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SOME GENERAL SUMMATION RESULTS WITH APPLICATIONS TO ARCTANGENT-TYPE SERIES

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Abstract. This article extends the scopes of the telescopic summation and Abel summation by parts methods by introducing tuning parameters and sequences of functions. In a substantial part, these general results are applied to arctangent-type series. With this approach, we re-examine known results from a fresh viewpoint and establish numerous new ones. An emphasis is placed on the elaboration of innovative unified formulas that are applicable to various scenarios. The exact values of new arctangent-type series, including some defined with the binomial coefficients or Fibonacci sequence, are obtained. We thus contribute to the advancement of telescopic techniques and enriches our understanding of arctangent-type series.

1. Introduction

In the field of mathematical analysis, the study of series has an important place due to its wide applications in diverse fields, including physics, engineering, computer science, and statistics. Among the classical methods frequently used in the manipulation and evaluation of series, there are the telescopic summation and Abel summation by parts methods. Through thorough rearrangement and grouping, the telescopic summation method exploits the cancellation of terms in a series to simplify its computation. On the other hand, the Abel summation by parts method consists of using the telescopic summation method to transform the series of products of terms into other series of more manageable products of terms, still with the aim of simplifying the computation. For a diverse presentation of these methods, we refer to [1], [7], [15], [17], and [8]. However, there is research to be carried out in this direction, aimed at exploring the manipulation and evaluation of sophisticated series.

In this article, we first establish extensions of the telescopic summation and Abel summation by parts methods. They distinguish from the former versions by including two tuning parameters, denoted as ℓ and m, and also consider sequences of functions depending on a variable, denoted by x. The finite and infinite versions of these results are given. Subsequently, we use them to examine various arctangent-type series, i.e., series involving the arctangent function, denoted

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as $\arctan(x)$, in the main term. We rediscover known results, including some presented in [5], and establish numerous new ones. We thus improve our mathematical understanding of this special kind of series. Among our new results, those described below stand out as significant enough to support this claim at this preliminary stage.

- For any $s > 0$ and $x > 0$, we will prove that

$$\text{"}\sum_{k=1}^{+\infty} \arctan\left\{\frac{s(s+1)x}{s^2x^2 + k^2(s+1)^2 + k(s^2-1) - s}\right\} = \arctan(x)\text{"}$$

and

$$\text{"}\sum_{k=1}^{+\infty} \arctan\left\{\frac{x(s+1)\sqrt{s}}{\left[sx^2 + \sqrt{[k(s+1)+s][k(s+1)-1]}\right]\left[\sqrt{k(s+1)+s} + \sqrt{k(s+1)-1}\right]}\right\}$$
$$= \arctan(x)\text{"}.$$

Thanks to our knowledge of the arctangent function, these formulas are easily manageable. In particular, some known examples appear with specific values of s and x, and numerous new ones can be derived. Thus, these general results can be viewed as unified formulas.

- For any $\ell \in \mathbb{N} \setminus \{0\}$ and $x > 0$, we may also mention the following arctangent-type series involving binomial coefficients denoted as $\binom{k}{\ell}$:

$$\text{"}\sum_{k=\ell}^{+\infty} \arctan\left[\frac{\binom{k}{\ell-1}x}{\binom{k}{\ell}\binom{k+1}{\ell} + x^2}\right] = \arctan(x)\text{"}.$$

The innovation is the presence of these particular coefficients and also the use of the initial value at ℓ.

- Furthermore, with reference to the Fibonacci sequence denoted as $(F_n)_{n\in\mathbb{N}}$, the following innovative formula will be demonstrated:

$$\text{"}\sum_{k=1}^{+\infty} k \arctan\left(\frac{F_{2k+2}}{F_{2k+2}^2 + 2}\right) = \frac{\pi}{4}\text{"}.$$

One of its features is the k in factor of the main arctangent term. To the best of our knowledge, this kind of modification has not been examined in the literature. In this sense, we also contribute to the arctangent-type series dealing with the Fibonacci sequence, following the spirit of the works in [14], [2], [10], and [11].

In addition to that, the arctangent function satisfies some interesting analytical properties that do not seem to be much exploited in the context of series. Among them, we can mention the following arctangent formula involving the square root: " $\arctan(x) = (1/2)\arctan\left\{x/\left[1+\sqrt{1+x^2}\right]\right\}$ ". In a sense, we fill this gap; certain techniques based on these properties are developed and illustrated by attractive examples.

The remainder of this article is organized as follows: In Section 2, our general versions of the telescopic summation and Abel summation by parts methods are presented, along with their detailed proofs. Section 3 provides substantial application work for these results; a plethora of arctangent-type series are established and discussed. A conclusion is given in Section 4.

2. General results

We begin this section by providing an extended version of the Abel summation by parts method, from which we will derive a new telescopic summation method.

2.1. Extended Abel summation by parts method.

The proposition below presents an extended version of the Abel summation by parts method. The main novelties of this extension will be discussed later, after the end of the proof.

Proposition 2.1. *Let $m \in \mathbb{N} \setminus \{0\}$, $\ell \in \mathbb{N}$, $\Xi_\ell = \{k \in \mathbb{N};\ k \geq \ell\}$, and $[f_k(x)]_{k\in\Xi_\ell}$ and $[g_k(x)]_{k\in\Xi_\ell}$ be two sequences of functions, with x in their domains of definition. In this setting, we distinguish between two formulas that extend the scope of the Abel summation by parts method: a finite series formula and an infinite series formula.*

(1) *Finite series formula: for any $n \in \Xi_\ell$, we have*

$$\sum_{k=\ell}^{n} g_k(x)[f_{k+m}(x) - f_k(x)]$$
$$= \sum_{v=0}^{m-1} f_{n+v+1}(x)g_{n+v+1}(x) - \sum_{v=\ell}^{m+\ell-1} f_v(x)g_v(x) - \sum_{k=\ell}^{n} f_{k+m}(x)[g_{k+m}(x) - g_k(x)].$$

(2) *Infinite series formula: under the assumptions that $\lim_{n\to+\infty} f_n(x)g_n(x)$ is finite and the involved series converge, we have*

$$\sum_{k=\ell}^{+\infty} g_k(x)\left[f_{k+m}(x) - f_k(x)\right]$$
$$= m \lim_{n\to+\infty} f_n(x)g_n(x) - \sum_{v=\ell}^{m+\ell-1} f_v(x)g_v(x) - \sum_{k=\ell}^{+\infty} f_{k+m}(x)[g_{k+m}(x) - g_k(x)].$$

Proof.

(1) The proof is based on arranging decompositions, suitable groupings of the terms, the standard telescopic summation method, and changes of indexes. The details are given below. As the first step, for any $n \in \Xi_\ell$,

we can write

$$\begin{aligned}
&\sum_{k=\ell}^{n}[f_{k+m}(x)g_{k+m}(x)-f_k(x)g_k(x)]=\sum_{k=\ell}^{n}[f_{k+m}(x)g_{k+m}(x)-0-f_k(x)g_k(x)]\\
&=\sum_{k=\ell}^{n}\left\{f_{k+m}(x)g_{k+m}(x)-\sum_{u=1}^{m-1}[f_{k+u}(x)g_{k+u}(x)-f_{k+u}(x)g_{k+u}(x)]-f_k(x)g_k(x)\right\}\\
&=\sum_{k=\ell}^{n}\sum_{v=0}^{m-1}[f_{k+v+1}(x)g_{k+v+1}(x)-f_{k+v}(x)g_{k+v}(x)]\\
&=\sum_{v=0}^{m-1}\sum_{k=\ell}^{n}[f_{k+v+1}(x)g_{k+v+1}(x)-f_{k+v}(x)g_{k+v}(x)]\\
&=\sum_{v=0}^{m-1}[f_{n+v+1}(x)g_{n+v+1}(x)-f_{\ell+v}(x)g_{\ell+v}(x)]\\
&=\sum_{v=0}^{m-1}f_{n+v+1}(x)g_{n+v+1}(x)-\sum_{v=0}^{m-1}f_{\ell+v}(x)g_{\ell+v}(x)\\
&=\sum_{v=0}^{m-1}f_{n+v+1}(x)g_{n+v+1}(x)-\sum_{v=\ell}^{m+\ell-1}f_v(x)g_v(x). \qquad (2.1)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{k=\ell}^{n}[f_{k+m}(x)g_{k+m}(x)-f_k(x)g_k(x)]\\
&=\sum_{k=\ell}^{n}\{f_{k+m}(x)[g_{k+m}(x)-g_k(x)]+g_k(x)[f_{k+m}(x)-f_k(x)]\}\\
&=\sum_{k=\ell}^{n}f_{k+m}(x)[g_{k+m}(x)-g_k(x)]+\sum_{k=\ell}^{n}g_k(x)[f_{k+m}(x)-f_k(x)]. \qquad (2.2)
\end{aligned}$$

Therefore, by combining Equations (2.1) and (2.2), we get

$$\begin{aligned}
&\sum_{v=0}^{m-1}f_{n+v+1}(x)g_{n+v+1}(x)-\sum_{v=\ell}^{m+\ell-1}f_v(x)g_v(x)\\
&=\sum_{k=\ell}^{n}f_{k+m}(x)[g_{k+m}(x)-g_k(x)]+\sum_{k=\ell}^{n}g_k(x)[f_{k+m}(x)-f_k(x)].
\end{aligned}$$

This can be rewritten as

$$\sum_{k=\ell}^{n} g_k(x)[f_{k+m}(x) - f_k(x)]$$
$$= \sum_{v=0}^{m-1} f_{n+v+1}(x)g_{n+v+1}(x) - \sum_{v=\ell}^{m+\ell-1} f_v(x)g_v(x) - \sum_{k=\ell}^{n} f_{k+m}(x)[g_{k+m}(x) - g_k(x)]. \tag{2.3}$$

The desired finite series formula is obtained.

