for all $p \in S_f(T_1, T_2)$.
(ii) $\lim_{n\to\infty} d(\theta_n, S_f(T_1, T_2))$ exists.

Proof. Let $p \in S_f(T_1, T_2)$. Using (1.9), (3.1), (3.2) and (2.2), we have

$$\begin{aligned}
d(\eta_n, p) &= d\left(\frac{b_n}{1-a_n}T_1^n\theta_n \oplus \frac{c_n}{1-a_n}T_2^n\theta_n, p\right) \\
&\leq \frac{b_n}{1-a_n}d(T_1^n\theta_n, p) + \frac{c_n}{1-a_n}d(T_2^n\theta_n, p) \\
&\leq \frac{b_n}{1-a_n}[d(\theta_n, p) + \alpha_n] + \frac{c_n}{1-a_n}[d(\theta_n, p) + \beta_n] \\
&\leq d(\theta_n, p) + \frac{\alpha_n b_n}{1-a_n} + \frac{\beta_n c_n}{1-a_n} \\
&\leq d(\theta_n, p) + H_n + K_n.
\end{aligned}$$

this gives

$$d(\eta_n, p) \leq d(\theta_n, p) + H_n + K_n. \tag{3.3}$$

Again using (1.9), (2.2) and (3.1)-(3.3), we have

$$\begin{aligned} d(\theta_{n+1},p) &= d\left((1-a_n)T_2^n\theta_n \oplus a_nT_1^n\eta_n, p\right) \\ &\leq (1-a_n)d(T_2^n\theta_n,p)+a_nd(T_1^n\eta_n,p) \\ &\leq (1-a_n)[d(\theta_n,p)+\beta_n]+a_n[d(\eta_n,p)+\alpha_n] \\ &\leq (1-a_n)[d(\theta_n,p)+\beta_n]+a_n[d(\theta_n,p)+H_n+K_n+\alpha_n] \\ &\leq d(\theta_n,p)+(1-a_n)\beta_n+a_n\alpha_n+a_n\frac{\alpha_nb_n}{1-a_n}+a_n\frac{\beta_nc_n}{1-a_n}, \end{aligned}$$

this gives

$$d(\theta_{n+1},p) \leq d(\theta_n,p)+(1-a_n)\beta_n+a_n\alpha_n+a_n\frac{\alpha_nb_n}{1-a_n}+a_n\frac{\beta_nc_n}{1-a_n} \tag{3.4}$$

taking infimum over all $p \in S_f(T_1,T_2)$, we get

$$d(\theta_{n+1},p) \leq d(\theta_n,S_f(T_1,T_2))+(1-a_n)\beta_n+a_n\alpha_n+a_n\frac{\alpha_nb_n}{1-a_n}+a_n\frac{\beta_nc_n}{1-a_n}. \tag{3.5}$$

Since by hypothesis of the lemma $\sum_{n=1}^{\infty}\alpha_n < \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$, it follows from Lemma 2.6, relation (3.4),(3.5) that $\lim_{n\to\infty} d(\theta_n,p)$ and $\lim_{n\to\infty} d(\theta_n,S_f(T_1,T_2))$ exists. □

Lemma 3.2. *Let D be a nonempty closed , convex subset of complete $CAT(0)$ space X and $T_1,T_2 : D \to D$ be two asymptotically nonexpansive mappings in the intermediate sense with $S_f(T_1,T_2) = S_f(T_1)\cap S_f(T_2) \neq \emptyset$. Suppose that the sequence $\{\theta_n\}$ is defined by the iteration process (1.9) and α_n and β_n are taken as in Lemma 3.1. Suppose that $\{a_n\},\{b_n\}$ and $\{c_n\}$ are real sequences in $[l,m]$ for some $l,m \in (0,1)$. If $d(\theta,T_1\theta) \leq d(T_2\theta,T_1\theta)$ for all $\theta \in D$, then $\lim_{n\to\infty} d(\theta_n,T_1\theta_n)=0$ and $\lim_{n\to\infty} d(\theta_n,T_2\theta_n)=0$.*

