APPROXIMATION OF COMMON FIXED POINT OF TWO
ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE
INTERMEDIATE SENSE FOR A NEW ITERATION PROCESS
IN CAT(0) SPACES

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Abstract. In this paper, we establish strong and \(\Delta\)-convergence for a new
iteration process containing two asymptotically nonexpansive mappings in the
intermediate sense which is broader than the class of asymptotically nonexpan-
sive mappings in the context of \(\text{CAT}(0)\) spaces. Our results extend, generalize
and improve many well-known results in the literature.

1. Introduction and preliminaries

Consider \((X, d)\) as a metric space. A geodesic path between \(\theta \in X\) and \(\eta \in X\)
(or, to put it another way, a geodesic from \(\theta\) to \(\eta\)) is a map \(r\) from \([0, l]\) to \(X\)
with \(r(0) = \theta\), \(r(l) = \eta\) and \(d(r(t), r(t_0)) = \left| t - t_0 \right|\), for any \(t, t_0 \in [0, l]\). Thus \(r\)
is an isometry and \(d(\theta, \eta) = l\). The image of \(r\) is a geodesic (or metric) segment
that joins \(\theta\) and \(\eta\). When geodesic is unique, it is denoted by \([\theta, \eta]\).

The space \((X, d)\) is said to be a geodesic space if any two points in \(X\) are
connected by a geodesic, and \(X\) is said to be uniquely geodesic if there is exactly
one geodesic between \(\theta\) and \(\eta\) for any \(\theta, \eta \in X\). If \(D\) contains every geodesic
segment connecting any two points, the subset \(D \subseteq X\) is convex.

A geodesic triangle \(\triangle(\theta_1, \theta_2, \theta_3)\) in a geodesic metric space \((X, d)\) is made up of
three points \(\theta_1, \theta_2, \theta_3 \in X\) (the vertices of \(\triangle\)), with a geodesic segment connecting
each pair of vertices (the edge of \(\triangle\)). A comparison triangle for the geodesic trian-
gle \(\triangle(\theta_1, \theta_2, \theta_3)\) in \((X, d)\) is a triangle \(\bar{\triangle}(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3) = \triangle(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)\) in Euclidean
space \(\mathbb{R}^2\) such that \(d_{\mathbb{R}^2}(\bar{\theta}_i, \bar{\theta}_j) = d(\theta_i, \theta_j)\) for \(i, j \in \{1, 2, 3\}\) [16].

Let \(\bar{\triangle}\) be a comparison triangle for a geodesic triangle \(\triangle\) in \(X\). The \(\triangle\) is
satisfy the \(\text{CAT}(0)\) inequality if \(\forall \theta, \eta \in \triangle\) and all comparison points \(\bar{\theta}, \bar{\eta} \in \bar{\triangle}\)
such that

\[
d(\theta, \eta) \leq d_{\mathbb{R}^2}((\bar{\theta}, \bar{\eta})
\]

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termediate sense; \(\Delta\)-convergence; strong convergence; iteration process; fixed point; common
fixed point.
A complete $\text{CAT}(0)$ space is often called Hadamard space [13]. Let $\theta, \eta, \zeta$ be points of $X$ and $\eta_0$ be the midpoint of segment $[\eta, \zeta]$, denoted by $\frac{\eta \oplus \zeta}{2}$, then the $\text{CAT}(0)$ inequality gives
\[
d^2(\theta, \eta_0) \leq \frac{1}{2}d^2(\theta, \eta) + \frac{1}{2}d^2(\theta, \zeta) - \frac{1}{4}d^2(\eta, \zeta).
\]
This is called $(CN)$ inequality of Bruhat and Tits [3]. A geodesic space is said to be $\text{CAT}(0)$ space if and only if it satisfies the $(CN)$ inequality [16].

Fixed point theory in $\text{CAT}(0)$ spaces was first studied by Kirk [9]. He proved that every nonexpansive mapping defined on a closed bounded convex subset of a complete $\text{CAT}(0)$ space always had a fixed point.

Let $D$ be a non-empty subset of a $\text{CAT}(0)$ space $X$ and let $T : D \to D$ be a mapping. The Mann iteration [14] process is defined by the sequence $\{\theta_n\}$,
\[
\begin{align*}
\theta_1 & \in D \\
\theta_{n+1} & = (1 - a_n)\theta_n + a_nT\theta_n, \ n \in \mathbb{N},
\end{align*}
\]
where $\{a_n\}$ is a sequence in $(0, 1)$.

The Ishikawa iteration [8] process is defined by the sequence $\{\theta_n\}$,
\[
\begin{align*}
\theta_1 & \in D \\
\theta_{n+1} & = (1 - a_n)\theta_n + a_nT\eta_n, \\
\eta_n & = (1 - b_n)\theta_n + b_nT\theta_n, \ n \in \mathbb{N},
\end{align*}
\]
where $\{a_n\}$ and $\{b_n\}$ are the sequences in $(0, 1)$. This iteration process reduces to the Mann iteration process when $b_n = 0$ for all $n \in \mathbb{N}$.

In 2007 Agarwal, O’Regan and Sahu [1] introduced the S-iteration process in Banach space,
\[
\begin{align*}
\theta_1 & \in D \\
\theta_{n+1} & = (1 - a_n)\theta_n + a_nT\eta_n, \\
\eta_n & = (1 - b_n)\theta_n + b_nT\theta_n, \ n \in \mathbb{N},
\end{align*}
\]
where $\{a_n\}$ and $\{b_n\}$ are the sequences in $(0, 1)$. Note that (1.3) is independent of (1.2) (and hence of (1.1)).

In 1991 Schu [23], The modified Mann iteration process which is a generalization of the Mann iteration process,
\[
\begin{align*}
\theta_1 & \in D \\
\theta_{n+1} & = (1 - a_n)\theta_n + a_nT^n\theta_n, \ n \in \mathbb{N},
\end{align*}
\]
where $\{a_n\}$ is a sequence in $(0, 1)$.

