

DEFERRED CONULL FK-SPACES

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ABSTRACT. In this paper, the concepts of deferred conullity for an FK-space X containing ϕ_1 , and deferred wedgeness are defined. We also define deferred semi-conservative FK-space and give its relation to these concepts. On the other hand, we have examined their relationship to ordinary conullity and Cesàro conullity. Furthermore, we have defined some deferred distinguished subspaces of an FK-space X . Also, we obtain some results for a summability domain Y_A to be deferred conull FK-space.

1. INTRODUCTION AND PRELIMINARIES

The classification of conservative matrices as conull or coregular was defined by Wilansky [18]. The mentioned classification was extended to all FK-spaces Jürimäe [21] and Snyder [17]. Then, some results of Sember [15] were improved by Bennett [3] for conull FK-spaces. Finally, İnce [13] studied (strongly) Cesàro conull FK-spaces, Dağadur [8] continued to work on C_λ -conull FK-spaces and to give some characterizations.

In this paper, we expand upon the work of the above authors by using deferred Cesàro mean under the weaker conditions.

In 1932, Agnew [1] defined the deferred Cesàro mean $D_{p,q}$ of the sequences x by

$$(D_{p,q}x)_n = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of nonnegative integers satisfying the conditions $p(n) < q(n)$ and $\lim_{n \rightarrow \infty} q(n) = \infty$. $D_{p,q}$ is clearly regular for any choice of $\{p(n)\}$ and $\{q(n)\}$.

w denotes the spaces of all complex valued sequences, any vector subspace X of w is a sequence space. A sequence space X with a complete, metrizable, locally convex topology τ is called FK-space if the inclusion map $i : (X, \tau) \rightarrow w$, $i(x) = x$ is continuous when w is endowed with the topology of coordinatewise convergence. An FK-space whose topology is normable is called a BK space.

By c , l_∞ we denote the spaces of all convergent sequences and bounded sequences, respectively. These are FK-spaces under $\|x\| = \sup_n |x_n|$. $bw = \{x \in w :$

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$\sum_{n=1}^{\infty} |x_n - x_{n+1}| < \infty$ the spaces of all summable sequences of bounded variation; $cs = \{x \in w : \sup_k |\sum_{n=1}^k x_n| < \infty\}$, the spaces of all summable sequences. $l = \{x \in w : \sum_n |x_n| < \infty\}$, the spaces of all absolute summable sequences.

Throughout this study e denotes the sequences of ones; δ^j ($j = 1, 2, \dots$) the sequence with the one in the j -th position; ϕ the linear span of δ^j 's. The linear span of ϕ and e is denoted by ϕ_1 . A sequence x in a locally convex sequence space X is said the property AK if $x^{(n)} \rightarrow x$ in X where $x^{(n)} = \sum_{k=1}^n x_k \delta^k$. An FK-space X is called conservative if $c \subset X$. Also, an FK-space X is called semi-conservative if $X^f \subset cs$, where $X^f = \{\{f(\delta^k)\} : f \in X'\}$ (see [19]; [18]). We recall (see [10, 11]) that the σ -dual of a subset X of w is defined to be

$$\begin{aligned} X^\sigma &= \left\{ x \in w : \lim_n \frac{1}{n} \sum_{k=1}^n \sum_{j=1}^k x_j y_j \text{ exists for all } y \in X \right\} \\ &= \{x \in w : x.y \in \sigma s \text{ for all } y \in X\}, \end{aligned}$$

where $x.y = (x_n y_n)$.

Following Jürimäe [21] and Snyder [17] we say that an FK-space (X, τ) containing ϕ_1 is a conull space if $e - e^{(n)} \rightarrow 0$ (weakly) in X . It is strongly conull space if $e - e^{(n)} \rightarrow 0$ in X . (X, τ) is a K-space containing ϕ and $\delta^k \rightarrow 0$ in X then (X, τ) is called a wedge space [3]. It is weak wedge space if $\delta^k \rightarrow 0$ (weak) in X . Bennett [3] gave a relationship between (strongly) conull and (weak) wedge FK-spaces. Also, an FK-space (X, τ) containing ϕ_1 is a Cesàro conull space if $e - \frac{1}{n} \sum_{k=1}^n e^{(k)} \rightarrow 0$ (weakly) in X , and it is strongly Cesàro conull space if $e - \frac{1}{n} \sum_{k=1}^n e^{(k)} \rightarrow 0$ in X [13].