(2) Since $\lim_{n\to+\infty} f_{n+v+1}(x)g_{n+v+1}(x) = \lim_{n\to+\infty} f_n(x)g_n(x)$, which is finite, it follows from Equation (2.3) that

$$\sum_{k=\ell}^{+\infty} g_k(x)[f_{k+m}(x) - f_k(x)] = \lim_{n\to+\infty} \left\{ \sum_{k=\ell}^{n} g_k(x)[f_{k+m}(x) - f_k(x)] \right\}$$
$$= \sum_{v=0}^{m-1} \left[\lim_{n\to+\infty} f_{n+v+1}(x)g_{n+v+1}(x) \right] - \sum_{v=\ell}^{m+\ell-1} f_v(x)g_v(x)$$
$$- \lim_{n\to+\infty} \left\{ \sum_{k=\ell}^{n} f_{k+m}(x)[g_{k+m}(x) - g_k(x)] \right\}$$
$$= m \lim_{n\to+\infty} f_n(x)g_n(x) - \sum_{v=\ell}^{m+\ell-1} f_v(x)g_v(x) - \sum_{k=\ell}^{+\infty} f_{k+m}(x)[g_{k+m}(x) - g_k(x)].$$

The stated infinite series formula is thus established.

This ends the proof. □

Let us now discuss the novelty of this proposition. In comparison to the classical Abel summation by parts method, the obtained results are more general in several aspects. First, the presence of the integer m offers a new level of flexibility: it is classically taken at $m = 0$ or $m = 1$ in the standard version. This is an important fact, as shown later in an arctangent-type series setting. The same remark holds for the presence of ℓ; we can modulate it depending on the context, whereas it is generally fixed as $\ell = 0$ or $\ell = 1$ in the standard version. In addition, the consideration of sequences of functions justifying the presence of x is of conceptual importance for further uses, in particular with the aim of modulating x or differentiation or integration with respect to this variable.

On the other hand, the choices of the sequences of functions $[f_k(x)]_{k\in\Xi_\ell}$ and $[g_k(x)]_{k\in\Xi_\ell}$ are determinant. However, at least one of these sequences must be "not too sophisticated" to make our extended Abel summation by parts method useful. In this article, we mainly focus on various $[f_k(x)]_{k\in\Xi_\ell}$, but with $[g_k(x)]_{k\in\Xi_\ell}$ such that $g_k(x) = 1$ or $g_k(x) = k$ for any $k \in \Xi_\ell$.

2.2. **Extended telescopic summation method.** A consequence of Proposition 2.1 is the result below, which can be viewed as an extension of the classical telescopic summation method.

Proposition 2.2. *Let $m \in \mathbb{N} \setminus \{0\}$, $\ell \in \mathbb{N}$, $\Xi_\ell = \{k \in \mathbb{N};\ k \geq \ell\}$, and $[f_k(x)]_{k \in \Xi_\ell}$ be a sequence of functions, with x in their domain of definition. In this setting, we distinguish between two formulas that extend the scope of the telescopic summation method: a finite series formula and an infinite series formula.*

(1) *Finite series formula: for any $n \in \Xi_\ell$, we have*

$$\sum_{k=\ell}^{n}[f_{k+m}(x) - f_k(x)] = \sum_{v=0}^{m-1} f_{n+v+1}(x) - \sum_{v=\ell}^{m+\ell-1} f_v(x).$$

(2) *Infinite series formula: under the assumptions that $\lim_{n\to+\infty} f_n(x)$ is finite and the involved series converge, we have*

$$\sum_{k=\ell}^{+\infty}[f_{k+m}(x) - f_k(x)] = m \lim_{n\to+\infty} f_n(x) - \sum_{v=\ell}^{m+\ell-1} f_v(x).$$

Proof. The proof of these two results is in fact a direct application of the two items in Proposition 2.1, respectively. It is enough to apply them with $g_k(x) = 1$ for any $k \in \Xi_\ell$. Let us mention that the last summation term vanished simply because $g_{k+m}(x) - g_k(x) = 1 - 1 = 0$ for any $k \in \Xi_\ell$. □

These general results can be applied in diverse settings, aiming to ease the analysis and computation of sophisticated series. In the section below, we illustrate their interest in the context of the arctangent-type series.

3. Applications

3.1. **On the arctangent function.** We now briefly review the arctangent function and some of its main properties. As the prime information, the arctangent function, denoted as $\arctan(x)$, is the inverse of the tangent function, i.e., $\arctan[\tan(x)] = \tan[\arctan(x)] = x$ (when x is not an odd multiple of $\pi/2$). In some senses, it maps real numbers to angles in the range $(-\pi/2, \pi/2)$, i.e., the domain of $\arctan(x)$ is $\mathbb{R}$ and its range is $(-\pi/2, \pi/2)$. It is a continuous odd function, i.e., $\arctan(-x) = -\arctan(x)$ for any $x \in \mathbb{R}$, implying that $\arctan(0) = 0$. It has the following notable limits:

$$\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}, \quad \lim_{x\to+\infty} \arctan(x) = \frac{\pi}{2}.$$

Some special values involving π are listed below. We have

$$\arctan\left[\sqrt{3}\right] = \frac{\pi}{3}, \quad \arctan(1) = \frac{\pi}{4}, \quad \arctan\left[\frac{1}{\sqrt{3}}\right] = \frac{\pi}{6},$$

$$\arctan\left[\frac{1}{1+\sqrt{2}}\right] = \frac{\pi}{8}, \quad \arctan\left[\frac{1}{2+\sqrt{3}}\right] = \frac{\pi}{12}$$

and

$$\arctan\left\{\sqrt{2}\sqrt{2+\sqrt{2}} - \left[1+\sqrt{2}\right]\right\} = \frac{\pi}{16}.$$

The derivative of $\arctan(x)$ is $1/(1+x^2)$ and its primitive is $x\arctan(x)-(1/2)\log(1+x^2)+C$, where C denotes a generic constant. Its Taylor series expansion is

$$\arctan(x)=\sum_{k=0}^{+\infty}(-1)^k\frac{x^{2k+1}}{2k+1},$$

which is valid for $x\in[-1,1]$. As a direct consequence, the following equivalence holds: $\arctan(x)\sim x$ when $x\to 0$. The arctangent function also satisfies several remarkable formulas. For instance, we have

$$\arctan(x)+\arctan\left(\frac{1}{x}\right)=\begin{cases}\frac{\pi}{2} & \text{if } x>0,\\ -\frac{\pi}{2} & \text{if } x<0.\end{cases}$$

Furthermore, for any $x\in\mathbb{R}$, the arctangent function is related to itself as follows:

$$\arctan(x)=2\arctan\left[\frac{x}{1+\sqrt{1+x^2}}\right].$$

Another important formula is the so-called "arctangent addition formula" specified as

$$\arctan(a)+\arctan(b)=\arctan\left(\frac{a+b}{1-ab}\right)+c\pi, \tag{3.1}$$

where

$$c=\begin{cases}0 & \text{if } ab<1,\\ 1 & \text{if } ab>1 \text{ with } \ a \text{ (and } b\text{) } >0,\\ -1 & \text{if } ab>1 \text{ with } a \text{ (and } b\text{) } <0.\end{cases}$$

There are also some connections with other known inverse trigonometric functions, such as

$$\arctan(x)=\arcsin\left[\frac{x}{\sqrt{1+x^2}}\right] \tag{3.2}$$

and, for $x\geq 0$,

$$\arctan(x)=\arccos\left[\frac{1}{\sqrt{1+x^2}}\right], \tag{3.3}$$

where $\arcsin(x)$ and $\arccos(x)$ denote the inverse of the sine and cosine functions, i.e., the arcsine and arccosine functions, respectively.

These properties make the arctangent function a valuable tool for diverse calculus purposes. In the rest of this article, following the spirit of the works in [5] and [6], we aim to apply the general results in Propositions 2.1 and 2.2 to exhibit a wide panel of notable arctangent-type series.

3.2. **First arctangent-type series results.**

3.2.1. *A general result.* The result below is about a series involving the arctangent function, adjustable sequence of numbers and tunable parameters. It can be viewed as a generalization of [5, Theorem 2.1 and equation (2.12)] or [6, Theorem 1].

Proposition 3.1. *Let* $m \in \mathbb{N}\backslash\{0\}$, $\ell \in \mathbb{N}$, $\Xi_\ell = \{k \in \mathbb{N};\ k \geq \ell\}$, $x \in \mathbb{R}\backslash\{0\}$ *and* $(u_k)_{k\in\Xi_\ell}$ *such that* $\inf_{k\in\Xi_\ell} u_{k+m}u_k x^2 > -1$. *Under the asumption that* $\lim_{n\to+\infty} u_n$ *exists (it can be infinite) and the involved series converge, we have*

$$\sum_{k=\ell}^{+\infty} \arctan\left[\frac{(u_{k+m}-u_k)x}{1+u_{k+m}u_k x^2}\right] = m\arctan(\lim_{n\to+\infty} u_n x) - \sum_{v=\ell}^{m+\ell-1} \arctan(u_v x).$$

Proof. The proof is based on a standard arctangent decomposition and Proposition 2.2. Thanks to Equation (3.1), for any a and b such that $ab > -1$, we have

$$\arctan\left(\frac{a-b}{1+ab}\right) = \arctan(a) - \arctan(b). \tag{3.4}$$

Therefore, under the assumption $u_{k+m}u_k x^2 \geq \inf_{k\in\Xi_\ell} u_{k+m}u_k x^2 > -1$ for any $k \in \Xi_\ell$, we have

$$\arctan\left[\frac{(u_{k+m}-u_k)x}{1+u_{k+m}u_k x^2}\right] = \arctan(u_{k+m}x) - \arctan(u_k x).$$

By applying this equation and the item numbered 2 in Proposition 2.2 with $f_k(x) = \arctan(u_k x)$, we can write

$$\begin{aligned}
&\sum_{k=\ell}^{+\infty} \arctan\left[\frac{(u_{k+m}-u_k)x}{1+u_{k+m}u_k x^2}\right] = \sum_{k=\ell}^{+\infty}[\arctan(u_{k+m}x) - \arctan(u_k x)] \\
&= \sum_{k=\ell}^{+\infty}[f_{k+m}(x) - f_k(x)] = m\lim_{n\to+\infty} f_n(x) - \sum_{v=\ell}^{m+\ell-1} f_v(x) \\
&= m\lim_{n\to+\infty}\arctan(u_n x) - \sum_{v=\ell}^{m+\ell-1}\arctan(u_v x) \\
&= m\arctan(\lim_{n\to+\infty} u_n x) - \sum_{v=\ell}^{m+\ell-1}\arctan(u_v x).
\end{aligned}$$

The desired formula is obtained. □

This proposition will be at the center of the coming results, with varying values on ℓ and m, and diverse assumptions on $(u_k)_{k\in\Xi_\ell}$.