Proof. Using (1.9) and (2.3), we have

$$\begin{aligned} &d^2(\eta_n,p) = d^2\left(\tfrac{b_n}{1-a_n}T_1^n\theta_n \oplus \tfrac{c_n}{1-a_n}T_2^n\theta_n, p\right) \\ &\leq \frac{b_n}{1-a_n}d^2(T_1^n\theta_n,p)+\frac{c_n}{1-a_n}d^2(T_2^n\theta_n,p)-\frac{b_nc_n}{(1-a_n)^2}d^2(T_2^n\theta_n,T_1^n\theta_n) \\ &\leq \frac{b_n}{1-a_n}[d(\theta_n,p)+\alpha_n]^2+\frac{c_n}{1-a_n}[d(\theta_n,p)+\beta_n]^2-\frac{b_nc_n}{(1-a_n)^2}d^2(T_2^n\theta_n,T_1^n\theta_n) \\ &\leq \frac{b_n}{1-a_n}[d^2(\theta_n,p)+\alpha_n^2+2\alpha_nd(\theta_n,p)]+\frac{c_n}{1-a_n}[d^2(\theta_n,p)+\beta_n^2+2\beta_nd(\theta_n,p)]- \\ &\frac{b_nc_n}{(1-a_n)^2}d^2(T_2^n\theta_n,T_1^n\theta_n) \\ &\leq d^2(\theta_n,p)+\frac{b_n\alpha_n^2}{1-a_n}+2\frac{\alpha_nb_n}{1-a_n}d(\theta_n,p)+\frac{c_n\beta_n^2}{1-a_n}+2\frac{c_n\beta_n}{1-a_n}d(\theta_n,p) \\ &-\frac{b_nc_n}{(1-a_n)^2}d^2(T_2^n\theta_n,T_1^n\theta_n) \end{aligned}$$

$\leq d^2(\theta_n, p) + A_n + B_n - \dfrac{b_n c_n}{(1-a_n)^2} d^2(T_2^n\theta_n, T_1^n\theta_n),$
this gives

$$d^2(\eta_n, p) \leq d^2(\theta_n, p) + A_n + B_n - \frac{b_n c_n}{(1-a_n)^2} d^2(T_2^n\theta_n, T_1^n\theta_n), \tag{3.6}$$

where $A_n = \dfrac{b_n\alpha_n^2}{1-a_n} + 2\dfrac{\alpha_n b_n}{1-a_n} d(\theta_n, p),$
$B_n = \dfrac{c_n\beta_n^2}{1-a_n} + 2\dfrac{c_n\beta_n}{1-a_n} d(\theta_n, p).$
Since by hypothesis $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$.
Again using (1.9), (2.3) and (3.6), we have

$d^2(\theta_{n+1}, p) = d^2\left((1-a_n)T_2^n\theta_n \oplus a_n T_1^n\eta_n, p\right)$

$\leq (1-a_n)d^2(T_2^n\theta_n, p) + a_n d^2(T_1^n\theta_n, p) - a_n(1-a_n)d^2(T_1^n\eta_n, T_2^n\theta_n)$

$\leq a_n[d(\eta_n, p) + \alpha_n]^2 + (1-a_n)[d(\theta_n, p) + \beta_n]^2 - a_n(1-a_n)d^2(T_1^n\eta_n, T_2^n\theta_n)$

$\leq a_n[d^2(\eta_n, p) + \alpha_n^2 + 2\alpha_n d(\eta_n, p)] + (1-a_n)[d^2(\theta_n, p) + \beta_n^2 + 2\beta_n d(\theta_n, p)] - a_n(1-a_n)d^2(T_1^n\eta_n, T_2^n\theta_n)$