2. MAIN RESULTS

In this section in addition to the work of [13] and [8] for any FK space, we extend the notion of conullity with deferred Cesàro means.

Let us first define the following space

$$\sigma_p^q[s] := \left\{ x \in w : \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k x_j \text{ exists} \right\}.$$

The space is BK spaces with the norm

$$\|x\| = \sup_n \left| \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k x_j \right|.$$

We defined d -dual of a subset X of w as

$$X^d = \{x \in w : x.y \in \sigma_p^q[s] \text{ for all } y \in X\}.$$

$X \subset X^{\nu\nu}$ for every set of sequences X , with $\nu = f, \sigma, d$.

Let X be a locally convex sequence space. If for all $x \in X$,

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \rightarrow x \text{ in } X,$$

then x is said to have the property $\sigma_p^q[K]$.

Definition 2.1. An FK-space X is called deferred semiconservative FK-space if $X^f \subset \sigma_p^q[s]$. i.e.; $X \supset \phi$ and for every $f \in X'$ the following limit is exists

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k f(\delta^j) \right\}.$$

Definition 2.2. Let $X \supset \phi_1$ be an FK-space and

$$\zeta^n := e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)}. \quad (2.1)$$

If $\zeta^n \rightarrow 0$ in X then X is called strongly deferred conull FK-space, where $e^{(k)} := \sum_{j=1}^k \delta^j$. If the convergence holds in the weak topology in (2.1) then X is called deferred conull FK-space. Hence X is deferred conull iff

$$f(e) = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k f(\delta^j), \quad \forall f \in X'.$$

It is clear from the above definitions that if X is deferred conull FK-space then it is deferred semiconservative FK-space.

Theorem 2.3. Let $X \supset \phi_1$ be an FK-space and $\left\{ \frac{p(n)}{q(n)-p(n)} \right\}$ be a bounded sequence. If X is Cesàro conull FK-space, then it is deferred conull FK-space.

Proof. Let X be Cesàro conull. Then for each $f \in X'$, we have

$$f(e) = \lim_n \frac{1}{n} \sum_{k=1}^n \sum_{j=1}^k f(\delta^j).$$

Let $s_k(f) = \sum_{j=1}^k f(\delta^j)$. So, $(s_k(f))$ is Cesàro summable to $f(e)$. Since $\left\{ \frac{p(n)}{q(n)-p(n)} \right\}$ is bounded, it is deferred summable to same value [1, Theorem 4.1]. Hence, X is deferred conull FK-space. \square

Theorem 2.4. Let $X \supset \phi_1$ be an FK-space and $q(n) = n$. If X is deferred conull FK-space, then it is Cesàro conull FK-space.

Proof. Let X be deferred conull FK-space. Then

$$f(e) = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k f(\delta^j), \quad \forall f \in X'.$$

Let $s_k(f)$ be as in Theorem 2.3. So, $(s_k(f))$ is deferred summable to $f(e)$. Since $q(n) = n$, it is Cesàro summable to same value [1, Theorem 6.1]. Thus X is Cesàro conull FK-space. \square

Theorem 2.5. Let X be a deferred semiconservative FK-space. Then X is conull FK-space iff it is deferred conull FK-space.

Proof. Necessity is clear. Sufficiency. Assume that X is deferred conull FK-space. Then,

$$f(e) - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k f(\delta^j) = \chi(f) + \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} f(\delta^j). \quad (2.2)$$

where $\chi(f) = f(e) - \sum_{j=1}^{\infty} f(\delta^j)$. Since X is deferred semiconservative we have $(f(\delta^j)) \in \sigma_p^q[s]$; so, the second term of the right side of the equation (2.2) is convergent to 0 as $n \rightarrow \infty$. By hypothesis, the left side of the equation (2.2) is convergent to 0 as $n \rightarrow \infty$. This implies that $\chi(f) = 0$, i.e. X is conull FK-space. \square

Recall that, A matrix A is called conservative is $c_A \supset c$ (i.e. $Ax \in c$ whenever $x \in c$). By taking $X = c_A$, we get the following theorem.