Remark 3.2. By considering the relations between the arctangent function and the arcsine and arccosine functions as described in Equations (3.2) and (3.3), respectively, we can use Proposition 3.1 to derive general results on arcsine-type and arccosine-type series.

3.2.2. *Specific results with a tunable sequence.* The result below determines general arctangent-type series that are constantly equal to $\pi/2$.

Proposition 3.3. *For any $x \in \mathbb{R} \setminus \{0\}$ and $(u_k)_{k\in\mathbb{N}}$ such that $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 > -1$, $u_0 = 0$ and $\lim_{n\to+\infty} u_n = +\infty$, we have*

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1} - u_k)x}{1 + u_{k+1}u_k x^2}\right] = \frac{\pi}{2}.$$

Proof. By applying Proposition 3.1 with $\ell = 0$ and $m = 1$, using $u_0 = 0$ and $\lim_{n\to+\infty} u_n = +\infty$, since $\lim_{t\to+\infty} \arctan(t) = \pi/2$ and $\arctan(0) = 0$, we obtain

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1} - u_k)x}{1 + u_{k+1}u_k x^2}\right] = 1 \times \arctan(\lim_{n\to+\infty} u_n x) - \sum_{v=0}^{0} \arctan(u_v x)$$
$$= \frac{\pi}{2} - \arctan(0) = \frac{\pi}{2}.$$

The stated formula is proved. □

This result is significant as it demonstrates a highly adjustable series equating to the constant $\pi/2$ consistently. Indeed, we can find an infinite number of sequences $(u_k)_{k\in\mathbb{N}}$ satisfying the required assumptions, and we can choose an infinite number of values for x. Some examples of such sequences involving multiple parameters are $u_k = bk^a$, with $a > 0$, and $b > 0$; $u_k = d[\log(1 + ak^c)]^b$, with $a > 0$, $b > 0$, $c > 0$, and $d > 0$; $u_k = d(e^{ak^c} - 1)^b$, with $a > 0$, $b > 0$, $c > 0$, and $d > 0$, and various combinations of them (sum, product, convex linear, etc.), among others.

For instance, by considering $u_k = k$, for any $x \in \mathbb{R} \setminus \{0\}$, we have $u_0 = 0$, $\lim_{n\to+\infty} u_n = +\infty$ and $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 = 0 > -1$, and it follows from Proposition 3.3 that

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{x}{1 + k(k+1)x^2}\right] = \sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1} - u_k)x}{1 + u_{k+1}u_k x^2}\right] = \frac{\pi}{2}. \tag{3.5}$$

As another specific example, by considering the polynomial $u_k = k(k-1)$, for any $x \in \mathbb{R} \setminus \{0\}$, we have $u_0 = 0$, $\lim_{n\to+\infty} u_n = +\infty$ and $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 = 0 > -1$, and Proposition 3.3 gives

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{2kx}{1 + k^2(k^2 - 1)x^2}\right] = \sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1} - u_k)x}{1 + u_{k+1}u_k x^2}\right] = \frac{\pi}{2}.$$

As a last example beyond polynomial-type sequences, we can consider $u_k = \log(1 + k)$. For any $x \in \mathbb{R} \setminus \{0\}$, we have $u_0 = 0$, $\lim_{n\to+\infty} u_n = +\infty$ and $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 = 0 > -1$, and it follows from Proposition 3.3 that

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{\log[1 + 1/(1+k)]x}{1 + \log(2+k)\log(1+k)x^2}\right] = \sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1} - u_k)x}{1 + u_{k+1}u_k x^2}\right] = \frac{\pi}{2}.$$

The result below identifies arctangent-type series that are equal to $-\arctan(x)$ (or, upon multiplication with -1, $\arctan(x)$), giving more flexibility than the previous proposition thanks to the presence of x in the final computation.

Proposition 3.4. *For any $x \in \mathbb{R} \setminus \{0\}$ and $(u_k)_{k\in\mathbb{N}}$ such that $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 > -1$, $u_0 = 1$ and $\lim_{n\to+\infty} u_n = 0$, we have*

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_k x^2}\right] = -\arctan(x),$$

or, equivalently,

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(u_k-u_{k+1})x}{1+u_{k+1}u_k x^2}\right] = \arctan(x).$$

Proof. By applying Proposition 3.1 with $\ell = 0$ and $m = 1$, using $u_0 = 1$, $\lim_{n\to+\infty} u_n = 0$, and $\arctan(0) = 0$, we obtain

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_k x^2}\right] = 1 \times \arctan(\lim_{n\to+\infty} u_n x) - \sum_{v=0}^{0} \arctan(u_v x)$$
$$= \arctan(0\times x) - \arctan(u_0 \times x) = -\arctan(x).$$

The equivalent formula is obtained by multiplying both parts by -1 and using the odd property of the arctangent function. The desired formulas are obtained. □

Again, we can find an infinite number of sequences $(u_k)_{k\in\mathbb{N}}$ satisfying the required assumptions, and we can choose an infinite number of values for x. Some examples of such sequences involving multiple parameters are $u_k = 1/(1+bk^a)$, with $a > 0$, and $b > 0$; $u_k = [\log(e + ak^c)]^{-b}$, with $a > 0$, $b > 0$, and $c > 0$; $u_k = e^{-ak^b}$, with $a > 0$, and $b > 0$, and various combinaisons of them, among others.

For instance, by considering $u_k = e^{-k}$, for any $x \in \mathbb{R} \setminus \{0\}$, we have $u_0 = 1$, $\lim_{n\to+\infty} u_n = 0$ and $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 \geq 0 > -1$, and it follows from Proposition 3.4 that

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(1-e)e^k x}{e^{2k+1}+x^2}\right] = \sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_k x^2}\right] = -\arctan(x),$$

or, equivalently,

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(e-1)e^k x}{e^{2k+1}+x^2}\right] = \arctan(x).$$

Two more specific examples of this proposition are presented below.

- An important example arises when considering the famous Fibonacci sequence. We recall that this sequence is indicated as $F_0 = 0$, $F_1 = 1$, and, for any $p \in \mathbb{N} \setminus \{0, 1\}$, $F_p = F_{p-1} + F_{p-2}$ (implying that $F_2 = 1$, among others). We also have $\lim_{p\to+\infty} F_p = +\infty$. As a result, the sequence $(u_k)_{k\in\mathbb{N}}$ defined by

$$u_k = \frac{1}{F_{2k+2}}$$

satisfies the required assumptions. Indeed, for any $x \in \mathbb{R} \setminus \{0\}$, we have $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 \geq 0 > -1$, $u_0 = 1/F_2 = 1$, and $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty}(1/F_{2n+1}) = 0$. Therefore, we can apply Proposition 3.4 in this setting, and find that

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(F_{2k+2} - F_{2k+4})x}{F_{2k+4}F_{2k+2} + x^2}\right] = \sum_{k=0}^{+\infty} \arctan\left[\frac{(1/F_{2k+4} - 1/F_{2k+2})x}{1 + (1/F_{2k+4})(1/F_{2k+2})x^2}\right]$$
$$= \sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1} - u_k)x}{1 + u_{k+1}u_k x^2}\right] = -\arctan(x).$$

This formula can also be derived from the work in [2]. By applying it with $x = 1$, we refind the so-called Lehmer identity (see [13]). Indeed, by noticing that $F_{2k+3} = F_{2k+4} - F_{2k+2}$ and using the Cassini identity, i.e., for any $p \in \mathbb{N} \setminus \{0, 1\}$, $F_p^2 = F_{p+1}F_{p-1} + (-1)^{p-1}$, with $p = 2k+3$, which gives $F_{2k+3}^2 = F_{2k+4}F_{2k+2} + 1$, we get

$$\sum_{k=0}^{+\infty} \arctan\left(\frac{1}{F_{2k+3}}\right) = \sum_{k=0}^{+\infty} \arctan\left(\frac{F_{2k+3}}{F_{2k+3}^2}\right) = \sum_{k=0}^{+\infty} \arctan\left(\frac{F_{2k+4} - F_{2k+2}}{F_{2k+4}F_{2k+2} + 1}\right)$$
$$= -\sum_{k=0}^{+\infty} \arctan\left(\frac{F_{2k+2} - F_{2k+4}}{F_{2k+4}F_{2k+2} + 1}\right) = -[-\arctan(1)] = \frac{\pi}{4}. \tag{3.6}$$

- Let us now show how binomial coefficients can be involved in elegant arctangent-type series. We recall that, for any $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$ such that $k \geq \ell$, the "ℓ among k" binomial coefficient is defined by $\binom{k}{\ell} = k!/[\ell!(k-\ell)!]$. With this notation in mind, for any $\ell \in \mathbb{N} \setminus \{0\}$, let us consider and study the following special series:

$$\sum_{k=\ell}^{+\infty} \arctan\left[\frac{\binom{k}{\ell-1}x}{\binom{k}{\ell}\binom{k+1}{\ell} + x^2}\right].$$