$\leq a_n[d^2(\eta_n, p) + L_n] + (1-a_n)[d^2(\theta_n, p) + M_n] - a_n(1-a_n)d^2(T_1^n\eta_n, T_2^n\theta_n)$

$\leq a_n d^2(\eta_n, p) + (1-a_n)d^2(\theta_n, p) + a_n L_n + (1-a_n)M_n - a_n(1-a_n)d^2(T_1^n\eta_n, T_2^n\theta_n)$

$\leq a_n[d^2(\theta_n, p) + A_n + B_n - \dfrac{b_n c_n}{(1-a_n)^2} d^2(T_2^n\theta_n, T_1^n\theta_n)] + (1-a_n)d^2(\theta_n, p) + a_n L_n + (1-a_n)M_n - a_n(1-a_n)d^2(T_1^n\eta_n, T_2^n\theta_n)$

$\leq a_n d^2(\theta_n, p) + a_n A_n + a_n B_n - \dfrac{a_n b_n c_n}{(1-a_n)^2} d^2(T_2^n\theta_n, T_1^n\theta_n) + (1-a_n)d^2(\theta_n, p) + a_n L_n + (1-a_n)M_n - a_n(1-a_n)d^2(T_1^n\eta_n, T_2^n\theta_n)$

$\leq d^2(\theta_n, p) + P_n + Q_n - \dfrac{a_n b_n c_n}{(1-a_n)^2} d^2(T_2^n\theta_n, T_1^n\theta_n). - a_n(1-a_n)d^2(T_1^n\eta_n, T_2^n\theta_n),$

this gives

$$d^2(\theta_{n+1}, p) \leq d^2(\theta_n, p) + P_n + Q_n - \frac{a_n b_n c_n}{(1-a_n)^2} d^2(T_2^n\theta_n, T_1^n\theta_n). - a_n(1-a_n)d^2(T_1^n\eta_n, T_2^n\theta_n), \tag{3.7}$$

where $L_n = \alpha_n^2 + 2\alpha_n d(\eta_n, p), M_n = \beta_n^2 + 2\beta_n d(\theta_n, p), P_n = a_n A_n + a_n B_n, Q_n = a_n L_n + (1-a_n)M_n$.
Since by hypothesis $\sum \alpha_n < \infty, \sum \beta_n < \infty$, it follows that $\sum P_n < \infty, \sum Q_n <$

∞.

Now from (3.7) we have,

$$\frac{a_n b_n c_n}{(1-a_n)^2} d^2(T_2^n\theta_n, T_1^n\theta_n) \le [d^2(\theta_n,p) - d^2(\theta_{n+1},p)] + Q_n + P_n$$

$$d^2(T_2^n\theta_n, T_1^n\theta_n) \le \frac{(1-a_n)^2}{a_n b_n c_n}[d^2(\theta_n,p) - d^2(\theta_{n+1},p)] + \frac{(1-a_n)^2}{a_n b_n c_n} Q_n + \frac{(1-a_n)^2}{a_n b_n c_n} P_n$$

$$d^2(T_2^n\theta_n, T_1^n\theta_n) \le \frac{(1-m)^2}{l^3}[d^2(\theta_n,p) - d^2(\theta_{n+1},p)] + \frac{(1-m)^2}{l^3} Q_n + \frac{(1-m)^2}{l^3} a_n P_n. \tag{3.8}$$