In 1994, Tan and Xu [26], studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process,
\[
\begin{align*}
\theta_1 & \in D \\
\theta_{n+1} & = (1 - a_n)\theta_n + a_nT\eta_n, \\
\eta_n & = (1 - b_n)\theta_n + b_nT\theta_n, \ n \in \mathbb{N},
\end{align*}
\]
where the sequences \( \{a_n\} \) and \( \{b_n\} \) are in \( (0, 1) \). This iteration process reduces to the modified Mann iteration process when \( b_n = 0 \) for all \( n \in \mathbb{N} \). Recently, Agarwal, O'Regan and Sahu [1] introduced the modified S-iteration process in a Banach space,

\[
\begin{align*}
\theta_1 &\in D \\
\theta_{n+1} &= (1 - a_n)T^n\theta_n + a_n T^n \eta_n, \\
\eta_n &= (1 - b_n)\theta_n + b_n T^n \theta_n, \quad n \in \mathbb{N},
\end{align*}
\]

(1.6)

where the sequences \( \{a_n\} \) and \( \{b_n\} \) are in \( (0, 1) \). Note that (1.6) is independent of (1.5) (and hence of (1.4)).

Sahin and Basarir [21] modified iteration process (1.6) in a CAT(0) space as follows. Let \( D \) be a nonempty closed convex subset of a complete CAT(0) space \( X \) and \( T : D \to D \) be an asymptotically quasi-nonexpansive mapping with \( S_f(T) = \{ \theta \in K : T\theta = \theta \} \neq \emptyset \). Suppose that \( \{\theta_n\} \) is a sequence generated iteratively by

\[
\begin{align*}
\theta_1 &\in D \\
\theta_{n+1} &= (1 - a_n)T^n\theta_n \oplus a_n T^n \eta_n, \\
\eta_n &= (1 - b_n)\theta_n \oplus b_n T^n \theta_n, \quad n \in \mathbb{N},
\end{align*}
\]

(1.7)

where \( \{a_n\} \) and \( \{b_n\} \) are sequences such that \( 0 \leq a_n, b_n \leq 1 \) for all \( n \in \mathbb{N} \).

Consider \( D \) to be a nonempty closed, convex subset of a complete CAT(0) space \( X \) and \( T_1, T_2 : D \to D \) be two asymptotically quasi-nonexpansive mappings in the intermediate sense with \( S_f(T_1, T_2) = S_f(T_1) \cap S_f(T_2) \neq \emptyset \). Suppose that \( \{\theta_n\} \) is a sequence generated iteratively by

\[
\begin{align*}
\theta_1 &\in D \\
\theta_{n+1} &= (1 - a_n)T_1^n\theta_n \oplus a_n T_2^n \eta_n, \\
\eta_n &= (1 - b_n)T_2^n \theta_n \oplus b_n T_1^n \theta_n, \quad n \in \mathbb{N},
\end{align*}
\]

(1.8)

where the sequences \( \{a_n\} \) and \( \{b_n\} \), throughout the paper, are such that \( 0 \leq a_n, b_n \leq 1 \) for all \( n \geq 1 \).

\[
\begin{align*}
\theta_1 &\in D \\
\theta_{n+1} &= (1 - a_n)T_1^n\theta_n \oplus a_n T_2^n \eta_n, \\
\eta_n &= (\frac{b_n}{1-a_n})T_2^n \theta_n \oplus \frac{c_n}{1-a_n}T_1^n \theta_n, \quad n \in \mathbb{N},
\end{align*}
\]

(1.9)

where \( a_n, b_n, c_n \in (0, 1) \) and \( a_n + b_n + c_n = 1 \).

2. Preliminaries

Let us recall some definitions and known results in the existing literature on this concept. Goebel and Kirk [6] proposed the concept of an asymptotically nonexpansive mapping in 1972. Many authors have studied the iterative approximation problems for asymptotically nonexpansive and asymptotically quasi-nonexpansive
mappings in a Banach space and a CAT(0) space. Many authors studied the iterative approximation problem for asymptotically quasi-nonexpansive mappings in Banach space and a CAT(0) space\cite{11, 20, 19, 22, 24}.

**Definition 2.1.** Let $(X, d)$ be a metric space and $D$, its nonempty subset. Let $T : D \to D$ be a mapping. A point $\theta \in D$ is called a fixed point of $T$ if $T\theta = \theta$. We will also denote by $S_f(T)$ the set of fixed points of $T$, that is, $S_f(T) = \{\theta \in K : T\theta = \theta\}$.

**Definition 2.2.** Let $(X, d)$ be a metric space and $D$ its nonempty subset. Let $T_1, T_2 : D \to D$ be mappings. A point $\theta \in D$ is called a common fixed point of $T_1$ and $T_2$ if $T_1\theta = T_2\theta = \theta$, and $S_f(T_1, T_2) = \{\theta \in K : T_1\theta = T_2\theta = \theta\}$ is the set of common fixed points of $T_1$ and $T_2$.

**Definition 2.3.** Let $(X, d)$ be a CAT(0) space and $D$ be its nonempty subset of $X$ in CAT(0) space. Then $T : D \to D$ is said to be

1. nonexpansive, if $d(T\theta, T\eta) \leq d(\theta, \eta)$, for all $\theta, \eta \in D$;
2. uniformly $L$-Lipschitzian, if there exists a $L \in (0, \infty)$ such that $d(T^n\theta, T^n\eta) \leq Ld(\theta, \eta)$ for all $\theta, \eta \in D$ and $n \geq 1$;
3. asymptotically nonexpansive, if there exists a sequence $u_n \in [0, \infty)$ with the property $\lim_{n \to \infty} u_n = 0$ and such that $d(T^n(\theta), T^n\eta) \leq (1 + u_n)d(\theta, \eta)$, for all $\theta, \eta \in D$;
4. semi-compact, if for a sequence $\{\theta_n\}$ in $D$ with $\lim_{n \to \infty} d(\theta_n, T\theta_n) = 0$, there exists a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\theta_{n_k} \to p \in D$;
5. asymptotically quasi-nonexpansive type, if $S_f(T) \neq \emptyset$ and $\lim_{n \to \infty} \sup \{\sup \{d(T^n\theta, p) - d(\theta, p)\}, \theta \in D, p \in S_f(T)\} \leq 0$.