By using the triangle of Pascal to the numerator term, i.e., $\binom{k}{\ell-1} = \binom{k+1}{\ell} - \binom{k}{\ell}$, and performing the change of indexes "$k = k + \ell$", we get

$$\sum_{k=\ell}^{+\infty} \arctan\left[\frac{\binom{k}{\ell-1}x}{\binom{k}{\ell}\binom{k+1}{\ell} + x^2}\right] = \sum_{k=\ell}^{+\infty} \arctan\left[\frac{\left[\binom{k+1}{\ell} - \binom{k}{\ell}\right]x}{\binom{k}{\ell}\binom{k+1}{\ell} + x^2}\right]$$
$$= \sum_{k=\ell}^{+\infty} \arctan\left\{\frac{\left[1/\binom{k}{\ell} - 1/\binom{k+1}{\ell}\right]x}{1 + \left[1/\binom{k+1}{\ell}\right]\left[1/\binom{k}{\ell}\right]x^2}\right\}$$
$$= \sum_{k=0}^{+\infty} \arctan\left\{\frac{\left[1/\binom{k+\ell}{\ell} - 1/\binom{k+\ell+1}{\ell}\right]x}{1 + \left[1/\binom{k+\ell+1}{\ell}\right]\left[1/\binom{k+\ell}{\ell}\right]x^2}\right\}.$$

In the light of this expression, with regard to Proposition 3.4, let us consider the sequence $(u_k)_{k\in\mathbb{N}}$ defined by

$$u_k = \frac{1}{\binom{k+\ell}{\ell}}.$$

It satisfies the required assumptions. Indeed, for any $x \in \mathbb{R} \setminus \{0\}$, we have $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 \geq 0 > -1$, $u_0 = 1/\binom{\ell}{\ell} = 1$ and $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty}[1/\binom{n+\ell}{\ell}] = 0$. A direct application of the proposition gives

$$\sum_{k=\ell}^{+\infty} \arctan\left[\frac{\binom{k}{\ell-1}x}{\binom{k}{\ell}\binom{k+1}{\ell}+x^2}\right] = -\sum_{k=0}^{+\infty} \arctan\left\{\frac{\left[1/\binom{k+\ell+1}{\ell} - 1/\binom{k+\ell}{\ell}\right]x}{1+\left[1/\binom{k+\ell+1}{\ell}\right]\left[1/\binom{k+\ell}{\ell}\right]x^2}\right\}$$
$$= -\left\{\sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_k x^2}\right]\right\} = -\left[-\arctan(x)\right] = \arctan(x).$$

In particular, by choosing $x = 1$, we get

$$\sum_{k=\ell}^{+\infty} \arctan\left[\frac{\binom{k}{\ell-1}}{\binom{k}{\ell}\binom{k+1}{\ell}+1}\right] = \arctan(1) = \frac{\pi}{4}. \tag{3.7}$$

The result below exhibits arctangent-type series that are equal to $\arctan(x)$, characterized by a distinct configuration on the sequence $(u_k)_{k\in\mathbb{N}}$ compared to Proposition 3.4.

Proposition 3.5. *For any $x \in \mathbb{R}\backslash 0$ and $(u_k)_{k\in\mathbb{N}}$ such that $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 > -1$, $u_0 = 0$ and $\lim_{n\to+\infty} u_n = 1$, we have*

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_k x^2}\right] = \arctan(x).$$

Proof. By applying Proposition 3.1 with $\ell = 0$ and $m = 1$, using $u_0 = 0$, $\lim_{n\to+\infty} u_n = 1$, and $\arctan(0) = 0$, we obtain

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_k x^2}\right] = 1 \times \arctan(\lim_{n\to+\infty} u_n x) - \sum_{v=0}^{0} \arctan(u_v x)$$
$$= \arctan(x) - \arctan(0) = \arctan(x).$$

The desired series expansion is proved. □

Again, we can determine an infinite number of sequences $(u_k)_{k\in\mathbb{N}}$ satisfying the required assumptions. As a fresh viewpoint, let us make a bridge between these assumptions and some well-known functions in probability theory. We can consider

$$u_k = G(k),$$

where $G(x)$ denotes the cumulative distribution function of any lifetime distribution, i.e., with support $[0, +\infty)$ (and the imposed initial value $G(0) = 0$). Examples of such functions are $G(x) = (1 - e^{-ax^b})^c$, with $a > 0$, $b > 0$, and

$c > 0$, which corresponds to the cumulative distribution function of the exponentiated Weibull distribution (see [16]); $G(x) = (x/a)^{bc}/\left[1 + (x/a)^b\right]^c$, with $a > 0$, $b > 0$, and $c > 0$, which is the cumulative distribution function of the Dagum distribution (see [9]); $G(x) = 1 - e^{-a(e^{bx}-1)}$, with $a > 0$, and $b > 0$, which represents the cumulative distribution function of the Gompertz distribution (see [12]); $G(x) = 1 - a^b/(a + e^{cx} - 1)^b$, with $a > 0$, $b > 0$, and $c > 0$, which corresponds to the cumulative distribution function of the gamma-Gompertz distribution (see [4]), and various combinaisons of them, among others.

As a specific example, by considering $u_k = k/(k+1)$, i.e., $u_k = G(k)$, where $G(x)$ is the cumulative distribution function of the Dagum distribution with $a = 1$, $b = 1$ and $c = 1$, for any $x \in \mathbb{R} \setminus \{0\}$, we have $u_0 = 0$, $\lim_{n\to+\infty} u_n = 1$ and $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 = 0 > -1$, and it follows from Proposition 3.5 that

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{x}{(k+1)[k(x^2+1)+2]}\right] = \sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_k x^2}\right] = \arctan(x).$$

Another example can be presented by selecting $u_k = 1 - e^{-k}$, i.e., $u_k = G(k)$, where $G(x)$ is the cumulative distribution function of the exponentiated Weibull distribution with $a = 1$, $b = 1$, and $c = 1$, corresponding to the cumulative distribution function of the standard exponential distribution. For any $x \in \mathbb{R} \setminus \{0\}$, we have $u_0 = 0$, $\lim_{n\to+\infty} u_n = 1$ and $\inf_{k\in\mathbb{N}} u_{k+1}u_k x^2 = 0 > -1$, and Proposition 3.5 gives

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{(e-1)e^k x}{[1-(1+e)e^k]x^2 + e^{2k+1}(x^2+1)}\right] = \sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_k x^2}\right]$$
$$= \arctan(x).$$

We end this part with a remark, discussing how we can take advantage of the standard properties of the arctangent function to derive original examples of arctangent-type series based on the previous results.

Remark 3.6. Thanks to the following known formula:

$$\arctan\left[\frac{x}{1+\sqrt{1+x^2}}\right] = \frac{1}{2}\arctan(x), \quad x \in \mathbb{R}, \tag{3.8}$$

and Propositions 3.3, 3.4, and 3.5, we can generate new arctangent-type series involving a square root. For instance, in the setting of these propositions, by taking $x = 1$, we get the following relation:

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{u_{k+1}-u_k}{\sqrt{(u_{k+1}^2+1)(u_k^2+1)} + u_{k+1}u_k + 1}\right]$$
$$= \sum_{k=0}^{+\infty} \arctan\left[\frac{(u_{k+1}-u_k)/(1+u_{k+1}u_k)}{1+\sqrt{1+(u_{k+1}-u_k)^2/(1+u_{k+1}u_k)^2}}\right]$$
$$= \frac{1}{2}\sum_{k=0}^{+\infty} \arctan\left(\frac{u_{k+1}-u_k}{1+u_{k+1}u_k}\right).$$

Based on this, some examples are described below, omitting technical details for brevity.

- By considering $u_k = k$, thanks to Equation (3.5), we have

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{1}{\sqrt{[(k+1)^2+1](k^2+1)}+k(k+1)+1}\right]$$
$$= \frac{1}{2}\sum_{k=0}^{+\infty} \arctan\left[\frac{1}{1+k(k+1)}\right] = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}.$$

- With the use of the Fibonacci sequence, by considering (in an indirect manner) $u_k = 1/F_{2k+2}$ and Equation (3.6), we have

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{1}{F_{2k+3}+\sqrt{F_{2k+3}^2+1}}\right] = \sum_{k=0}^{+\infty} \arctan\left[\frac{1/F_{2k+3}}{1+\sqrt{1+1/F_{2k+3}^2}}\right]$$
$$= \frac{1}{2}\sum_{k=0}^{+\infty} \arctan\left(\frac{1}{F_{2k+3}}\right) = \frac{1}{2}\times\frac{\pi}{4} = \frac{\pi}{8}.$$

- With the use of binomial coefficients, by considering (in an indirect manner again) $u_k = 1/\binom{k+\ell}{\ell}$ and Equation (3.7), we have

$$\sum_{k=\ell}^{+\infty} \arctan\left[\frac{\binom{k}{\ell-1}}{\sqrt{\binom{k}{\ell-1}^2+\left[1+\binom{k}{\ell}\binom{k+1}{\ell}\right]^2}+\binom{k}{\ell}\binom{k+1}{\ell}+1}\right]$$
$$= \frac{1}{2}\sum_{k=\ell}^{+\infty} \arctan\left[\frac{\binom{k}{\ell-1}}{\binom{k}{\ell}\binom{k+1}{\ell}+1}\right] = \frac{1}{2}\times\frac{\pi}{4} = \frac{\pi}{8}.$$

A wide panel of other new examples using this technique can be presented.

3.2.3. *Specific results with tunable parameters.* The result below proposes an adaptable arctangent-type series thanks to the presence of several tuning parameters.