Now again from (3.7) we get,

$$d^2(\theta_{n+1},p) \le d^2(\theta_n,p) + P_n + Q_n - \alpha_n(1-\alpha_n) d^2(T_1^n\eta_n, T_2^n\theta_n)$$

$$d^2(T_1^n\eta_n, T_2^n\theta_n) \le \frac{1}{\alpha_n(1-\alpha_n)}[d^2(\theta_n,p) + d^2(\theta_{n+1},p)] + \frac{1}{\alpha_n(1-\alpha_n)} P_n + \frac{1}{\alpha_n(1-\alpha_n)} Q_n$$

$$\le \frac{1}{\alpha_n(1-\alpha_n)}[d^2(\theta_{n+1},p) + d^2(\theta_n,p)] + \frac{1}{\alpha_n(1-\alpha_n)} R_n$$

$$\le \frac{1}{l(1-m)}[d^2(\theta_{n+1},p) + d^2(\theta_n,p)] + \frac{1}{l(1-m)} R_n,$$

this gives

$$d^2(T_1^n\eta_n, T_2^n\theta_n) \le \frac{1}{l(1-m)}[d^2(\theta_{n+1},p) + d^2(\theta_n,p)] + \frac{1}{l(1-m)} R_n, \tag{3.9}$$

where $R_n = P_n + Q_n$.
Since $P_n \to 0$ as $n \to \infty$ and $Q_n \to 0$ as $n \to \infty$ so $R_n \to 0$ as $n \to \infty$ and $d(\theta_n, p)$ is convergent therefore on taking limit as $n \to \infty$ in (3.8), (3.9) we get

$$\lim_{n\to\infty} d^2(T_2^n\theta_n, T_1^n\theta_n) = 0. \tag{3.10}$$

and

$$\lim_{n\to\infty} d^2(T_1^n\eta_n, T_2^n\theta_n) = 0. \tag{3.11}$$

Now

$$\begin{aligned} d(T_2^n\theta_n, \theta_n) &\le d(T_2^n\theta_n, T_1^n\theta_n) + d(T_1^n\theta_n, \theta_n) \\ &\le d(T_2^n\theta_n, T_1^n\theta_n) + d(T_1^n\theta_n, T_2^n\theta_n) \\ &\le 2d(T_2^n\theta_n, T_1^n\theta_n) \to 0 \text{ as } n \to \infty \end{aligned}$$

this gives

$$d(T_2^n\theta_n, \theta_n) \le 2d(T_2^n\theta_n, T_1^n\theta_n) \to 0 \text{ as } n \to \infty \tag{3.12}$$

by (3.10) and (3.12), we obtain

$$\lim_{n\to\infty} d(T_1^n\theta_n, \theta_n) = 0. \tag{3.13}$$

Again note that

$$\begin{aligned} d(\theta_{n+1}, T_2^n\theta_n) &= d((1-a_n)T_2^n\theta_n \oplus a_nT_1^n\eta_n, T_2^n\theta_n) \\ &\leq a_n(T_1^n\eta_n, T_2^n\theta_n) \to 0 \ as \ n \to \ \infty, \end{aligned}$$

this gives

$$d(\theta_{n+1}, T_2^n\theta_n) \leq a_n(T_1^n\eta_n, T_2^n\theta_n). \tag{3.14}$$

By (3.12) and (3.14), we get

$$d(\theta_{n+1}, \theta_n) \leq d(\theta_{n+1}, T_2^n, \theta_n) + d(T_2^n\theta_n, \theta_n) \to 0 \ as \ n \to \infty. \tag{3.15}$$

Let $\phi_n = d(T^n\theta_n, \theta_n)$ by (3.12) we have $\phi_n \to 0 \ as \ n \to \infty$.

We have

$$\begin{aligned} d(\theta_n, T_2\theta_n) &\leq d(\theta_n, \theta_{n+1}) + d(\theta_{n+1}, T_2\theta_{n+1}) + d(T_2^{n+1}\theta_{n+1}, T_2^{n+1}\theta_n) + d(T_2^{n+1}\theta_n, T_2\theta_n) \\ &\leq d(\theta_n, \theta_{n+1}) + d(\theta_{n+1}, T_2\theta_{n+1}) + \phi_{n+1} + d(\theta_{n+1}, \theta_n) + \beta_{n+1} + d(T_2^{n+1}\theta_n, T_2\theta_n) \\ &\leq 2d(\theta_n, \theta_{n+1}) + \phi_{n+1} + \beta_{n+1} + d(T_2^{n+1}\theta_n, T_2\theta_n) \\ &\leq 2d(\theta_n, \theta_{n+1}) + \phi_{n+1} + \beta_{n+1} + d(T_2^{n+1}\theta_n, T_2\theta_n) \to 0 \ as \ n \to \infty, \end{aligned} \tag{3.16}$$