**Definition 2.4.** \cite{10} A sequence $\{\theta_n\}$ in CAT(0) space $X$ is said to be $\Delta-$converge to $\theta \in X$, if $x$ is unique asymptotic center of $\{\theta_n\}$ for every subsequence $\{u_n\}$ of $\{\theta_n\}$.

In this case we write $\Delta - \lim_{n \to \infty} \theta_n = \theta$ and call $\theta$ the $\Delta$ limit of $\{\theta_n\}$.

**Definition 2.5.** \cite{7} Let $D$ be closed, convex subset of a CAT(0) space $X$. A bounded sequence $\{\theta_n\}$ in $D$ is said to converge weakly to $q \in D$ iff $\phi(q) = \inf_{\theta \in D} \phi(\theta), \phi(q) = \lim \sup_{n \to \infty} d(\theta_n, \theta)$.

Note that $\{\theta_n\} \to q$ iff $A_D\{\theta_n\} = \{q\}$.

In 1993, Bruck, Kuczumow, and Reich\cite{2} introduced a notion of asymptotically nonexpansive mapping in the intermediate sense. A mapping $T : D \to D$ is said to be asymptotically nonexpansive in the intermediate sense provided that $T$ is uniformly continuous and

$$\limsup_{n \to \infty} \sup_{\theta, \eta \in D} \left\{ d(T^n\theta, T^n\eta) - d(\theta, \eta) \right\} \leq 0$$

From the above definition, it follows that an asymptotically nonexpansive mapping must be asymptotically nonexpansive mapping in the intermediate sense.

**Lemma 2.1.** \cite{18} Let $X$ be a CAT(0) space.
(i) Let \( \theta, \eta \in X \), for each \( t \in [0, 1] \), there exists a unique point \( \zeta \in [\theta, \eta] \) such that
\[
d(\theta, \zeta) = td(\theta, \eta), \quad d(\eta, \zeta) = (1 - t)d(\theta, \eta).
\]
We use the notation \((1 - t)\theta \oplus t\eta\) for unique point \( \zeta \) satisfying (2.1)

(ii) For all \( t \in [0, 1] \) and \( \theta, \eta, \zeta \in X \)
\[
d(((1 - t)\theta \oplus t\eta), \zeta) \leq (1 - t)d(\theta, \zeta) + td(\eta, \zeta).
\]

(iii) For any \( t \in [0, 1] \) and \( \theta, \eta, \zeta \in X \)
\[
d^2(\zeta, t\theta \oplus (1 - t)\eta) \leq td^2(\zeta, \theta) + (1 - t)d^2(\zeta, \eta) - t(1 - t)d^2(\theta, \eta).
\]

Let \( \{\theta_n\} \) be a bounded sequence in closed, convex subset \( D \) of a \( CAT(0) \)-space \( X \). For any \( \theta \) in \( X \), set \( r(\theta, \{\theta_n\}) = \lim_{n \to \infty} \sup d(\theta, \theta_n) \). The asymptotic radius \( r(\{\theta_n\}) \) of \( \{\theta_n\} \) is given by \( r(\{\theta_n\}) = \inf \{r(\theta, \{\theta_n\}) : \theta \in X \} \) and the asymptotic centre \( A(\{\theta_n\}) \) of \( \{\theta_n\} \) is the set \( A(\{\theta_n\}) = \{ \theta \in X : r(\{\theta_n\}) = r(\theta, \{\theta_n\}) \} \), to be known that, \( A(\{\theta_n\}) \) consists of exactly one point in \( CAT(0) \) space. In this paper, the symbol “\( \rightharpoonup \)” for weak convergence.

**Lemma 2.2.** [15] Given \( \{\theta_n\} \subset X \) such that \( \{\theta_n\} \) \( \Delta \)-converges to \( \theta \) and given \( \eta \in X \) with \( \eta \neq \theta \), then
\[
\lim_{n \to \infty} \sup d(\theta_n, \theta) < \lim_{n \to \infty} \sup d(\theta_n, \eta)
\]

The above condition is known as Opial property in Banach space.

**Lemma 2.3.** [17] Let \( \{\theta_n\} \) be a bounded sequence in \( CAT(0) \) space \( X \), and let \( D \) be a closed convex subset of \( X \) which contains \( \{\theta_n\} \). Then
(i) \( \Delta \lim \theta_n = \theta \) implies \( \theta_n \rightharpoonup \theta \).
(ii) The convergence of (i) is true if \( \{\theta_n\} \) is regular.

**Lemma 2.4.** [4] If \( \{\theta_n\} \) be a bounded sequence in \( CAT(0) \) space \( X \), with \( A(\{\theta_n\}) = \{\theta\} \), and \( \{u_n\} \) is a subsequence of \( \{\theta_n\} \), with \( A(\{u_n\}) = \{u\} \), and the sequence \( \{d(\theta_n, u)\} \) converges, then \( u = \theta \)

**Lemma 2.5.** [5] If \( D \) is a closed convex subset of a \( CAT(0) \) space \( X \) and if \( \{\theta_n\} \) is bounded sequence in \( D \), then the asymptotically center of \( \{\theta_n\} \) is in \( D \)

**Lemma 2.6.** [25] Suppose \( \{a_n\} \) and \( \{b_n\} \) are two non-negative sequences of real numbers such that \( a_{n+1} \leq a_n + b_n \), for all \( n \geq 1 \). If \( \sum_{n=1}^{\infty} b_n \) converges then \( \lim_{n \to \infty} a_n \) exists.

**Lemma 2.7.** [10] Let \( X \) be a complete \( CAT(0) \) space, \( D \) be a nonempty closed convex subset of \( X \). If \( T : D \to D \) is an asymptotically nonexpansive mapping in the intermediate sense, then \( T \) has a fixed point.