Proposition 3.7. *For any $x \in \mathbb{R}\setminus\{0\}$, $\ell \in \mathbb{N}$, $r > 0$ and $s > -1$ such that $r(s+1) \in \mathbb{N}\setminus\{0\}$ and*

$$\left(\frac{\ell}{r}+s\right)\left(\frac{\ell}{r}-1\right)x^2 > -1$$

(which is obvious if $\ell \geq r$), we have

$$\sum_{k=\ell}^{+\infty} \arctan\left[\frac{r^2(s+1)x}{r^2+(k+rs)(k-r)x^2}\right] = \frac{\pi}{2}r(s+1) - \sum_{v=\ell}^{r(s+1)+\ell-1} \arctan\left[\left(\frac{v}{r}-1\right)x\right].$$

Proof. For any $x \in \mathbb{R}\backslash\{0\}$, $\ell \in \mathbb{N}$, $r > 0$ and $s > -1$ such that $r(s+1) \in \mathbb{N}\backslash\{0\}$, we can write

$$\sum_{k=\ell}^{+\infty} \arctan\left[\frac{r^2(s+1)x}{r^2 + (k+rs)(k-r)x^2}\right] = \sum_{k=\ell}^{+\infty} \arctan\left\{\frac{[(k/r+s)-(k/r-1)]x}{1+(k/r+s)(k/r-1)x^2}\right\}$$
$$= \sum_{k=\ell}^{+\infty} \arctan\left[\frac{(u_{k+r(s+1)} - u_k)x}{1 + u_{k+r(s+1)}u_k x^2}\right],$$

where

$$u_k = \frac{k}{r} - 1.$$

In the setting of Proposition 3.1, we have $m = r(s+1)$,

$$\inf_{k \in \Xi_\ell} u_{k+m}u_k x^2 \geq \left(\frac{\ell}{r} + s\right)\left(\frac{\ell}{r} - 1\right)x^2 > -1$$

and

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{n}{r} - 1 = +\infty.$$

The required conditions are satisfied; Proposition 3.1 implies that

$$\sum_{k=\ell}^{+\infty} \arctan\left[\frac{r^2(s+1)x}{r^2+(k+rs)(k-r)x^2}\right] = m\arctan(\lim_{n\to+\infty} u_n x) - \sum_{v=\ell}^{m+\ell-1} \arctan(u_v x)$$
$$= \frac{\pi}{2}r(s+1) - \sum_{v=\ell}^{r(s+1)+\ell-1} \arctan\left[\left(\frac{v}{r} - 1\right)x\right].$$

This ends the proof. □

Under this simplified and general statement, this result is new. It can be used to generate a wide panel of new arctangent-type series but also recover some known ones. Some (new) examples focusing on the case $s = -1/2$ are described below.

- By selecting $\ell = 0$, $r = 2$, and $x = 1$, we have

$$\sum_{k=0}^{+\infty} \arctan\left(\frac{2}{k^2 - 3k + 6}\right) = \frac{\pi}{2} - \sum_{v=0}^{0} \arctan\left(\frac{v}{2} - 1\right) = \frac{\pi}{2} + \arctan(1)$$
$$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}.$$

- By taking $\ell = 0$, $r = 2$, and $x = \sqrt{3}$, we have

$$\sum_{k=0}^{+\infty} \arctan\left[\frac{2\sqrt{3}}{3k^2 - 9k + 10}\right] = \frac{\pi}{2} - \sum_{v=0}^{0} \arctan\left[\left(\frac{v}{2} - 1\right)\sqrt{3}\right] = \frac{\pi}{2} + \arctan\left[\sqrt{3}\right]$$
$$= \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}.$$

- By choosing $\ell = 1$, $r = 4$, and $x = 2$, we find that

$$\sum_{k=1}^{+\infty} \arctan\left(\frac{4}{k^2 - 6k + 12}\right) = \pi - \sum_{v=1}^{2} \arctan\left[\left(\frac{v}{4} - 1\right) 2\right]$$
$$= \pi + \arctan\left(\frac{3}{2}\right) + \arctan(1) = \pi + \arctan\left(\frac{3}{2}\right) + \frac{\pi}{4}$$
$$= \frac{5\pi}{4} + \arctan\left(\frac{3}{2}\right),$$

with $\arctan(3/2) \approx 0.9827$.

Other applications of Proposition 3.7 focusing on the case $s = 0$ are described below, with some connected to existing arctangent-type series.

- By selecting $\ell = 1$, $r = 1$, and $x = 1$, we have

$$\sum_{k=1}^{+\infty} \arctan\left(\frac{1}{k^2 - k + 1}\right) = \frac{\pi}{2} - \sum_{v=1}^{1} \arctan(v - 1) = \frac{\pi}{2} - \arctan(0) = \frac{\pi}{2}. \quad (3.9)$$

Let us notice that, with a change of index, we obtain

$$\sum_{k=0}^{+\infty} \arctan\left(\frac{1}{k^2 + k + 1}\right) = \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{k^2 - k + 1}\right) = \frac{\pi}{2}.$$

By removing the first term in $k = 0$ equal to $\arctan(1) = \pi/4$, this series is equal to $\pi/4$ and corresponds to [5, Equation (2.7)]. With such calculus techniques, a wide panel of other arctangent-type series can be determined.

- By taking $\ell = 0$, $r = 1$, and $x = 1$, we get

$$\sum_{k=0}^{+\infty} \arctan\left(\frac{1}{k^2 - k + 1}\right) = \frac{\pi}{2} - \sum_{v=0}^{0} \arctan(v - 1) = \frac{\pi}{2} + \arctan(1) = \frac{3\pi}{4}.$$

In fact, it corresponds to the previous series with the addition of the term in $k = 0$, i.e., $\arctan(1)$.

- By choosing $\ell = 1$, $r = 2$, and $x = 2/\sqrt{3}$, we find that

$$\sum_{k=1}^{+\infty} \arctan\left[\frac{2\sqrt{3}}{k^2 - 2k + 3}\right] = \pi - \sum_{v=1}^{2} \arctan\left[\left(\frac{v}{2} - 1\right) \frac{2}{\sqrt{3}}\right]$$
$$= \pi + \arctan\left[\frac{1}{\sqrt{3}}\right] - \arctan(0) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}.$$

- By selecting $\ell = 1$, $r = 1$, and $x = 1/\sqrt{2}$, we have

$$\sum_{k=1}^{+\infty} \arctan\left[\frac{\sqrt{2}}{k^2 - k + 2}\right] = \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left[(v - 1)\frac{1}{\sqrt{2}}\right] = \frac{\pi}{2} - \arctan(0) = \frac{\pi}{2}.$$

Some other examples focusing on the case $s = 1$ are presented below.

- By taking $\ell = 1$, $r = 1$, and $x = 1$, we obtain

$$\sum_{k=1}^{+\infty} \arctan\left(\frac{2}{k^2}\right) = \pi - \sum_{v=1}^{2} \arctan(v-1) = \pi - \arctan(0) - \arctan(1)$$
$$= \pi - 0 - \frac{\pi}{4} = \frac{3\pi}{4}. \tag{3.10}$$

This case was also considered in [3], and in [5, Example 2.3].

- By choosing $\ell = 1$, $r = 1/2$, and $x = 1$, we get

$$\sum_{k=1}^{+\infty} \arctan\left(\frac{1}{2k^2}\right) = \frac{\pi}{2} - \sum_{v=1}^{1} \arctan(2v-1) = \frac{\pi}{2} - \arctan(1)$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \tag{3.11}$$

This was also obtained in [5, Equation (3.9)].

At this step, let us formulate a comment on how the previous results and other similar results can be combined thanks to the properties of the arctangent function to create innovative arctangent-type series.

Remark 3.8. We must keep in mind that the properties of the arctangent function can also be applied in the series form. Some examples are presented below.

– Owing to Equations (3.4), (3.10) and (3.11), we establish that

$$\sum_{k=1}^{+\infty} \arctan\left[\frac{3k^2}{2(k^4+1)}\right] = \sum_{k=1}^{+\infty} \arctan\left\{\frac{2/k^2 - 1/(2k^2)}{1+(2/k^2)[1/(2k^2)]}\right\}$$
$$= \sum_{k=1}^{+\infty} \left[\arctan\left(\frac{2}{k^2}\right) - \arctan\left(\frac{1}{2k^2}\right)\right] = \sum_{k=1}^{+\infty} \arctan\left(\frac{2}{k^2}\right) - \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{2k^2}\right)$$
$$= \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}.$$

– Owing to Equations (3.8) and (3.11), we have

$$\sum_{k=1}^{+\infty} \arctan\left[\sqrt{4k^4+1} - 2k^2\right] = \sum_{k=1}^{+\infty} \arctan\left[\frac{1/(2k^2)}{1+\sqrt{1+[1/(2k^2)]^2}}\right]$$
$$= \frac{1}{2}\sum_{k=1}^{+\infty} \arctan\left(\frac{1}{2k^2}\right) = \frac{1}{2} \times \frac{\pi}{4} = \frac{\pi}{8}.$$

Similarly, based on Equation (3.10), we get

$$\sum_{k=1}^{+\infty} \arctan\left[\frac{2}{\sqrt{k^4+4}+k^2}\right] = \sum_{k=1}^{+\infty} \arctan\left[\frac{2/k^2}{1+\sqrt{1+(2/k^2)^2}}\right]$$
$$= \frac{1}{2}\sum_{k=1}^{+\infty} \arctan\left(\frac{2}{k^2}\right) = \frac{1}{2} \times \frac{3\pi}{4} = \frac{3\pi}{8}.$$

To the best of our knowledge, these examples are new, and so much more can be produced with similar techniques.

We go on presenting examples on the case $s = 1$.

- By selecting $\ell = 1$, $r = 1/2$, and $x = \sqrt{3}$, we have

$$\sum_{k=1}^{+\infty} \arctan\left[\frac{\sqrt{3}}{6k^2 - 1}\right] = \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left[(2v-1)\sqrt{3}\right] = \frac{\pi}{2} - \arctan\left[\sqrt{3}\right] = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

- By taking $\ell = 1$, $r = 1/2$, and $x = 1/\left[1 + \sqrt{2}\right]$, we get

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{1-\sqrt{2}}{\left[4\sqrt{2} - 6\right]k^2 - \sqrt{2} + 1}\right\} = \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left[(2v-1)\frac{1}{1+\sqrt{2}}\right] = \frac{\pi}{2} - \arctan\left[\frac{1}{1+\sqrt{2}}\right] = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8}.$$

- By choosing $\ell = 1$, $r = 1/2$, and $x = 1/\left[2 + \sqrt{3}\right]$, we obtain

$$\sum_{k=1}^{+\infty} \arctan\left[\frac{7 + 4\sqrt{3}}{2\left[2+\sqrt{3}\right]k^2 + 7\sqrt{3} + 12}\right] = \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left[(2v-1)\frac{1}{2+\sqrt{3}}\right] = \frac{\pi}{2} - \arctan\left[\frac{1}{2+\sqrt{3}}\right] = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}.$$

- By selecting $\ell = 1$, $r = 1$, and $x = 1/\sqrt{2}$, we have

$$\sum_{k=1}^{+\infty} \arctan\left[\frac{2\sqrt{2}}{k^2 + 1}\right] = \pi - \sum_{v=1}^{2} \arctan\left[(v-1)\frac{1}{\sqrt{2}}\right] = \pi - \arctan(0) - \arctan\left[\frac{1}{\sqrt{2}}\right] = \pi - \arctan\left[\frac{1}{\sqrt{2}}\right] = \frac{\pi}{2} + \arctan\left[\sqrt{2}\right],$$

with $\arctan\left[\sqrt{2}\right] \approx 0.9553$.