this gives

$$d(\theta_n, T_2\theta_n) \leq 2d(\theta_n, \theta_{n+1}) + \phi_{n+1} + \beta_{n+1} + d(T_2^{n+1}\theta_n, T_2\theta_n) \to 0 \ as \ n \to \infty \tag{3.16}$$

By (3.12), (3.15), $d_{n+1} \to 0$ as $n \ \to \infty$ and the uniform continuity of T_2. Similarly we can prove that $\lim_{n\to\infty} d(\theta_n, T_1\theta_n) = 0$.

□

Theorem 3.3. *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X, and let $T_1, T_2 : D \to D$ be two asymptotically nonexpansive mappings in the intermediate sense with $S_f(T_1, T_2) \neq \emptyset$. Suppose that θ_n is defined by the iteration process (1.9) and α_n and β_n are taken as in Lemma 3.1. Suppose that $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0, 1)$. Then the sequence $\{\theta_n\}$ is $\Delta-$convergent to a point of $S_f(T_1, T_2)$.*

Proof. We first show that $w_w(\{\theta_n\}) \subseteq S_f(T_1, T_2)$ Let $v \in w_w(\{\theta_n\})$, then there exists a subsequence $\{v_n\}$ of $\{\theta_n\}$ such that $A(\{\theta_n\}) = \{v\}$. By Lemma 2.5, there exists a subsequence $\{w_n\}$ of $\{v_n\}$ such that $\Delta - \lim_{n\to\infty} w_n = w \in D$. By Lemma 2.10, $w \in S_f(T_2)$ and $w \in S_f(T_1)$ and so $w \in S_f(T_1, T_2)$. By Lemma 3.1 $\lim_{n\to\infty} d(\theta_n, S_f(T_1, T_2))$ exists, so by Lemma 2.4 we have $v = w$, i.e., $w_w(\{\theta_n\}) \subseteq S_f(T_1, T_2)$.
To show that $\{\theta_n\}$ is $\Delta-$converges to a point in $S_f(T_1, T_2)$, it is sufficient to show that $w_w(\{\theta_n\})$ consists of exactly one point.
Let $\{v_n\}$ be a subsequence of $\{\theta_n\}$ with $A(\{v_n\}) = \{v\}$, and $A(\{\theta_n\}) = \{\theta\}$ for some $v \in W_w(\{\theta_n\}) \subseteq S_f(T_1, T_2)$ and $\{d(\theta_n, w)\}$ converges. By lemma 2.4, we have $\theta = w \in S_f(T_1, T_2)$. Thus $w_w(\{\theta_n\}) = \{\theta_n\}$. This shows that $\{\theta_n\}$ is $\Delta-$convergent to a point of $S_f(T_1, T_2)$. □

Theorem 3.4. *Let D be a nonempty closed, convex subset of a complete $CAT(0)$ space X and let $T_1, T_2 : D \to D$ be two asymptotically nonexpansive mappings in the intermediate sense such that $S_f(T_1, T_2) = S_f(T_1) \cap S_f(T_2) \neq \emptyset$. Suppose that*

$\{\theta_n\}$ is defined by the iteration process (1.9) , α_n and β_n be taken as in Lemma 3.1. Suppose that $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0,1)$. If $\liminf_{n\to\infty} d(\theta_n, S_f(T_1,T_2)) = 0$ and $\limsup_{n\to\infty} d(\theta_n, S_f(T_1,T_2)) = 0$, Where $d(\theta, S_f(T_1,T_2)) = \lim_{p\in S_f(T_1,T_2)} d(\theta, p)$. Then the sequence $\{\theta_n\}$ converges strongly to a point in $S_f(T_1,T_2)$.