Corollary 2.6. *Let A be a conservative matrix. Then c_A is conull FK-space iff it is deferred conull FK-space.*

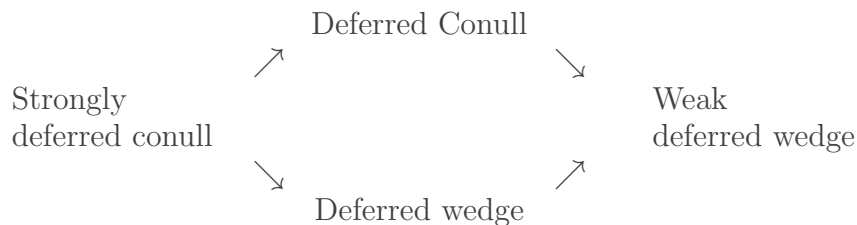
Definition 2.7. Let (X, τ) be a K space containing ϕ . (X, τ) is called deferred wedge FK-space if

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \delta^k \rightarrow 0 \quad (n \rightarrow \infty) \text{ in } X;$$

and (X, τ) is called weak deferred wedge FK-space, if for all $f' \in X$

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} f'(\delta^k) \rightarrow 0 \quad (n \rightarrow \infty).$$

There is a relationship between deferred conullity and deferred wedgeness as follows.



Now let us consider the following one-to-one and onto mapping

$$T : w \rightarrow w, \quad Tx = \left(x_1, x_1 + x_2, \dots, \sum_{k=1}^n x_k, \dots \right),$$

and

$$T^{-1}x = (x_1, x_2 - x_1, \dots, x_n - x_{n-1}, \dots) \quad (\text{see [3, 5]}).$$

With the help of the mapping $T^{-1}(x)$, a relation between conull and wedge spaces can be established as follows.

Lemma 2.8. *Let (X, τ) be an FK-space. Then*

i) X is strongly deferred conull FK-space if and only if $T^{-1}(X)$ is deferred wedge FK-space.

ii) X is deferred conull FK-space if and only if $T^{-1}(X)$ is weak deferred wedge FK-space.

Proof. The FK topology of X can be given a sequence of seminorms $\{r(n)\}$. Then $T^{-1}(X)$ can be topologized by $t_n(x) = r_n(Tx)$, ($n = 1, 2, \dots$), so that it also becomes an FK-space [12].

Necessity of (ii). Let X be a deferred conull FK-space. Observe that $T : (T^{-1}(X), \tau') \rightarrow (X, \tau)$ is a topological isomorphism [12]. Since X is deferred conull, $\zeta^n \rightarrow 0$ (weakly) in X . Because $T^{-1} : (X, \tau) \rightarrow (T^{-1}(X), \tau')$ is continuous, it is weakly continuous. So,

$$T^{-1} \left(e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k \delta^j \right) = \frac{1}{q(n) - p(n)} \sum_{j=p(n)+1}^{q(n)} \delta^j \rightarrow 0 \text{ (weakly)}$$

in $T^{-1}(X)$. So $T^{-1}(X)$ is weak deferred wedge space.

Sufficiency. It is enough to observe that $T : (T^{-1}(X), \tau') \rightarrow (X, \tau)$ is weakly continuous and then

$$T \left(\frac{1}{q(n) - p(n)} \sum_{j=p(n)+1}^{q(n)} \delta^j \right) = e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k \delta^j.$$

□

Theorem 2.9. *Let $X \supset \phi_1$ be an FK-space. If X is $\sigma_p^q[K]$ space then it is deferred conull FK-space. The converse is not true.*

Proof. Let X be a $\sigma_p^q[K]$ space. Then we obtain $\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \rightarrow x$ ($n \rightarrow \infty$) for each $x \in X$. In particular, let $x = e \in X$. Therefore, X is deferred conull FK-space, so, it is not $\sigma_p^q[K]$ space. □

Theorem 2.10. *Let X be a conservative FK-space and consider the propositions below:*

- i) X is a $\sigma_p^q[K]$ space*
 - ii) X is strongly deferred conull FK-space*
 - iii) X is deferred conull FK-space*
 - iv) $e \in \bar{\phi}$.*
- Then (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) holds.*

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) is obtained from their definitions.