- By taking $\ell = 1$, $r = 1$, and $x = 1/2$, we have

$$\sum_{k=1}^{+\infty} \arctan\left(\frac{4}{k^2 + 3}\right) = \pi - \sum_{v=1}^{2} \arctan\left[(v-1)\frac{1}{2}\right] = \pi - \arctan(0) - \arctan\left(\frac{1}{2}\right) = \pi - \arctan\left(\frac{1}{2}\right) = \frac{\pi}{2} + \arctan(2),$$

with $\arctan(2) \approx 1.1071$.

An example focusing on the case $s = 2$ is proposed below. By selecting $\ell = 1$, $r = 1/3$, and $x = \sqrt{3}/2$, we have

$$\sum_{k=1}^{+\infty} \arctan\left[\frac{6\sqrt{3}}{27k^2 + 9k - 2}\right] = \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left[(3v-1)\frac{\sqrt{3}}{2}\right] = \frac{\pi}{2} - \arctan\left[\sqrt{3}\right]$$
$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

A general case concerning the choice of s is investigated in the proposition below.

Proposition 3.9. *For any $s > 0$ and $x > 0$, we have*

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{s(s+1)x}{s^2 + [k^2(s+1)^2 + k(s^2-1) - s]x^2}\right\} = \arctan\left(\frac{1}{x}\right),$$

or, equivalently,

$$\sum_{k=1}^{+\infty} \arctan\left[\frac{s(s+1)x}{s^2x^2 + k^2(s+1)^2 + k(s^2-1) - s}\right] = \arctan(x).$$

Proof. By applying Proposition 3.7 with $\ell = 1$, $r = 1/(s+1)$ with $s > 0$ and $x = y/s$ with $y > 0$, noticing that

$$\left(\frac{\ell}{r} + s\right)\left(\frac{\ell}{r} - 1\right)x^2 = (2s+1)\frac{y^2}{s} > 0 > -1,$$

we have

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{s(s+1)y}{s^2 + [k^2(s+1)^2 + k(s^2-1) - s]y^2}\right\}$$
$$= \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left\{[(s+1)v - 1]\frac{y}{s}\right\} = \frac{\pi}{2} - \arctan\left\{[(s+1) - 1]\frac{y}{s}\right\}$$
$$= \frac{\pi}{2} - \arctan(y) = \arctan\left(\frac{1}{y}\right).$$

The second result follows by substituting y by $1/y$. Then we put $y = x$ in the two formulas for the sake of uniformity in the notations. The proposition is demonstrated. □

3.2.4. *Specific results involving the square root and tunable parameters.* Similarly to Proposition 3.7, the result below proposes to exploit the generality of Proposition 3.1 to exhibit an original arctangent-type series involving the square root of some crucial terms.

Proposition 3.10. *For any $x \in \mathbb{R} \setminus \{0\}$, $\ell \in \mathbb{N}$, $r > 0$ and $s > -1$ such that $r(s+1) \in \mathbb{N} \setminus \{0\}$ and $\ell \geq r$, we have*

$$\sum_{k=\ell}^{+\infty} \arctan\left\{\frac{x(s+1)r\sqrt{r}}{\left[r + x^2\sqrt{(k+rs)(k-r)}\right]\left[\sqrt{k+rs} + \sqrt{k-r}\right]}\right\}$$
$$= \frac{\pi}{2}r(s+1) - \sum_{v=\ell}^{r(s+1)+\ell-1} \arctan\left[x\sqrt{\left(\frac{v}{r} - 1\right)}\right].$$

Proof. For any $x \in \mathbb{R} \setminus \{0\}$, $\ell \in \mathbb{N}$, $r > 0$ and $s > -1$ such that $r(s+1) \in \mathbb{N} \setminus \{0\}$ and $\ell \geq r$, we can write

$$\sum_{k=\ell}^{+\infty} \arctan\left\{\frac{x(s+1)r\sqrt{r}}{\left[r + x^2\sqrt{(k+rs)(k-r)}\right]\left[\sqrt{k+rs} + \sqrt{k-r}\right]}\right\}$$
$$= \sum_{k=\ell}^{+\infty} \arctan\left\{\frac{\left[(k/r+s) - (k/r-1)\right]x}{\left[1 + x^2\sqrt{k/r+s}\sqrt{k/r-1}\right]\left[\sqrt{k/r+s} + \sqrt{k/r-1}\right]}\right\}$$
$$= \sum_{k=\ell}^{+\infty} \arctan\left\{\frac{\left[\sqrt{k/r+s} - \sqrt{k/r-1}\right]x}{1 + x^2\sqrt{k/r+s}\sqrt{k/r-1}}\right\}$$
$$= \sum_{k=\ell}^{+\infty} \arctan\left\{\frac{(u_{k+r(s+1)} - u_k)x}{1 + u_{k+r(s+1)}u_k x^2}\right\},$$

where

$$u_k = \sqrt{\frac{k}{r} - 1}.$$

In the setting of Proposition 3.1, we have $m = r(s+1)$,

$$\inf_{k \in \Xi_\ell} u_{k+m}u_k x^2 \geq x^2\sqrt{\frac{\ell}{r} + s}\sqrt{\frac{\ell}{r} - 1} \geq 0 > -1$$

and

$$\lim_{n \to +\infty} u_n = \lim_{n \to +\infty} \sqrt{\frac{n}{r} - 1} = +\infty,$$

implying that

$$\sum_{k=\ell}^{+\infty} \arctan\left\{\frac{x(s+1)r\sqrt{r}}{\left[r + x^2\sqrt{(k+rs)(k-r)}\right]\left[\sqrt{k+rs} + \sqrt{k-r}\right]}\right\}$$
$$= m\arctan(\lim_{n \to +\infty} u_n x) - \sum_{v=\ell}^{m+\ell-1} \arctan(u_v x)$$
$$= \frac{\pi}{2}r(s+1) - \sum_{v=\ell}^{r(s+1)+\ell-1} \arctan\left[x\sqrt{\frac{v}{r} - 1}\right].$$

This completes the proof. □

To the best of our knowledge, the obtained formula innovates because of its generality and the presence of square root terms, which are rarely considered in the context of arctangent-type series.

Some examples of Proposition 3.10 are given below; all of them are new. The first ones focusing on the case $s = -1/2$ are described below.

- By selecting $\ell = 2$, $r = 2$, and $x = 1$, we have

$$\sum_{k=2}^{+\infty} \arctan\left[\frac{\sqrt{2}}{\left[2+\sqrt{k^2-3k+2}\right]\left[\sqrt{k-1}+\sqrt{k-2}\right]}\right]$$
$$= \frac{\pi}{2} - \sum_{v=2}^{2} \arctan\left[\sqrt{\frac{v}{2}-1}\right] = \frac{\pi}{2} - \arctan(0) = \frac{\pi}{2}.$$

- By taking $\ell = 3$, $r = 2$, and $x = \sqrt{2}$, we get

$$\sum_{k=3}^{+\infty} \arctan\left[\frac{1}{\left[1+\sqrt{k^2-3k+2}\right]\left[\sqrt{k-1}+\sqrt{k-2}\right]}\right]$$
$$= \frac{\pi}{2} - \sum_{v=3}^{3} \arctan\left[\sqrt{2}\sqrt{\frac{v}{2}-1}\right] = \frac{\pi}{2} - \arctan(1)$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Other examples considering the case $s = 0$ are developed below.

- By choosing $\ell = 1$, $r = 1$, and $x = 1$, we have

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{1}{\left[1+\sqrt{k(k-1)}\right]\left[\sqrt{k}+\sqrt{k-1}\right]}\right\} = \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left[\sqrt{v-1}\right]$$
$$= \frac{\pi}{2} - \arctan(0) = \frac{\pi}{2}.$$

- By selecting $\ell = 1$, $r = 1$, and $x = 1/\sqrt{2}$, we have

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{\sqrt{2}}{\left[2+\sqrt{k(k-1)}\right]\left[\sqrt{k}+\sqrt{k-1}\right]}\right\} = \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left[\frac{1}{\sqrt{2}}\sqrt{v-1}\right]$$
$$= \frac{\pi}{2} - \arctan(0) = \frac{\pi}{2}.$$

Some examples focusing on the case $s = 1$ are presented below.

- By taking $\ell = 1$, $r = 1$, and $x = 1$, we have

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{2}{\left[1+\sqrt{k^2-1}\right]\left[\sqrt{k+1}+\sqrt{k-1}\right]}\right\} = \pi - \sum_{v=1}^{2} \arctan\left[\sqrt{v-1}\right]$$
$$= \pi - \arctan(0) - \arctan(1) = \pi - 0 - \frac{\pi}{4} = \frac{3\pi}{4}.$$

- By selecting $\ell = 1$, $r = 1/2$, and $x = 1$, we have

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{2}{\left[1+\sqrt{4k^2-1}\right]\left[\sqrt{2k+1}+\sqrt{2k-1}\right]}\right\}$$
$$= \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left[\sqrt{2v-1}\right] = \frac{\pi}{2} - \arctan(1)$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

- By choosing $\ell = 1$, $r = 1/2$, and $x = \sqrt{3}$, we have

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{2\sqrt{3}}{\left[1+3\sqrt{4k^2-1}\right]\left[\sqrt{2k+1}+\sqrt{2k-1}\right]}\right\}$$
$$= \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left[\sqrt{3}\sqrt{2v-1}\right] = \frac{\pi}{2} - \arctan\left[\sqrt{3}\right]$$
$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

Two examples for the case $s = 2$ are examined below.