Proof. From (3.5), we have

$$d(\theta_{n+1}, p) \leq d(\theta_n, S_f(T_1,T_2)) + (1-a_n)\beta_n + a_n\alpha_n + a_n\frac{\alpha_n b_n}{1-a_n} + a_n\frac{\beta_n c_n}{1-a_n},$$

where $p \in S_f(T_1,T_2)$. Since by hypothesis of the theorem $\sum_{n=1}^{\infty}\alpha_n < \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$ by Lemma 2.6 and $\liminf_{n\to\infty} d(\theta_n, S_f(T_1,T_2)) = 0$ or $\limsup_{n\to\infty} d(\theta_n, S_f(T_1,T_2)) = 0$, gives that $\lim_{n\to\infty} d(\theta_n, S_f(T_1,T_2)) = 0$.
Now, we show that $\{\theta_n\}$ is Cauchy sequence in D.
From (3.4) and by hypothesis $0 < l < a_n, b_n, c_n < m < 1$, we have

$$d(\theta_{n+p}, q) \leq d(\theta_{n+p-1}, q) + \frac{m^2}{1-m}\alpha_{n+p-q} + \frac{m^2}{1-m}\beta_{n+p-q} + (1-l)\beta_{n+p-q} + m\alpha_{n+p-q}$$

......

$$\leq d(\theta_n, q) + \frac{m^2}{1-m}\sum_{k=n}^{n+p-1}\alpha_k + \frac{m^2}{1-m}\sum_{k=n}^{n+p-1}\beta_k + (1-l)\sum_{k=n}^{n+p-1}\beta_k + m\sum_{k=n}^{n+p-1}\alpha_k$$

$$\leq d(\theta_n, q) + \left(\frac{m}{1-m}\right)\sum_{k=n}^{n+p-1}\alpha_k + \left(\frac{m^2}{1-m} + (1-l)\right)\sum_{k=n}^{n+p-1}\beta_k,$$

for $n, p \in \mathbb{N}\&\ q \in S_f(T_1,T_2)$.
Since $\lim_{n\to\infty} d(\theta_n, S_f(T_1,T_2)) = 0$, therefore for any $\epsilon > 0$, there exists a natural number n_0 such that $d(\theta_n, S_f(T_1,T_2)) < \frac{\epsilon}{12}$, $\sum_{k=n}^{n+p-1}\alpha_k < \left[\frac{(1-m)\epsilon}{6m}\right]$ and $\sum_{k=n}^{n+p-1}\beta_k < \left(\frac{(1-m)\epsilon}{6(m^2+(1-m)(1-l))}\right)$ for all $n \geq n_0$. So we can find $p^* \in S_f(T_1,T_2)$ such that $d(\theta_0, p^*) < \left(\frac{\epsilon}{6}\right)$. Hence for all $n \geq n_0, p \geq 1$, we have

$$d(\theta_{n+p}, \theta_n) \leq 2d(\theta_{n_0}, p^*) + 2\frac{m}{1-m}\sum_{k=n_0}^{n+p-1}\alpha_k + 2\left[\frac{m^2}{1-m} + (1-l)\right]\sum_{k=n_0}^{n+p-1}\beta_k$$

$$\leq 2\left(\frac{\epsilon}{6}\right) + 2\left(\frac{m}{1-m}\right)\left[\frac{(1-m)\epsilon}{6m}\right] + 2\left[\frac{m^2}{1-m} + (1-l)\right]\left(\frac{(1-m)\epsilon}{6(m^2+(1-m)(1-l))}\right)$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + 2\left[\frac{m^2+(1-l)(1-m)}{1-m}\right]\left(\frac{(1-m)\epsilon}{6(m^2+(1-m)(1-l))}\right)$$