(iii) \Rightarrow (iv). If $f = 0$, on ϕ , then $f(e) = \lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k f(\delta^j) = 0$. Hence, from Hahn Banach Theorem, $e \in \bar{\phi}$ is obtained. □

$z^{-1}.X = \{x : z.x \in X\}$ is an FK space [18]. Using this expression, the following proposition is obtained.

Proposition 2.11. *Let (X, q) be an FK- $\sigma_p^q[K]$ space and $z \in w$. Then $z^{-1}.X$ is also a $\sigma_p^q[K]$ space.*

Taking $X = \sigma_p^q[s]$ in Proposition 2.11 we get

$$\begin{aligned} z^{-1}.\sigma_p^q[s] &= \{x : zx \in \sigma_p^q[s]\} \\ &= \left\{ x : \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k z_j x_j \text{ exists} \right\} \\ &= z^d. \end{aligned}$$

So we have Theorem 2.12 and Theorem 2.13.

Theorem 2.12. *If $z \in w$, then z^d is a $\sigma_p^q[K]$ space.*

Theorem 2.13. *If $z \in \sigma_p^q[s]$, then z^d is strongly deferred conull FK-space.*

Proof. If $z \in \sigma_p^q[s]$, then $e \in z^{-1}\sigma_p^q[s] = z^d$. From Theorem 2.12, z^d is a $\sigma_p^q[K]$ space. Hence, we have $e - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)} \rightarrow 0$, $n \rightarrow \infty$. This completes the proof. \square

Theorem 2.14. *i) An FK-space that contains a (strongly) deferred conull FK-space must be a (strongly) deferred conull FK-space.*

ii) A closed subspace, containing ϕ_1 , of a (strongly) deferred conull FK space is a (strongly) deferred conull FK-space

iii) A countable intersection of (strongly) deferred conull FK-spaces is a (strongly) deferred conull FK-spaces.

The proof is easily obtained from the elementary properties of FK-spaces [18].

Theorem 2.15. *If X be a deferred conull FK-space, then $l_\infty \cap X$ is nonseperable subspace of l_∞ .*

Proof. c is not deferred conull FK-space, and hence, by Theorem 2.14(ii) implies that $c \cap X$ is not closed in X . By Theorem 8 of Bennett [3], desired result is obtained. \square

3. SOME DEFERRED DISTINGUISHED SUBSPACES OF X

In this section we will define the distinguished spaces $D_p^q W$, $D_p^q S$ and $D_p^q F^+$ of an FK-space X , and give their relation with the notions of (strongly) deferred conullity and deferred semiconservativity.

Definition 3.1. Let $X \supset \phi$ be an FK-space.

$$\begin{aligned} D_p^q W &:= D_p^q W(X) = \left\{ x \in X : \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \rightarrow x \text{ (weakly) in } X \right\} \\ &= \left\{ x \in X : f(x) = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k x_j f(\delta^j) \text{ for all } f \in X \right\}, \end{aligned}$$

$$\begin{aligned}
D_p^q S &:= D_p^q S(X) = \left\{ x \in X : \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \rightarrow x \right\} \\
&= \{ x \in X : x \text{ has } \sigma_p^q[K] \text{ in } X \} \\
&= \left\{ x \in X : x = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k x_j \delta^j \right\}.
\end{aligned}$$

So, X is a $\sigma_p^q[K]$ -space if and only if $D_p^q S = X$.