- By selecting $\ell = 1$, $r = 1/3$, and $x = \sqrt{3/2}$, we have

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{3\sqrt{6}}{\left[2+3\sqrt{9k^2+3k-2}\right]\left[\sqrt{3k+2}+\sqrt{3k-1}\right]}\right\}$$
$$= \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left[\sqrt{\frac{3}{2}}\sqrt{3v-1}\right] = \frac{\pi}{2} - \arctan\left[\sqrt{3}\right]$$
$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

- By taking $\ell = 4$, $r = 1/3$, and $x = 1$, we obtain

$$\sum_{k=4}^{+\infty} \arctan\left\{\frac{3}{\left[1+\sqrt{9k^2+3k-2}\right]\left[\sqrt{3k+2}+\sqrt{3k-1}\right]}\right\}$$
$$= \frac{\pi}{2} - \sum_{v=4}^{4} \arctan\left[\sqrt{3v-1}\right] = \frac{\pi}{2} - \arctan\left[\sqrt{11}\right],$$

  with $\arctan\left[\sqrt{11}\right] \approx 1.2779$.

A general case concerning the choice of s is investigated in the proposition below.

Proposition 3.11. *For any $s > 0$ and $x > 0$, we have*

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{x(s+1)\sqrt{s}}{\left[s+x^2\sqrt{[k(s+1)+s][k(s+1)-1]}\right]\left[\sqrt{k(s+1)+s}+\sqrt{k(s+1)-1}\right]}\right\}$$
$$= \arctan\left(\frac{1}{x}\right),$$

or, equivalently,

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{x(s+1)\sqrt{s}}{\left[sx^2+\sqrt{[k(s+1)+s][k(s+1)-1]}\right]\left[\sqrt{k(s+1)+s}+\sqrt{k(s+1)-1}\right]}\right\}$$
$$= \arctan(x).$$

Proof. By applying Proposition 3.10 with $\ell = 1$, $r = 1/(s+1)$ with $s > 0$ and $x = y/\sqrt{s}$ with $y > 0$, noticing that

$$\left(\frac{\ell}{r}+s\right)\left(\frac{\ell}{r}-1\right)x^2 = (2s+1)y^2 > 0 > -1,$$

we have

$$\sum_{k=1}^{+\infty} \arctan\left\{\frac{x(s+1)\sqrt{s}}{\left[s+x^2\sqrt{[k(s+1)+s][k(s+1)-1]}\right]\left[\sqrt{k(s+1)+s}+\sqrt{k(s+1)-1}\right]}\right\}$$
$$= \frac{\pi}{2} - \sum_{v=1}^{1} \arctan\left\{\frac{y}{\sqrt{s}}\sqrt{(s+1)v-1}\right\}$$
$$= \frac{\pi}{2} - \arctan\left\{\frac{y}{\sqrt{s}}\sqrt{(s+1)-1}\right\} = \frac{\pi}{2} - \arctan(y) = \arctan\left(\frac{1}{y}\right).$$

The second formula follows by substituting y by $1/y$. Then we put $y = x$ for the sake of uniformity in the statements. This ends the proof. □

The subsection below is about a more deep use of the general Abel summation by parts method described in Proposition 2.1, still with a focus on applications to arctangent-type series.

3.3. Second arctangent-type series result.

3.3.1. *A general result.* The result below is about specific summation by parts satisfied by arctangent-type series.

Proposition 3.12. *Let* $m \in \mathbb{N} \setminus \{0\}$, $\ell \in \mathbb{N}$, $\Xi_\ell = \{k \in \mathbb{N};\ k \geq \ell\}$, $x \in \mathbb{R} \setminus \{0\}$ *and* $(u_k)_{k\in\Xi_\ell}$ *such that* $\inf_{k\in\Xi_\ell} u_{k+m}u_kx^2 > -1$. *Under the assumptions that* $\lim_{n\to+\infty} n\arctan(u_nx)$ *exists and is finite, and the involved series converge, we have*

$$\sum_{k=\ell}^{+\infty} k\arctan\left[\frac{(u_{k+m}-u_k)x}{1+u_{k+m}u_kx^2}\right] = m\lim_{n\to+\infty} n\arctan(u_nx)$$
$$- \sum_{v=\ell}^{m+\ell-1}(v-m)\arctan(u_vx) - m\sum_{k=\ell}^{+\infty}\arctan(u_kx).$$

In the special case $\ell = m = 1$, *the following simplified form holds:*

$$\sum_{k=1}^{+\infty} k\arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_kx^2}\right] = \lim_{n\to+\infty} n\arctan(u_nx) - \sum_{k=1}^{+\infty}\arctan(u_kx).$$

Proof. As in the proof of Proposition 3.1, Equation (3.4) implies that, under the assumption $u_{k+m}u_kx^2 \geq \inf_{k\in\Xi_\ell} u_{k+m}u_kx^2 > -1$ for any $k \in \Xi_\ell$,

$$\arctan\left[\frac{(u_{k+m}-u_k)x}{1+u_{k+m}u_kx^2}\right] = \arctan(u_{k+m}x) - \arctan(u_kx).$$

Thus, by applying this equation and the extended version of the Abel summation described by the item numbered 2 in Proposition 2.1 with the following sequences of functions:

$$f_k(x) = \arctan(u_kx), \quad g_k(x) = k,$$

noticing that $g_{k+m}(x) - g_k(x) = (k+m) - k = m$, we have

$$\sum_{k=\ell}^{+\infty} k \arctan\left[\frac{(u_{k+m}-u_k)x}{1+u_{k+m}u_kx^2}\right] = \sum_{k=\ell}^{+\infty} k\left[\arctan(u_{k+m}x) - \arctan(u_kx)\right]$$
$$= \sum_{k=\ell}^{+\infty} g_k(x)\left[f_{k+m}(x) - f_k(x)\right]$$
$$= m \lim_{n\to+\infty} f_n(x)g_n(x) - \sum_{v=\ell}^{m+\ell-1} f_v(x)g_v(x) - \sum_{k=\ell}^{+\infty} f_{k+m}(x)[g_{k+m}(x) - g_k(x)]$$
$$= m \lim_{n\to+\infty} n\arctan(u_nx) - \sum_{v=\ell}^{m+\ell-1} v\arctan(u_vx) - m\sum_{k=\ell}^{+\infty}\arctan(u_{k+m}x)$$
$$= m \lim_{n\to+\infty} n\arctan(u_nx) - \sum_{v=\ell}^{m+\ell-1} v\arctan(u_vx) - m\sum_{k=\ell+m}^{+\infty}\arctan(u_kx)$$
$$= m \lim_{n\to+\infty} n\arctan(u_nx) - \sum_{v=\ell}^{m+\ell-1} v\arctan(u_vx)$$
$$- m\left[\sum_{k=\ell}^{+\infty}\arctan(u_kx) - \sum_{k=\ell}^{m+\ell-1}\arctan(u_kx)\right]$$
$$= m \lim_{n\to+\infty} n\arctan(u_nx) - \sum_{v=\ell}^{m+\ell-1} (v-m)\arctan(u_vx) - m\sum_{k=\ell}^{+\infty}\arctan(u_kx).$$

To complete, in the special case $\ell = m = 1$, we have

$$\sum_{v=\ell}^{m+\ell-1}(v-m)\arctan(u_vx) = \sum_{v=1}^{1}(v-1)\arctan(u_vx) = 0\times\arctan(u_1x) = 0,$$

giving the desired formula. The proposition is proved. □

In the above proof, the choice of $g_k(x) = k$ is motivated to derive a manageable formula; other choices are possible, depending on what we want to demonstrate. Thus, thanks to this result, we can determine original arctangent-type series of the following form:

$$\sum_{k=\ell}^{+\infty} k \arctan\left[v_k(x)\right],$$

where $[v_k(x)]_{k\in\Xi_\ell}$ denotes a precise sequence of functions. To the best of our knowledge, this type of series has not received much attention in literature.

Remark 3.13. As commented in Remark 3.2 but for Proposition 3.1, based on Proposition 3.12 and Equations (3.2) and (3.3), we can derive general results on arcsine-type and arccosine-type series.

3.3.2. *Examples.* Let us now illustrate Proposition 3.12 with some concrete examples, including one using the Fibonacci sequence.

- By taking $\ell = 1$, $m = 1$, and $u_k = 2/k^2$, using the equivalence $\arctan(t) \sim t$ when $t \to 0$, we have

$$\sum_{k=1}^{+\infty} k \arctan\left[\frac{2(2k+1)x}{k^2(k+1)^2+4x^2}\right] = -\sum_{k=1}^{+\infty} k \arctan\left[-\frac{2(2k+1)x}{k^2(k+1)^2+4x^2}\right]$$
$$= -\sum_{k=1}^{+\infty} k \arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_k x^2}\right] = -\left\{\lim_{n\to+\infty} n \arctan(u_n x) - \sum_{k=1}^{+\infty} \arctan(u_k x)\right\}$$
$$= -\left\{\lim_{n\to+\infty} n \arctan\left(\frac{2}{n^2}x\right) - \sum_{k=1}^{+\infty} \arctan\left(\frac{2}{k^2}x\right)\right\}$$
$$= -\left\{0 - \sum_{k=1}^{+\infty} \arctan\left(\frac{2}{k^2}x\right)\right\} = \sum_{k=1}^{+\infty} \arctan\left(\frac{2}{k^2}x\right).$$

In particular, by taking $x = 1$, it follows from Equation (3.10) that

$$\sum_{k=1}^{+\infty} k \arctan\left[\frac{2(2k+1)}{k^2(k+1)^2+4}\right] = \sum_{k=1}^{+\infty} \arctan\left(\frac{2}{k^2}\right) = \frac{3\pi}{4}. \tag{3.12}$$

As another example, by taking $x = 1/4$, it follows from Equation (3.11) that

$$\sum_{k=1}^{+\infty} k \arctan\left[\frac{2(2k+1)}{4k^2(k+1)^2+1}\right] = \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{2k^2}\right) = \frac{\pi}{4}. \tag{3.13}$$

The remark below presents some techniques for deriving innovative series based on the results above.