$$= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

Theorem 3.5. *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X, and let $T_1, T_2 : D \to D$ be two asymptotically nonexpansive mappings in the intermediate sense with $S_f(T_1,T_2) \neq \emptyset$. Suppose that $\{\theta_n\}$ is defined by the iteration process (1.9) and α_n and β_n are taken as in Lemma 3.1. Suppose that $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0,1)$. If T_1, T_2 satisfy the following conditions:*
(i) $\lim_{n\to\infty} d(\theta_n, T_1\theta_n) = 0$ *and* $\lim_{n\to\infty} d(\theta_n, T_2\theta_n) = 0$
(ii) If the sequence $\{\zeta_n\}$ in D satisfies $\lim_{n\to\infty} d(\theta_n, T_1\zeta_n) = 0$ *and* $\lim_{n\to\infty} d(\zeta_n, T_2\zeta_n) = 0$

then $\liminf_{n\to\infty} d(\zeta_n, S_f(T_1,T_2)) = 0$ *or* $\limsup_{n\to\infty} d(\zeta_n, S_f(T_1,T_2)) = 0$.
Then the sequence $\{\theta_n\}$ *converges strongly to a point of* $S_f(T_1,T_2)$.

Proof. It following from the hypothesis that $\lim_{n\to\infty} d(\theta_n, T_1\theta_n) = 0$ and $\lim_{n\to\infty} d(\theta_n, T_2\theta_n) = 0$ from (ii), $\liminf_{n\to\infty} d(\zeta_n, S_f(T_1,T_2)) = 0$ or $\limsup_{n\to\infty} d(\zeta_n, S_f(T_1,T_2)) = 0$. Therefore, the sequence $\{\theta_n\}$ must converge strongly to a point in $S_f(T_1,T_2)$ by Theorem 3.3. □

Theorem 3.6. *Let* D *be a nonempty closed convex subset of a complete* $CAT(0)$ *space* X*, and let* $T_1, T_2 : D \to D$ *be two asymptotically nonexpansive mappings in the intermediate sense with* $S_f(T_1,T_2) \neq \emptyset$ *Suppose that* $\{\theta_n\}$ *is defined by the iteration process (1.9) and* α_n *and* β_n *are taken as in Lemma 3.1. Suppose that* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *are real sequences in* $[l,m]$ *for some* $l, m \in (0,1)$*. If either* T_1 *or* T_2 *is semi-compact, then the sequence* $\{\theta_n\}$ *converges strongly to a point of* $S_f(T_1,T_2)$.

Proof. Suppose that T_2 is semi-compact. By Lemma 3.2, we have $\lim_{n\to\infty} d(\theta_n, T_2\theta_n) = 0$. So there exists a subsequence $\{\theta_{n_j}\}$ of $\{\theta_n\}$ such that $\{\theta_{n_j}\} \to p \in D$. Now again Lemma 3.2 guarantees that $\lim_{n_j\to\infty} d(\theta_{n_j}, T\theta_{n_j}) = 0$ and so $d(p,Tp) = 0$. Similarly, we can show that $d(p,Tp) = 0$. Thus $p \in S_f(T_1,T_2)$. By (3.5), we have

$$d(\theta_{n+1}, p) \leq d(\theta_n, S_f(T_1,T_2)) + (1-a_n)\beta_n + a_n\alpha_n + a_n\frac{\alpha_n b_n}{1-a_n} + a_n\frac{\beta_n c_n}{1-a_n}$$

Since by hypothesis $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, by Lemma 2.6, $\lim_{n\to\infty} d(\theta_n, p)$ exists and $\theta_{n_j} \to p \in S_f(T_1,T_2)$ gives that $\theta_n \to p \in S_f(T_1,T_2)$. This shows that $\{\theta_n\}$ converges strongly to a point of $S_f(T_1,T_2)$. □

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