$$\begin{aligned}
D_p^q F^+ &:= D_p^q F^+(X) \\
&= \left\{ x \in X : \lim_n \left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\} \text{ is weakly Cauchy in } X \right\} \\
&= \left\{ x \in X : \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k x_j f(\delta^j) \text{ exists for all } f \in X' \right\} \\
&= \{ x \in X : \{x_n f(\delta^n)\} \in \sigma_p^q[s] \text{ for all } f \in X' \} = (X^f)^d.
\end{aligned}$$

$$D_p^q F = D_p^q F^+ \cap X$$

Theorem 3.2. *Let X be an FK-space $\supset \phi$, $z \in w$. Then*

i) $z \in D_p^q W$ if and only if $z^{-1}.X$ is deferred conull FK-space; in particular $e \in D_p^q W$ if and only if X is deferred conull FK-space.

ii) $z \in D_p^q S$ if and only if $z^{-1}.X$ is strongly deferred conull FK-space; in particular $e \in D_p^q S$ if and only if X is strongly deferred conull FK-space.

iii) $z \in D_p^q F^+$ if and only if $z^{-1}.X$ is deferred semiconservative FK-space; in particular $e \in D_p^q F^+$ if and only if X is deferred semiconservative FK-space.

Proof. (i) Necessity. Let $f \in (z^{-1}.X)'$. Then $f \in (z^{-1}.X)'$ iff $f(x) = \alpha x + g(z.x)$, where $\alpha \in \phi$, $g \in X'$ [18]. So we have

$$\begin{aligned}
&f \left(e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k \delta^j \right) = \alpha \left(e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k \delta^j \right) \\
&+ g \left(z \left(e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k \delta^j \right) \right) \\
&= \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} \alpha_j + g \left(z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k z_j \delta^j \right). \quad (3.1)
\end{aligned}$$

Since $\alpha \in \phi$, $\sum_{j=1}^{\infty} \alpha_j$ exists, so we have $\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} \alpha_j \rightarrow 0$ as $n \rightarrow \infty$. By hypothesis for all $g \in X'$, $g \left(z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k z_j \delta^j \right) \rightarrow 0$ as $n \rightarrow \infty$. Sufficiency is clear by (3.1).

(ii) Necessity. From Theorem 4.3.6 of [18], we consider the following seminorms of $z^{-1}.X$. Firstly,

$$t_i(\zeta^n) = \begin{cases} 0, & i \leq p(n) \\ \frac{i-p(n)}{q(n)-p(n)}, & p(n) < i < q(n) \\ 1, & i \geq q(n). \end{cases}$$

For each i , $t_i(\zeta^n) \rightarrow 0$ as $n \rightarrow \infty$. Also,

$$h(\zeta^n) = r(z.\zeta^n) = r\left(z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k z_j \delta^j\right).$$

Now the result follows at once.

(iii) Let $z^{-1}.X$ be deferred semiconservative FK-space. Since $(z^{-1}.X)^f \subset \sigma_p^q[s]$, we get $f \in (z^{-1}.X)'$. So the equality $f(\delta^n) = \alpha\delta^n + g(z_n\delta^n) = \alpha\delta^n + z_n g(\delta^n)$ is hold, where $\alpha \in \phi$, $g \in (z^{-1}.X)'$. Thus, since $\alpha \in \phi \subset \sigma_p^q[s]$ then $\{f(\delta)\} \in \sigma_p^q[s]$ if and only if $\{z_n g(\delta^n)\} \in \sigma_p^q[s]$, i.e. $z \in D_p^q F^+$. \square

4. SUMMABILITY DOMAINS AND APPLICATIONS

In this section we give simple conditions for a summability domain Y_A to be (strongly) deferred conull FK-space. At the end we give some examples. We are concerned with matrix transformations $y = Ax$, where $x, y \in w$, $A = \{a_{ij}\}_{i,j=1}^{\infty}$ is an infinite matrix with complex coefficients, and

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j \quad (i = 1, 2, \dots).$$