Remark 3.14. Naturally, new arctangent-type series can be derived by using the properties of the arctangent function. For instance, it follows from Equations (3.4), (3.12) and (3.13) that

$$\sum_{k=1}^{+\infty} k\arctan\left\{\frac{6(2k+1)(k^2+k-1)(k^2+k+1)}{k(k+1)\{k(k+1)[4k^2(k+1)^2+17]+16\}+8}\right\}$$
$$=\sum_{k=1}^{+\infty} k\arctan\left\{\frac{2(2k+1)/[k^2(k+1)^2+4]-2(2k+1)/[4k^2(k+1)^2+1]}{1+2(2k+1)/[k^2(k+1)^2+4]\times 2(2k+1)/[4k^2(k+1)^2+1]}\right\}$$
$$=\sum_{k=1}^{+\infty} k\left\{\arctan\left[\frac{2(2k+1)}{k^2(k+1)^2+4}\right]-\arctan\left[\frac{2(2k+1)}{4k^2(k+1)^2+1}\right]\right\}$$
$$=\sum_{k=1}^{+\infty} k\arctan\left[\frac{2(2k+1)}{k^2(k+1)^2+4}\right]-\sum_{k=1}^{+\infty} k\arctan\left[\frac{2(2k+1)}{4k^2(k+1)^2+1}\right]$$
$$=\frac{3\pi}{4}-\frac{\pi}{4}=\frac{\pi}{2}.$$

Let us go on presenting more examples of Proposition 3.12.

- By choosing $\ell=1$, $m=2$, and $u_k=2/k^2$, we obtain

$$\sum_{k=1}^{+\infty} k\arctan\left[\frac{8(k+1)x}{k^2(k+2)^2+4x^2}\right]=-\sum_{k=1}^{+\infty} k\arctan\left[-\frac{8(k+1)x}{k^2(k+2)^2+4x^2}\right]$$
$$=-\sum_{k=1}^{+\infty} k\arctan\left[\frac{(u_{k+2}-u_k)x}{1+u_k u_{k+2}x^2}\right]$$
$$=-\left\{2\lim_{n\to+\infty} n\arctan(u_n x)-\sum_{v=1}^{2}(v-2)\arctan(u_v x)-2\sum_{k=1}^{+\infty}\arctan(u_k x)\right\}$$
$$=-\left\{2\lim_{n\to+\infty} n\arctan\left(\frac{2}{n^2}x\right)+\arctan(2x)-0\times\arctan\left(\frac{1}{2}x\right)\right.$$
$$\left.-2\sum_{k=1}^{+\infty}\arctan\left(\frac{2}{k^2}x\right)\right\}$$
$$=-\left\{0+\arctan(2x)-2\sum_{k=1}^{+\infty}\arctan\left(\frac{2}{k^2}x\right)\right\}$$
$$=-\arctan(2x)+2\sum_{k=1}^{+\infty}\arctan\left(\frac{2}{k^2}x\right).$$

In particular, by taking $x=1$, it follows from Equation (3.10) that

$$\sum_{k=1}^{+\infty} k\arctan\left[\frac{8(k+1)}{k^2(k+2)^2+4}\right]=-\arctan(2)+2\sum_{k=1}^{+\infty}\arctan\left(\frac{2}{k^2}\right)$$
$$=-\arctan(2)+2\times\frac{3\pi}{4}=-\arctan(2)+\frac{3\pi}{2}.$$

We recall that $\arctan(2) \approx 1.1071$. As another example, by taking $x = 1/4$, it follows from Equation (3.11) that

$$\sum_{k=1}^{+\infty} k \arctan\left[\frac{8(k+1)}{4k^2(k+2)^2+1}\right] = -\arctan\left(\frac{1}{2}\right) + 2\sum_{k=1}^{+\infty} \arctan\left(\frac{1}{2k^2}\right)$$
$$= -\arctan\left(\frac{1}{2}\right) + 2 \times \frac{\pi}{4} = -\arctan\left(\frac{1}{2}\right) + \frac{\pi}{2} = \arctan(2).$$

- By selecting $\ell = 1$, $m = 1$, and $u_k = 1/(k^2 - k + 1)$, we have

$$\sum_{k=1}^{+\infty} k \arctan\left[\frac{2kx}{k^2(k^2+1)+x^2+1}\right] = -\sum_{k=1}^{+\infty} k \arctan\left[-\frac{2kx}{k^2(k^2+1)+x^2+1}\right]$$
$$= -\sum_{k=1}^{+\infty} k \arctan\left[\frac{(u_{k+1}-u_k)x}{1+u_{k+1}u_k x^2}\right]$$
$$= -\left\{\lim_{n\to+\infty} n \arctan(u_n x) - \sum_{k=1}^{+\infty} \arctan(u_k x)\right\}$$
$$= -\left\{\lim_{n\to+\infty} n \arctan\left(\frac{1}{n^2-n+1}x\right) - \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{k^2-k+1}x\right)\right\}$$
$$= -\left\{0 - \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{k^2-k+1}x\right)\right\} = \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{k^2-k+1}x\right).$$

In particular, by choosing $x = 1$, Equation (3.9) gives

$$\sum_{k=1}^{+\infty} k \arctan\left[\frac{2k}{k^2(k^2+1)+2}\right] = \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{k^2-k+1}\right) = \frac{\pi}{2}.$$

A notable example of Proposition 3.12 can be given with the use of the Fibonacci sequence. In this context, let us consider the sequence $(u_k)_{k\in\mathbb{N}}$ defined by

$$u_k = \frac{1}{F_{2k+1}}.$$

By taking $\ell = 1$ and $m = 1$, noticing that $F_{2k+2} = F_{2k+3} - F_{2k+1}$, using the Cassini identity, i.e., for any $p \in \mathbb{N} \setminus \{0, 1\}$, $F_p^2 = F_{p+1}F_{p-1} + (-1)^{p-1}$, with $p = 2k + 2$, which gives $F_{2k+2}^2 = F_{2k+3}F_{2k+1} - 1$, and $\lim_{n\to+\infty} n/F_{2n+1} = 0$, we get

$$\begin{aligned}
\sum_{k=1}^{+\infty} k \arctan\left(\frac{F_{2k+2}}{F_{2k+2}^2+2}\right) &= \sum_{k=1}^{+\infty} k \arctan\left[\frac{F_{2k+3}-F_{2k+1}}{F_{2k+1}F_{2k+3}+1}\right] \\
&= -\sum_{k=1}^{+\infty} k \arctan\left[\frac{1/F_{2k+3}-1/F_{2k+1}}{1+(1/F_{2k+3})(1/F_{2k+1})}\right] = -\sum_{k=1}^{+\infty} k \arctan\left[\frac{u_{k+1}-u_k}{1+u_{k+1}u_k}\right] \\
&= -\left\{\lim_{n\to+\infty} n \arctan(u_n) - \sum_{k=1}^{+\infty} \arctan(u_k)\right\} \\
&= -\left\{\lim_{n\to+\infty} n \arctan\left(\frac{1}{F_{2n+1}}\right) - \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{F_{2k+1}}\right)\right\} \\
&= -\left\{0 - \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{F_{2k+1}}\right)\right\} = \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{F_{2k+1}}\right).
\end{aligned}$$

It follows from Equation (3.6) that

$$\sum_{k=1}^{+\infty} k \arctan\left(\frac{F_{2k+2}}{F_{2k+2}^2+2}\right) = \sum_{k=1}^{+\infty} \arctan\left(\frac{1}{F_{2k+1}}\right) = \sum_{k=0}^{+\infty} \arctan\left(\frac{1}{F_{2k+3}}\right) = \frac{\pi}{4}.$$

We conclude this section by listing new formulas as additional applications of Proposition 3.12 under certain sequences and parameter configurations:

$$\sum_{k=1}^{+\infty} k \arctan\left[\frac{2\sqrt{3}(2k-1)}{(k-1)k[(k-1)k+4]+18}\right] = \frac{7\pi}{6},$$

$$\sum_{k=1}^{+\infty} k \arctan\left[\frac{2\sqrt{2}k}{k^4+3k^2+6}\right] = \frac{\pi}{2}$$

and

$$\sum_{k=1}^{+\infty} k \arctan\left\{\frac{3\sqrt{3}(2k+1)}{6k(k+1)[3k(k+1)-1]-1}\right\} = \frac{\pi}{6}.$$

The details are omitted for the sake of brevity. Other examples can be presented in a similar manner.

4. Conclusion

In this article, we have extended the applicability and understanding of the telescopic summation and Abel summation by parts methods through the introduction of tuning parameters and sequences of functions. By applying these general results to arctangent-type series, known results were re-examined, many new ones were established, and unified formulas were exhibited. In particular, several innovative arctangent-type series defined with the binomial numbers or with the Fibonacci sequence add an attractive dimension to this research. Among the logical perspectives, there is the investigation of general series of the following form:

$$" \sum_{k=\ell}^{+\infty} k^{\alpha} \arctan\left[v_k(x)\right] ",$$

where α denotes a tuning parameter beyond the (now) well-known cases $\alpha = 0$ and $\alpha = 1$, and $[v_k(x)]_{k \in \Xi_\ell}$ denotes a sequence of functions beyond the conventional form: $v_k(x) = [(u_{k+m} - u_k)x]/[1 + u_{k+m}u_k x^2]$. Such series can play an important role in probability and statistics in particular, giving direct formulas for the moments of discrete distributions of the arctangent type, which are certainly under-exploited modeling tools because of their overall lack of mastery. Although the techniques in this article may be adaptable to this challenge, further work is needed, representing a direction for future research.

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