**Lemma 2.8.** [10] Let \( X \) be a complete \( CAT(0) \) space, \( D \) be a nonempty closed convex subset of \( X \). If \( T : D \to D \) is an asymptotically nonexpansive mapping in the intermediate sense, then \( S_f(T) \) is closed and convex.
Lemma 2.9. [10] (Demiclosed principle) Let $D$ be a closed convex subset of a complete $\text{CAT}(0)$ space $X$ and $T : D \to D$ is an asymptotically nonexpansive mapping in the intermediate sense. If $\{\theta_n\}$ is a bounded sequence in $D$ such that $\lim_{n \to \infty} d(\theta_n, T\theta_n) = 0$ and $\{\theta_n\} \to w$, then $Tw = w$.

Lemma 2.10. [10] Let $D$ be a closed convex subset of a complete $\text{CAT}(0)$ space $X$ and $T : D \to D$ is an asymptotically nonexpansive mapping in the intermediate sense. If $\{\theta_n\}$ is bounded sequence in $D$ converging to $\theta$ and $\lim_{n \to \infty} d(\theta_n, T\theta_n) = 0$, then $\theta \in D$ and $T\theta = \theta$.

3. Main results

In this section, we prove the following lemmas using a new scheme (1.9) and also prove convergence results using these lemmas.

**Lemma 3.1.** Let $D$ be a closed and convex subset of complete $\text{CAT}(0)$ space $X$ and $T_1, T_2 : D \to D$ be two asymptotically nonexpansive mappings in the intermediate sense with $S_f(T_1, T_2) \neq \emptyset$. Suppose that $\{\theta_n\}$ is defined by the iteration (1.9). Put

$$
\alpha_n = \max \left\{ 0, \sup_{\theta, \eta \in D, n \geq 1} \left( d(T_1^n \theta, T_1^n \eta) - d(\theta, \eta) \right) \right\}
$$

and

$$
\beta_n = \max \left\{ 0, \sup_{\theta, \eta \in D, n \geq 1} \left( d(T_2^n \theta, T_2^n \eta) - d(\theta, \eta) \right) \right\}
$$

such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$. Suppose that $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0, 1)$. Then:

(i) $\lim_{n \to \infty} d(\theta_n, p)$ exists for all $p \in S_f(T_1, T_2)$.

(ii) $\lim_{n \to \infty} d(\theta_n, S_f(T_1, T_2))$ exists.

**Proof.** Let $p \in S_f(T_1, T_2)$. Using (1.9), (3.1), (3.2) and (2.2), we have

$$
d(\eta_n, p) = d \left( \frac{b_n}{1 - a_n} T_1^n \theta_n + \frac{c_n}{1 - a_n} T_2^n \theta_n, p \right)
$$

$$
\leq \frac{b_n}{1 - a_n} d(T_1^n \theta_n, p) + \frac{c_n}{1 - a_n} d(T_2^n \theta_n, p)
$$

$$
\leq \frac{b_n}{1 - a_n} \left[ d(\theta_n, p) + a_n \right] + \frac{c_n}{1 - a_n} \left[ d(\theta_n, p) + \beta_n \right]
$$

$$
\leq d(\theta_n, p) + \frac{\alpha_n b_n}{1 - a_n} + \frac{\beta_n c_n}{1 - a_n}
$$

$$
\leq d(\theta_n, p) + H_n + K_n.
$$

this gives

$$
d(\eta_n, p) \leq d(\theta_n, p) + H_n + K_n.
$$

(3.3)
Again using (1.9), (2.2) and (3.1)-(3.3), we have
\[ d(\theta_{n+1}, p) = d((1 - a_n)T^n_2\theta_n + a_nT^n_1\eta_n, p) \]
\[ \leq (1 - a_n)d(T^n_2\theta_n, p) + a_n d(T^n_1\eta_n, p) \]
\[ \leq (1 - a_n)[d(\theta_n, p) + \beta_n] + a_n[d(\eta_n, p) + \alpha_n] \]
\[ \leq (1 - a_n)[d(\theta_n, p) + \beta_n] + a_n[d(\theta_n, p) + H_n + K_n + \alpha_n] \]
\[ \leq d(\theta_n, p) + (1 - a_n)\beta_n + a_n\alpha_n + a_n\alpha_n b_n + a_n\frac{\beta_n c_n}{1 - a_n}, \] this gives
\[ d(\theta_{n+1}, p) \leq d(\theta_n, p) + (1 - a_n)\beta_n + a_n\alpha_n + a_n\frac{\alpha_n b_n}{1 - a_n} + a_n\frac{\beta_n c_n}{1 - a_n} \] (3.4)

taking infimum over all \( p \in S_f(T_1, T_2) \), we get
\[ d(\theta_{n+1}, p) \leq d(\theta_n, S_f(T_1, T_2)) + (1 - a_n)\beta_n + a_n\alpha_n + a_n\frac{\alpha_n b_n}{1 - a_n} + a_n\frac{\beta_n c_n}{1 - a_n}. \] (3.5)

Since by hypothesis of the lemma \( \sum_{n=1}^{\infty} \alpha_n < \infty \) and \( \sum_{n=1}^{\infty} \beta_n < \infty \), it follows from Lemma 2.6, relation (3.4),(3.5) that \( \lim_{n \to \infty} d(\theta_n, p) \) and \( \lim_{n \to \infty} d(\theta_n, S_f(T_1, T_2)) \) exists.

Lemma 3.2. Let \( D \) be a nonempty closed, convex subset of complete \( CAT(0) \) space \( X \) and \( T_1, T_2 : D \to D \) be two asymptotically nonexpansive mappings in the intermediate sense with \( S_f(T_1, T_2) = S_f(T_1) \cap S_f(T_2) \neq \emptyset \). Suppose that the sequence \( \{\theta_n\} \) is defined by the iteration process (1.9) and \( \alpha_n \) and \( \beta_n \) are taken as in Lemma 3.1. Suppose that \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are real sequences in \( [l, m] \) for some \( l, m \in (0, 1) \). If \( d(\theta, T_1\theta) \leq d(T_2\theta, T_1\theta) \) for all \( \theta \in D \), then \( \lim_{n \to \infty} d(\theta_n, T_1\theta_n) = 0 \) and \( \lim_{n \to \infty} d(\theta_n, T_2\theta_n) = 0 \).