The sequence $\{a_{ij}\}_{j=1}^{\infty}$ is called the i -th row of A and is denoted by a^i , ($i = 1, 2, \dots$); similarly, the j -th column of the matrix A , $\{a_{ij}\}_{i=1}^{\infty}$ is denoted by a^j , ($j = 1, 2, \dots$). For an FK-space Y , the summability domain Y_A is defined by

$$Y_A = \{x \in w : Ax \text{ exists and } Ax \in Y\}.$$

Also, Y_A is an FK-space under the seminorms $t_n(x) = |x_n|$, ($n = 1, 2, \dots$);

$$h_n(x) = \sup_m \left| \sum_{j=1}^m a_{nj} x_j \right| \quad (n = 1, 2, \dots) \quad \text{and} \quad (r \circ A)(x) = r(Ax) \quad [18].$$

Theorem 4.1. *Let Y be an FK-space and A be a matrix such that $\phi_1 \subset Y_A$. Then Y_A is a deferred conull FK-space iff*

$$A \left[e - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)} \right] \rightarrow 0 \quad (\text{weakly}) \text{ in } Y.$$

Proof. Necessity. Let Y_A be a deferred conull FK-space. Then $\forall f \in Y'_A$,

$$f(\zeta^n) \rightarrow 0, \quad (n \rightarrow \infty). \quad (4.1)$$

Let $f(x) = g(Ax)$, for $g \in Y'$. So by Theorem 4.4.2 of Wilansky [18], $f \in Y'_A$. Because of $f(\zeta^n) = g(A\zeta^n)$, the desired result is obtained from (4.1).

Sufficiency. Let $f \in Y'_A$. Again by Theorem 4.4.2 of Wilansky [18], $\forall f \in Y'_A$,
iff

$$f(x) = \sum_{k=1}^{\infty} \alpha_k x_k + g(Ax),$$

for all $x \in Y_A$, where $\alpha \in w_A^\beta = \{x : \sum_{n=1}^{\infty} x_n y_n \text{ convergent for all } y \in w_A\}$ and $g \in Y'$. Hence we have

$$f(\zeta^n) = \frac{1}{q(n) - p(n)} \sum_{j=p(n)+1}^{q(n)} \sum_{k=1}^{\infty} \alpha_k + g(A(\zeta^n)). \quad (4.2)$$

By hypothesis $e \in Y_A \subset w_A$. Then $\alpha \in w_A^\beta \subset e^\beta = cs$ which implies that

$$\lim_{n \rightarrow \infty} \frac{1}{q(n) - p(n)} \sum_{j=p(n)+1}^{q(n)} \sum_{k=1}^{\infty} \alpha_k = 0.$$

Also the second term on the right hand side of (4.2) tends to zero. This completed the proof. \square

Theorem 4.2. $z \in w$, Y is an FK-space and A is a matrix such that $\phi \subset Y_A$ i.e. the columns of A belong to Y . Then the following propositions are equivalent in Y_A .

- i) $z \in D_p^q W$,
- ii) $Az - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} Az^{(k)} \rightarrow 0$ (weakly) in Y ,
- iii) $Y_{A.z}$ is deferred conull FK-space,
- iv) $g(Az) = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k z_j g(a^j)$ for each $g \in Y'$, where

$$Az = (a_{ij} z_j), \quad Az^{(k)} = \sum_{j=1}^k z_j a^j \quad \text{and} \quad (Az^{(k)})_i = \sum_{j=1}^k a_{ij} z_j.$$

Proof. By previous theorem (i) \equiv (iii). To show that (iii) \equiv (iv), let $A.z := B$. Then by Theorem 4.1, Y_B is a deferred conull if and only if $\forall g \in Y'$,

$$g(Be) = \lim_{n \rightarrow \infty} \frac{1}{q(n) - p(n)} \sum_{j=p(n)+1}^{q(n)} g(Be^{(k)}).$$

Thus, since

$$Be = \left[\sum_{j=1}^{\infty} b_{ij} \right] = \left[\sum_{j=1}^{\infty} a_{ij} z_j \right] = Az,$$

then we have

$$(Be^{(k)})_i = \left[\sum_{j=1}^{\infty} a_{ij} z_j \right]_i \quad \text{and} \quad Be^{(k)} = \sum_{j=1}^k a^j z_j.$$

So for each $g \in Y'$, we obtained

$$g(Az) = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k z_j g(a^j).$$

Since $Az^{(k)} = \sum_{j=1}^k z_j a^j$, the proof of (ii) \equiv (iii) is clear. \square

Theorem 4.3. $z \in w$, (Y, r) is an FK-space and A is a matrix such that $\phi \subset Y_A$ i.e. the columns of A belong to Y . Then the following propositions are equivalent in Y_A .

