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SOME TURAN-TYPE INEQUALITIES FOR A CLASS OF RATIONAL FUNCTIONS WITH PRESCRIBED POLES

BASHIR AHMAD GANIE[1*], VARUN MOHAN[2] AND WALI MOHAMMAD SHAH[3]

Abstract. In this paper we prove some results for rational functions with t-fold zeros at the origin and thereby obtain generalizations and refinements of some known inequalities for rational functions. These results in particular yield the improvement and generalizations of some polynomial inequalities.

1. Introduction and preliminaries

Let $\mathcal{P}_n$ denote the class of all complex polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n and $P'(z)$ be the derivative of $P(z)$. Let D_k^- represent the set of all points which lie inside $T_k := \{z : |z| = k\}$, D_k^+ the set of points which lie outside T_k . Concerning the estimate of $|P'(z)|$ in terms of $|P(z)|$ for $z \in T_1$, Bernstein [5] proved the following :

Theorem 1.1. *If $P \in \mathcal{P}_n$, then*

$$\max_{z\in T_1} |P'(z)| \leq n \max_{z\in T_1} |P(z)|. \tag{1.1}$$

Inequality (1.1) is sharp and equality holds if $P(z)$ has all its zeros at the origin.
If we restrict the location of the zeros of a polynomial, then in a special case Turán [16] proved the following :

Theorem 1.2. *If all the zeros of $P \in \mathcal{P}_n$ lie in $T_1 \cup D_1^-$, then*

$$\max_{z\in T_1} |P'(z)| \geq \frac{n}{2} \max_{z\in T_1} |P(z)| \tag{1.2}$$

Inequality (1.2) is sharp and equality holds for $P(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.
There exists several extensions and generalizations of (1.2). However Rather et

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* Corresponding author.
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al. [13] proved the following generalization by taking the coefficients of a polynomial into consideration.

Theorem 1.3. *If all the zeros of a polynomial $P(z)$ lie in $T_k \cup D_k^-, k \leq 1$, , then for $z \in T_1$,*

$$|P'(z)| \geq \left\{ \frac{n}{1+k} + \frac{k(k^n|c_n| - |c_0|)}{(1+k)(k^n|c_n| + |c_0|)} \right\} |P(z)|. \tag{1.3}$$

Now let

$$\mathcal{R}_n = \mathcal{R}_n(\alpha_1, \alpha_2, ..., \alpha_n) := \left\{ \frac{P(z)}{w(z)} : P \in \mathcal{P}_n \right\},$$

where

$$w(z) = \prod_{j=1}^{n} (z - \alpha_j), \ \alpha_j \in \mathbb{C}, j = 1, 2, ..., n.$$

Thus $\mathcal{R}_n$ is the set of all rational functions with poles $\alpha_1, \alpha_2, ..., \alpha_n$ and with finite limit at ∞. Throughout this paper, we shall assume that the poles $\alpha_1, \alpha_2, ..., \alpha_n$ lie in D_1^+. We observe that the Blaschke product $B \in \mathcal{R}_n$, where

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{\alpha_j} z}{z - \alpha_j} \right) = \frac{w^*(z)}{w(z)},$$

with $w^*(z) = z^n \overline{w\left(\frac{1}{\bar{z}}\right)} = \prod_{j=1}^{n} (1 - \overline{\alpha_j} z)$. Also we note that $|B(z)| = 1$ for $z \in T_1$. Li, Mohapatra and Rodriguez [6] obtained Bernstein-type inequalities for rational functions $r \in \mathcal{R}_n$ with prescribed poles $\alpha_1, \alpha_2, ..., \alpha_n$ replacing z^n by $B(z)$. In fact, they proved following result.

Theorem 1.4. *If $r \in \mathcal{R}_n$ and all the zeros of r lie in $T_1 \cup D_1^-$, then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2} |B'(z)||r(z)|. \tag{1.4}$$

Inequality (1.4) is sharp and equality holds for the rational function $r(z) = aB(z) + b$, $|a| = |b| = 1$.
Aziz and Shah [3] generalized Theorem 1.4 by proving the following result.

Theorem 1.5. *If $r \in \mathcal{R}_n$ and all zeros of r lie in $T_k \cup D_k^-, k \leq 1$ then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2} \left\{ |B'(z) + \frac{n(1-k)}{1+k} \right\} |r(z)|. \tag{1.5}$$

Akther et al. [1] generalized Theorem 1.5 and proved the following result.

Theorem 1.6. *If $r \in \mathcal{R}_n$ has all zeros in $T_k \cup D_k^-, k \leq 1,$ with a t-fold zeros at the origin , then for every complex number $\beta, |\beta| \leq 1$ and for $z \in T_1$*

$$\left|zr'(z) + \frac{(m-t)\beta}{1+k}r(z)\right| \geq \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n) + 2m - n + 2(m-t)Re(\beta)}{1+k}\right\}|r(z)|. \tag{1.6}$$

By involving some coefficients of the polynomial $P(z)$, Mir et al. [8] generalized as well as improved inequality (1.4) by proving the following result.

Theorem 1.7. *If $r \in \mathcal{R}_n$ has all zeros in $T_k \cup D_k^-, k \leq 1,$ then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right\}|r(z)|. \tag{1.7}$$

These results were further refined and generalized by various authors (for references, see [12], [9], [10],[11]) .