Proof. Using (1.9) and (2.3), we have
\[ d^2(\eta_n, p) = d^2\left(\frac{b_n}{1 - a_n}T^n_1\theta_n + \frac{c_n}{1 - a_n}T^n_2\theta_n, p\right) \]
\[ \leq \frac{b_n}{1 - a_n}d^2(T^n_1\theta_n, p) + \frac{c_n}{1 - a_n}d^2(T^n_2\theta_n, p) - \frac{b_n c_n}{(1 - a_n)^2}d^2(T^n_2\theta_n, T^n_1\theta_n) \]
\[ \leq \frac{b_n}{1 - a_n}[d(\theta_n, p) + \alpha_n]^2 + \frac{c_n}{1 - a_n}[d(\theta_n, p) + \beta_n]^2 - \frac{b_n c_n}{(1 - a_n)^2}d^2(T^n_2\theta_n, T^n_1\theta_n) \]
\[ \leq \frac{b_n}{1 - a_n}[d^2(\theta_n, p) + 2\alpha_n d(\theta_n, p)] + \frac{c_n}{1 - a_n}[d^2(\theta_n, p) + \beta_n + 2\beta_n d(\theta_n, p)] - \frac{b_n c_n}{(1 - a_n)^2}d^2(T^n_2\theta_n, T^n_1\theta_n) \]
\[ \leq d^2(\theta_n, p) + \frac{b_n}{1 - a_n}d(\theta_n, p) + \frac{c_n}{1 - a_n}d(\theta_n, p) \]
\[ \leq d^2(\theta_n, p) + A_n + B_n = \frac{b_n c_n}{(1 - a_n)^2} d^2(T_2^n \theta_n, T_1^n \theta_n), \]

this gives

\[ d^2(\eta_n, p) \leq d^2(\theta_n, p) + A_n + B_n = \frac{b_n c_n}{(1 - a_n)^2} d^2(T_2^n \theta_n, T_1^n \theta_n), \]

where

\[ A_n = \frac{b_n \alpha_n^2}{1 - a_n} + 2 \frac{\alpha_n b_n}{1 - a_n} d(\theta_n, p), \]
\[ B_n = \frac{c_n \beta_n^2}{1 - a_n} + 2 \frac{c_n \beta_n}{1 - a_n} d(\theta_n, p). \]

Since by hypothesis \( \sum_{n=1}^{\infty} \alpha_n < \infty \) and \( \sum_{n=1}^{\infty} \beta_n < \infty \), it follows that \( \sum_{n=1}^{\infty} A_n < \infty \) and \( \sum_{n=1}^{\infty} B_n < \infty \).

Again using (1.9), (2.3) and (3.6), we have

\[ d^2(\theta_{n+1}, p) = d^2((1 - a_n)T_2^n \theta_n + a_n T_1^n \eta_n, p) \]
\[ \leq (1 - a_n)d^2(T_2^n \theta_n, p) + a_n d^2(T_1^n \eta_n, p) - a_n(1 - a_n)d^2(T_1^n \eta_n, T_2^n \theta_n) \]
\[ \leq a_n [d(\eta_n, p) + \alpha_n]^2 + (1 - a_n)[d(\theta_n, p) + \beta_n]^2 - a_n(1 - a_n)d^2(T_1^n \eta_n, T_2^n \theta_n) \]
\[ \leq a_n [d^2(\eta_n, p) + \alpha_n^2 + 2\alpha_n d(\eta_n, p)] + (1 - a_n)d^2(\theta_n, p) + \beta_n^2 + 2\beta_n d(\theta_n, p)] - a_n(1 - a_n)d^2(T_1^n \eta_n, T_2^n \theta_n) \]
\[ \leq a_n [d^2(\eta_n, p) + L_n] + (1 - a_n)d^2(\theta_n, p) + M_n - a_n(1 - a_n)d^2(T_1^n \eta_n, T_2^n \theta_n) \]
\[ \leq a_n d^2(\eta_n, p) + (1 - a_n)d^2(\theta_n, p) + a_n L_n + (1 - a_n)M_n - a_n(1 - a_n)d^2(T_1^n \eta_n, T_2^n \theta_n) \]
\[ \leq a_n d^2(\theta_n, p) + A_n + B_n - \frac{b_n c_n}{(1 - a_n)^2} d^2(T_2^n \theta_n, T_1^n \theta_n) ] + (1 - a_n)d^2(\theta_n, p) + a_n L_n + (1 - a_n)M_n - a_n(1 - a_n)d^2(T_1^n \eta_n, T_2^n \theta_n) \]
\[ \leq a_n d^2(\theta_n, p) + a_n A_n + a_n B_n - \frac{a_n b_n c_n}{(1 - a_n)^2} d^2(T_2^n \theta_n, T_1^n \theta_n) ] + (1 - a_n)d^2(\theta_n, p) + a_n L_n + (1 - a_n)M_n - a_n(1 - a_n)d^2(T_1^n \eta_n, T_2^n \theta_n) \]
\[ \leq d^2(\theta_n, p) + P_n + Q_n - \frac{a_n b_n c_n}{(1 - a_n)^2} d^2(T_2^n \theta_n, T_1^n \theta_n) ] - a_n(1 - a_n)d^2(T_1^n \eta_n, T_2^n \theta_n), \]

this gives

\[ d^2(\theta_{n+1}, p) \leq d^2(\theta_n, p) + P_n + Q_n - \frac{a_n b_n c_n}{(1 - a_n)^2} d^2(T_2^n \theta_n, T_1^n \theta_n) ] - a_n(1 - a_n)d^2(T_1^n \eta_n, T_2^n \theta_n), \]

where \( L_n = a_n + 2\alpha_n d(\eta_n, p), M_n = \beta_n + 2\beta_n d(\theta_n, p), P_n = a_n A_n + a_n B_n, Q_n = a_n L_n + (1 - a_n)M_n. \)