- i) $z \in D_p^q S$,
- ii) $Az - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} Az^{(k)} \rightarrow 0$ in Y ,
- iii) $Y_{A,z}$ is strongly deferred conull FK-space,
- iv) $Az = \lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k z_j a^j$ convergence in Y , where a^j is the k th column of A .

Proof. By Theorem 3.2, we obtain (i) \equiv (iii). Since $Az^{(k)} = \sum_{j=1}^k z_j a^j$, we have (ii) \equiv (iv). Let $z \in D_p^q S$. Since $z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k z_j \delta^j \rightarrow 0$, $n \rightarrow \infty$, and $A : Y_A \rightarrow Y$ is continuous so $A(z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)}) \rightarrow 0$, $n \rightarrow \infty$. This gives (i) \Rightarrow (ii).

By Theorem 4.3.8. of Wilansky [18], $(w_A, t \cup h)$ is an AK-space, so it is $\sigma_p^q[K]$ -space. Since $z \in Y_A \subset w_A$, for each i , we obtain

$$t_i \left(z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) \rightarrow 0 \text{ and } h_i \left(z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) \rightarrow 0.$$

By hypothesis

$$\begin{aligned} & (r \circ A) \left(z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) \\ &= r \left(Az - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} Az^{(k)} \right) \rightarrow 0, \end{aligned}$$

which proves theorem. \square

Corollary 4.4. Let (Y, q) is an FK-space and A is a matrix such that $\phi_1 \subset Y_A$. Then Y_A is strongly deferred conull FK-space if and only if

$$A \left[e - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)} \right] \rightarrow 0 \text{ in } Y.$$

Proof. Take $z = e$ in Theorem 4.3 ((ii) \equiv (iii)). \square

Our next result follows immediately from definitions.

Theorem 4.5. If weak convergence and strong convergence coincide in a locally convex FK-space Y , then for the Y_A , $D_p^q S = D_p^q W$.

Example 4.6. Theorem 4.5 applies when $Y = l, bv$ and bv_0 .

The following result is obtained by Theorem 4.5.

Theorem 4.7. *Let Y be an FK-space such that weakly convergent sequences are convergent in the FK topology and let A be a matrix. Then Y_A is deferred conull FK-space iff it is strongly deferred conull FK-space.*

Corollary 4.8. *Let $\phi_1 \subset c_A$. Then c_A is strongly deferred conull FK-space if and only if*

$$\limsup_n \sup_i \left| \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right| = 0.$$

Proof. Take $Y = c$ in Corollary 4.4. c_A is strongly deferred conull FK-space if and only if

$$A \left(e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)} \right) \rightarrow 0 \text{ in } c.$$

So, we get

$$\begin{aligned} & \lim_n \left\| A \left(e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k \delta^j \right) \right\|_{\infty} = 0 \\ \Leftrightarrow & \limsup_n \sup_i \left| \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right| = 0. \end{aligned}$$

This proves the result. □

Corollary 4.9. *Let $\phi \subset l_A$. Then l_A is (strongly) deferred conull space FK-space if and only if*

$$\lim_n \sum_{i=1}^{\infty} \left| \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right| = 0.$$

Proof. Let l_A be deferred conull FK-space. By Theorem 3.2(i) $e \in D_p^q W$. By Theorem 4.7 $e \in D_p^q W$ if and only if

$$\left(Ae - \sum_{k=p(n)+1}^{q(n)} Ae^{(k)} \right) \rightarrow 0 \text{ (weakly) , } n \rightarrow \infty \text{ in } l.$$

Thus,

$$\left\| A \left(e - \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k \delta^j \right) \right\|_l \rightarrow 0, n \rightarrow \infty.$$

So we have

$$\begin{aligned} & \lim_n \left\| \left(Ae - \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k a_{ij} \right) \right\|_l = 0 \\ \Leftrightarrow & \lim_n \sum_{i=1}^{\infty} \left| \left(Ae - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k a_{ij} \right) \right|_i = 0 \\ \Leftrightarrow & \lim_n \sum_{i=1}^{\infty} \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right| = 0. \end{aligned}$$

This proves the corollary. \square

Corollary 4.10. *Let $\phi \subset (bv)_A$. Then $(bv)_A$ is (strongly) deferred conull FK-space if and only if*

$$\begin{aligned} & \lim_n \left\{ \sum_{i=1}^{\infty} \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} (a_{ij} - a_{i+1,j}) \right| \right. \\ & \left. + \lim_i \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right| \right\} = 0. \end{aligned}$$

Proof. Let $(bv)_A$ is deferred conull FK-space. By Theorem 4.2, $e \in D_p^q W$. Since weakly and strongly convergence are equivalent in bv , $D_p^q S = D_p^q W$ in bv_A by Theorem 4.7. So, $e \in D_p^q S$. Thus

$$\left(Ae - \sum_{k=p(n)+1}^{q(n)} Ae^{(k)} \right) \rightarrow 0, \quad n \rightarrow \infty \text{ in } bv.$$

So we

$$\begin{aligned} & \lim_n \left\| \left(Ae - \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k a_{ij} \right) \right\|_{i \parallel_{bv}} = 0 \\ \Leftrightarrow & \lim_n \left\| \left(\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right) \right\|_{i \parallel_{bv}} = 0 \\ \Leftrightarrow & \lim_n \left\{ \sum_{i=1}^{\infty} \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} (a_{ij} - a_{i+1,j}) \right| \right. \\ & \left. + \lim_i \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right| \right\} = 0. \end{aligned}$$

This completes the proof. \square

Corollary 4.11. *Let $\phi \subset (l_\infty)_A$. Then $(l_\infty)_A$ is deferred conull FK-space if and only if*

$$i) \sup_{i,n} \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k a_{ij} \right| < \infty$$

ii) for any given $\varepsilon > 0$ and increasing sequences $p(n_s)$ and $q(n_s)$ of positive integers, there exists L such that $\sup_i \min_{1 \leq s \leq L} \left| \frac{1}{q(n_s)-p(n_s)} \sum_{k=p(n_s)+1}^{q(n_s)} \sum_{j=k+1}^{\infty} a_{ij} \right| < \varepsilon$.

Proof. Let us take $Y = l_\infty$ in Theorem 4.2. Then $(l_\infty)_A$ is deferred conull FK-space if and only if

$$Ae - \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k a_{ij} \rightarrow 0 \text{ (weakly) , } n \rightarrow \infty \text{ in } l_\infty.$$

This means that

$$\sup_n \left\| \left(\sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k a_{ij} \right) \right\|_{i \parallel_\infty} < \infty \Leftrightarrow \sup_{i,n} \left| \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^k a_{ij} \right| < \infty$$

and there exists L such that

$$\sup_i \min_{1 \leq s \leq L} \left| \frac{1}{q(n_s)-p(n_s)} \sum_{k=p(n_s)+1}^{q(n_s)} \sum_{j=1}^k a_{ij} - \sum_{j=1}^{\infty} a_{ij} \right| < \varepsilon$$

for $\varepsilon > 0$ and increasing sequences $p(n_s)$, $q(n_s)$ of positive integers [9]. Moreover

$$\left| \frac{1}{q(n_s)-p(n_s)} \sum_{k=p(n_s)+1}^{q(n_s)} \sum_{j=1}^k a_{ij} - \sum_{j=1}^{\infty} a_{ij} \right| = \left| \frac{1}{q(n_s)-p(n_s)} \sum_{k=p(n_s)+1}^{q(n_s)} \sum_{j=k+1}^{\infty} a_{ij} \right|.$$

Thus we get

$$\sup_i \min_{1 \leq s \leq L} \left| \frac{1}{q(n_s)-p(n_s)} \sum_{k=p(n_s)+1}^{q(n_s)} \sum_{j=k+1}^{\infty} a_{ij} \right| < \varepsilon.$$

The proof is completed. \square

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