Since by hypothesis \( \sum \alpha_n < \infty, \sum \beta_n < \infty \), it follows that \( \sum P_n < \infty, \sum Q_n < \)
\[ \infty. \]

Now from (3.7) we have,
\[ \frac{a_n b_n c_n}{(1 - a_n)^2} d^2(T^n_2 \theta_n, T^n_1 \theta_n) \leq [d^2(\theta_n, p) - d^2(\theta_{n+1}, p)] + Q_n + P_n \]
\[ d^2(T^n_2 \theta_n, T^n_1 \theta_n) \leq \frac{(1 - a_n)^2}{a_n b_n c_n} [d^2(\theta_n, p) - d^2(\theta_{n+1}, p)] + \frac{(1 - a_n)^2}{a_n b_n c_n} Q_n + \frac{(1 - a_n)^2}{a_n b_n c_n} P_n \]
\[ d^2(T^n_2 \theta_n, T^n_1 \theta_n) \leq \frac{(1 - m)^2}{l^3} [d^2(\theta_n, p) - d^2(\theta_{n+1}, p)] + \frac{(1 - m)^2}{l^3} Q_n + \frac{(1 - m)^2}{l^3} a_n P_n. \]  
(3.8)

Now again from (3.7) we get,
\[ d^2(\theta_{n+1}, p) \leq d^2(\theta_n, p) + P_n + Q_n - a_n (1 - a_n) d^2(T^n_1 \eta_n, T^n_2 \theta_n) \]
\[ d^2(T^n_1 \eta_n, T^n_2 \theta_n) \leq \frac{1}{\alpha_n (1 - \alpha_n)} [d^2(\theta_n, p) + d^2(\theta_{n+1}, p)] + \frac{1}{\alpha_n (1 - \alpha_n)} P_n + \frac{1}{\alpha_n (1 - \alpha_n)} Q_n \]
\[ \leq \frac{1}{\alpha_n (1 - \alpha_n)} [d^2(\theta_{n+1}, p) + d^2(\theta_n, p)] + \frac{1}{\alpha_n (1 - \alpha_n)} R_n \]
\[ \leq \frac{1}{l(1 - m)} [d^2(\theta_{n+1}, p) + d^2(\theta_n, p)] + \frac{1}{l(1 - m)} R_n, \]
this gives
\[ d^2(T^n_1 \eta_n, T^n_2 \theta_n) \leq \frac{1}{l(1 - m)} [d^2(\theta_{n+1}, p) + d^2(\theta_n, p)] + \frac{1}{l(1 - m)} R_n, \]  
(3.9)
where \( R_n = P_n + Q_n. \)
Since \( P_n \to 0 \) as \( n \to \infty \) and \( Q_n \to 0 \) as \( n \to \infty \) so \( R_n \to 0 \) as \( n \to \infty \) and \( d(\theta_n, p) \) is convergent therefore on taking limit as \( n \to \infty \) in (3.8), (3.9) we get
\[ \lim_{n \to \infty} d^2(T^n_2 \theta_n, T^n_1 \theta_n) = 0. \]  
(3.10)

and
\[ \lim_{n \to \infty} d^2(T^n_1 \eta_n, T^n_2 \theta_n) = 0. \]  
(3.11)

Now
\[ d(T^n_2 \theta_n, \theta_n) \leq d(T^n_2 \theta_n, T^n_1 \theta_n) + d(T^n_1 \theta_n, \theta_n) \]
\[ \leq d(T^n_2 \theta_n, T^n_1 \theta_n) + d(T^n_1 \theta_n, T^n_2 \theta_n) \]
\[ \leq 2d(T^n_2 \theta_n, T^n_1 \theta_n) \to 0 \text{ as } n \to \infty \]
this gives
\[ d(T^n_2 \theta_n, \theta_n) \leq 2d(T^n_2 \theta_n, T^n_1 \theta_n) \to 0 \text{ as } n \to \infty \]  
(3.12)
by (3.10) and (3.12), we obtain
\[ \lim_{n \to \infty} d(T^n_1 \theta_n, \theta_n) = 0. \]  
(3.13)
Again note that
\[ d(\theta_{n+1}, T_2^m \theta_n) = d((1 - a_n)T_2^m \theta_n \oplus a_n T_1^m \eta_n, T_2^m \theta_n) \leq a_n d(T_1^m \eta_n, T_2^m \theta_n) \to 0 \text{ as } n \to \infty, \]
this gives
\[ d(\theta_{n+1}, T_2^m \theta_n) \leq a_n d(T_1^m \eta_n, T_2^m \theta_n). \tag{3.14} \]

By (3.12) and (3.14), we get
\[ d(\theta_{n+1}, \theta_n) \leq d(\theta_{n+1}, T_2^m \theta_n) + d(T_2^m \theta_n, \theta_n) \to 0 \text{ as } n \to \infty. \tag{3.15} \]

Let \( \phi_n = d(T^n \theta_n, \theta_n) \) by (3.12) we have \( \phi_n \to 0 \text{ as } n \to \infty. \)

We have
\[ d(\theta_n, T_2 \theta_n) \leq d(\theta_n, \theta_{n+1}) + d(\theta_{n+1}, T_2 \theta_{n+1}) + d(T_2^m \theta_{n+1}, T_2^m \theta_n) \]
\[ \leq d(\theta_n, \theta_{n+1}) + d(\theta_{n+1}, T_2 \theta_{n+1}) + \phi_{n+1} \]
\[ \leq 2d(\theta_n, \theta_{n+1}) + \phi_{n+1} + \beta_{n+1} + d(T_2^m \theta_n, T_2 \theta_n) \]
\[ \leq 2d(\theta_n, \theta_{n+1}) + \phi_{n+1} + \beta_{n+1} + d(T_2^m \theta_n, T_2 \theta_n) \to 0 \text{ as } n \to \infty, \tag{3.16} \]
this gives
\[ d(\theta_n, T_2 \theta_n) \leq 2d(\theta_n, \theta_{n+1}) + \phi_{n+1} + \beta_{n+1} + d(T_2^m \theta_n, T_2 \theta_n) \to 0 \text{ as } n \to \infty \tag{3.16} \]

By (3.12), (3.15), \( d_{n+1} \to 0 \text{ as } n \to \infty \) and the uniform continuity of \( T_2 \). Similarly we can prove that \( \lim_{n \to \infty} d(\theta_n, T_1 \theta_n) = 0. \)

\[ \Box \]

**Theorem 3.3.** Let \( D \) be a nonempty closed convex subset of a complete CAT(0) space \( X \), and let \( T_1, T_2 : D \to D \) be two asymptotically nonexpansive mappings in the intermediate sense such that \( S_f(T_1, T_2) \neq \emptyset \). Suppose that \( \theta_n \) is defined by the iteration process (1.9) and \( a_n \) and \( b_n \) are taken as in Lemma 3.1. Suppose that \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are real sequences in \([l, m]\) for some \( l, m \in (0, 1) \). Then the sequence \( \{\theta_n\} \) is \( \Delta \)-convergent to a point of \( S_f(T_1, T_2) \).

**Proof.** We first show that \( w_w(\{\theta_n\}) \subseteq S_f(T_1, T_2) \) Let \( v \in w_w(\{\theta_n\}) \), then there exists a subsequence \( \{v_n\} \) of \( \{\theta_n\} \) such that \( A(\{\theta_n\}) = \{v\} \). By Lemma 2.5, there exists a subsequence \( \{w_n\} \) of \( \{v_n\} \) such that \( \Delta - \lim_{n \to \infty} w_n = w \in D \). By Lemma 2.10, \( w \in S_f(T_2) \) and \( w \in S_f(T_1) \) and so \( w \in S_f(T_1, T_2) \). By Lemma 3.1 \( \lim_{n \to \infty} d(\theta_n, S_f(T_1, T_2)) \) exists, so by Lemma 2.4 we have \( v = w \), i.e., \( w_w(\{\theta_n\}) \subseteq S_f(T_1, T_2) \).

To show that \( \{\theta_n\} \) is \( \Delta \)-converges to a point in \( S_f(T_1, T_2) \), it is sufficient to show that \( w_w(\{\theta_n\}) \) consists of exactly one point.

Let \( \{v_n\} \) be a subsequence of \( \{\theta_n\} \) with \( A(\{v_n\}) = \{v\} \), and \( A(\{\theta_n\}) = \{\theta\} \) for some \( v \in W_w(\{\theta_n\}) \subseteq S_f(T_1, T_2) \) and \( \{\theta_n, w\} \) converges. By Lemma 2.4, we have \( \theta = w \in S_f(T_1, T_2) \). Thus \( w_w(\{\theta_n\}) = \{\theta_n\} \). This shows that \( \{\theta_n\} \) is \( \Delta \)-convergent to a point of \( S_f(T_1, T_2) \). \[ \Box \]

**Theorem 3.4.** Let \( D \) be a nonempty closed, convex subset of a complete CAT(0) space \( X \) and let \( T_1, T_2 : D \to D \) be two asymptotically nonexpansive mappings in the intermediate sense such that \( S_f(T_1, T_2) = S_f(T_1) \cap S_f(T_2) \neq \emptyset \). Suppose that
\{\theta_n\} is defined by the iteration process (1.9), \(\alpha_n\) and \(\beta_n\) be taken as in Lemma 3.1. Suppose that \(\{a_n\}, \{b_n\}\) and \(\{c_n\}\) are real sequences in \([l, m]\) for some \(l, m \in (0, 1)\). If \(\lim \inf_{n \to \infty} d(\theta_n, S_f(T_1, T_2)) = 0\) and \(\lim \sup_{n \to \infty} d(\theta_n, S_f(T_1, T_2)) = 0\), where \(d(\theta, S_f(T_1, T_2)) = \lim_{p \in S_f(T_1, T_2)} d(\theta, p)\). Then the sequence \(\{\theta_n\}\) converges strongly to a point in \(S_f(T_1, T_2)\).

**Proof.** From (3.5), we have
\[
d(\theta_{n+1}, p) \leq d(\theta_n, S_f(T_1, T_2)) + (1 - a_n)\beta_n + a_n\alpha_n + a_n\frac{\beta_n c_n}{1 - a_n} + a_n\frac{\beta_n c_n}{1 - a_n},
\]
where \(p \in S_f(T_1, T_2)\). Since by hypothesis of the theorem \(\sum_{n=1}^{\infty} \alpha_n < \infty\) and \(\sum_{n=1}^{\infty} \beta_n < \infty\) by Lemma 2.6 and \(\lim \inf_{n \to \infty} d(\theta_n, S_f(T_1, T_2)) = 0\) or \(\lim \sup_{n \to \infty} d(\theta_n, S_f(T_1, T_2)) = 0\), gives that \(\lim_{n \to \infty} d(\theta_n, S_f(T_1, T_2)) = 0\).

Now, we show that \(\{\theta_n\}\) is Cauchy sequence in \(D\).

From (3.4) and by hypothesis \(0 < l < a_n, b_n, c_n < m < 1\), we have
\[
d(\theta_{n+p}, q) \leq d(\theta_{n+p-1}, q) + \frac{m^2}{1 - m} \alpha_{n+p-q} + \frac{m^2}{1 - m} \beta_{n+p-q} + (1 - l)\beta_{n+p-q} + m\alpha_{n+p-q}
\]

\[
\leq d(\theta_n, q) + \frac{m^2}{1 - m} \sum_{k=n+1}^{n+p-1} \alpha_k + \frac{m^2}{1 - m} \sum_{k=n+1}^{n+p-1} \beta_k + (1 - l)\sum_{k=n+1}^{n+p-1} \beta_k + m\sum_{k=n+1}^{n+p-1} \alpha_k
\]

\[
\leq d(\theta_n, q) + \left(\frac{m}{1 - m}\right)\sum_{k=n+1}^{n+p-1} \alpha_k + \left(\frac{m^2}{1 - m} + (1 - l)\right)\sum_{k=n+1}^{n+p-1} \beta_k,
\]

for \(n, p \in \mathbb{N}\) & \(q \in S_f(T_1, T_2)\).

Since \(\lim_{n \to \infty} d(\theta_n, S_f(T_1, T_2)) = 0\), therefore for any \(\epsilon > 0\), there exists a natural number \(n_0\) such that \(d(\theta_n, S_f(T_1, T_2)) < \frac{\epsilon}{12}\), \(\sum_{k=n_0}^{n+p-1} \alpha_k < \frac{(1 - m)\epsilon}{6m}\)

and \(\sum_{k=n_0}^{n+p-1} \beta_k < \left(\frac{(1 - m)\epsilon}{6(m^2 + (1 - m)(1 - l))}\right)\) for all \(n \geq n_0\). So we can find \(p^* \in S_f(T_1, T_2)\) such that \(d(\theta_0, p^*) < \left(\frac{\epsilon}{6}\right)\). Hence for all \(n \geq n_0, p \geq 1\), we have
\[
d(\theta_{n+p}, \theta_n) \leq 2d(\theta_n, p^*) + 2\frac{m}{1 - m} \sum_{k=n_0}^{n+p-1} \alpha_k + 2\left[\frac{m^2}{1 - m} + (1 - l)\right] \sum_{k=n_0}^{n+p-1} \beta_k
\]

\[
\leq 2\left(\frac{\epsilon}{6}\right) + 2\left(\frac{m}{1 - m}\right)\frac{(1 - m)\epsilon}{6m} + 2\left[\frac{m^2}{1 - m} + (1 - l)\right]\left(\frac{(1 - m)\epsilon}{6(m^2 + (1 - m)(1 - l))}\right)
\]

\[
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

\[\square\]

**Theorem 3.5.** Let \(D\) be a nonempty closed convex subset of a complete CAT(0) space \(X\), and let \(T_1, T_2 : D \to D\) be two asymptotically nonexpansive mappings in the intermediate sense with \(S_f(T_1, T_2) \neq \emptyset\). Suppose that \(\{\theta_n\}\) is defined by the iteration process (1.9) and \(\alpha_n\) and \(\beta_n\) are taken as in Lemma 3.1. Suppose that \(\{a_n\}, \{b_n\}\) and \(\{c_n\}\) are real sequences in \([l, m]\) for some \(l, m \in (0, 1)\). If \(T_1, T_2\) satisfy the following conditions:

(i) \(\lim_{n \to \infty} d(\theta_n, T_1\theta_n) = 0\) and \(\lim_{n \to \infty} d(\theta_n, T_2\theta_n) = 0\)

(ii) If the sequence \(\{\zeta_n\}\) in \(D\) satisfies \(\lim_{n \to \infty} d(\theta_n, \zeta_n) = 0\) and \(\lim_{n \to \infty} d(\zeta_n, T_2\zeta_n) = 0\)
then $\liminf_{n \to \infty} d(\zeta_n, S_f(T_1, T_2)) = 0$ or $\limsup_{n \to \infty} d(\zeta_n, S_f(T_1, T_2)) = 0$.

Then the sequence $\{\theta_n\}$ converges strongly to a point of $S_f(T_1, T_2)$.

**Proof.** It following from the hypothesis that $\lim_{n \to \infty} d(\theta_n, T_1\theta_n) = 0$ and $\lim_{n \to \infty} d(\theta_n, T_2\theta_n) = 0$ from (ii), $\liminf_{n \to \infty} d(\zeta_n, S_f(T_1, T_2)) = 0$ or $\limsup_{n \to \infty} d(\zeta_n, S_f(T_1, T_2)) = 0$. Therefore, the sequence $\{\theta_n\}$ must converge strongly to a point in $S_f(T_1, T_2)$ by Theorem 3.3.

**Theorem 3.6.** Let $D$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and let $T_1, T_2 : D \to D$ be two asymptotically nonexpansive mappings in the intermediate sense with $S_f(T_1, T_2) \neq \emptyset$ Suppose that $\{\theta_n\}$ is defined by the iteration process (1.9) and $\alpha_n$ and $\beta_n$ are taken as in Lemma 3.2. Suppose that $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0, 1)$. If either $T_1$ or $T_2$ is semi-compact, then the sequence $\{\theta_n\}$ converges strongly to a point of $S_f(T_1, T_2)$.

**Proof.** Suppose that $T_2$ is semi-compact. By Lemma 3.2, we have $\lim_{n \to \infty} d(\theta_n, T_2\theta_n) = 0$. So there exists a subsequence $\{\theta_{n_j}\}$ of $\{\theta_n\}$ such that $\{\theta_{n_j}\} \to p \in D$. Now again Lemma 3.2 guarantees that $\lim_{n \to \infty} d(\theta_{n_j}, T\theta_{n_j}) = 0$ and so $d(p, Tp) = 0$. Similarly, we can show that $d(p, Tp) = 0$. Thus $p \in S_f(T_1, T_2)$. By (3.5), we have

$$d(\theta_{n+1}, p) \leq d(\theta_n, S_f(T_1, T_2)) + (1 - a_n)\beta_n + a_n\alpha_n + a_n \frac{\alpha_n b_n}{1 - a_n} + a_n \frac{\beta_n c_n}{1 - a_n}.$$ 

Since by hypothesis $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, by Lemma 2.6, $\lim_{n \to \infty} d(\theta_n, p)$ exists and $\theta_{n_j} \to p \in S_f(T_1, T_2)$ gives that $\theta_n \to p \in S_f(T_1, T_2)$. This shows that $\{\theta_n\}$ converges strongly to a point of $S_f(T_1, T_2)$. 

**References**