2. Lemmas

For the proof of our results, we require the following lemmas.

Lemma 2.1. *If $z \in T_1$ then*

$$Re\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |B'(z)|}{2}.$$

The above lemma is due to Aziz and zargar [4].

Lemma 2.2. *If $(b_j)_{j=1}^m$ is a finite collection of real numbers such that $0 \leq b_j \leq 1$, $j = 1, 2, 3, ..., m,$ then*

$$\sum_{j=1}^{m} \frac{1 - b_j}{1 + b_j} \geq \frac{1 - \prod_{j=1}^m b_j}{1 + \prod_{j=1}^m b_j}.$$

This lemma is due to Rather et al. [13].

3. Main Results

As an improvement of Theorem 1.6, we first prove the following :

Theorem 3.1. *If $r \in \mathcal{R}_n$ be such that $r(z) = \dfrac{z^t h(z)}{w(z)}$ and all the zeros of $h(z) = \sum_{j=0}^{m-t} c_{t+j} z^j$ lie in $T_k \cup D_k^-, k \leq 1$, then for every complex number $\beta, |\beta| \leq 1$ and for $z \in T_1$*

$$\left| zr'(z) + \frac{(m-t)\beta}{1+k} r(z) \right| \geq$$
$$\frac{1}{2}\left\{ |B'(z)| + \frac{k(2t-n)+2m-n+2(m-t)Re(\beta)}{1+k} + \frac{2k(k^{m-t}|c_m|-|c_t|)}{(1+k)(k^{m-t}|c_m|+|c_t|)} \right\} |r(z)|.$$

Remark 3.2. Since all the zeros of $h(z)$ lie in $T_k \cup D_k^-, k \leq 1$, so $k^{m-t}|c_m| \geq |c_t|$. Therefore Theorem 3.1 is an improvement of Theorem 1.6.
For $\beta = 0, m = n,$,Theorem 3.1 gives the following refinement of Theorem 1.5.

Corollary 3.3. *If $r \in \mathcal{R}_n$, $r(z) = \dfrac{z^t h(z)}{w(z)}$ has all zeros in $T_k \cup D_k^-, k \leq 1$, with t-fold zeros at the origin, then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\left\{ |B'(z)| + \frac{2kt+n(1-k)}{1+k} + \frac{2k(k^{n-t}|c_n|-|c_t|)}{(1+k)(k^{n-t}|c_n|+|c_t|)} \right\} |r(z)|.$$

Taking $t = 0, m = n$, in Theorem 3.1, we get the following result.

Corollary 3.4. *If $r \in \mathcal{R}_n$, $r(z) = \dfrac{P(z)}{w(z)}$, where $P(z) = \sum_{j=0}^{n} a_j z^j$ has all zeros in $T_k \cup D_k^-, k \leq 1$, then for every complex number $\beta, |\beta| \leq 1$ and for $z \in T_1$*

$$\left| zr'(z) + \frac{n\beta}{1+k} r(z) \right| \geq$$
$$\frac{1}{2}\left\{ |B'(z)| + \frac{n(1-k+2Re(\beta))}{1+k} + \frac{2k(k^n|a_n|-|a_0|)}{(1+k)(k^n|a_n|+|a_0|)} \right\} |r(z)|.$$

Remark : If in Theorem 3.1 , we assume $r(z)$ has a pole of order n at $z = \alpha, |\alpha| > 1$, then

$$r(z) = \frac{P(z)}{(z-\alpha)^n}.$$

So that

$$r'(z) = \frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}},$$

where $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ denotes the polar derivative of $P(z)$ with respect to point α.
Also in this case

$$B(z) = \prod_{1}^{n} \left(\frac{1 - \overline{\alpha}z}{z - \alpha} \right) = \left(\frac{1 - \overline{\alpha}z}{z - \alpha} \right)^n.$$

This gives

$$B'(z) = \frac{n(1 - \overline{\alpha}z)^{n-1}(|\alpha|^2 - 1)}{(z - \alpha)^{n+1}}.$$

So that for $z \in T_1$,

$$|B'(z)| = \frac{n(|\alpha|^2 - 1)}{|z - \alpha|^2}.$$

Using these values in Theorem 3.1 , we get for $z \in T_1$

$$\left| \frac{-zD_\alpha P(z)}{(z - \alpha)^{n+1}} + \frac{(m - t)\beta P(z)}{(1 + k)(z - \alpha)^n} \right|$$
$$\geq \frac{1}{2} \left\{ \frac{n(|\alpha|^2 - 1)}{|z - \alpha|^2} + \frac{k(2t - n) + 2m - n + 2(m - t)Re(\beta)}{1 + k} + \frac{2k(k^{m-t}|c_m| - |c_t|)}{(1 + k)(k^{m-t}|c_m| + |c_t|)} \right\} \frac{|P(z)|}{|z - \alpha|^n}.$$

That is

$$\left| \frac{-zD_\alpha P(z)}{(z - \alpha)} + \frac{(m - t)\beta P(z)}{1 + k} \right|$$
$$\geq \frac{1}{2} \left\{ \frac{n(|\alpha|^2 - 1)}{|z - \alpha|^2} + \frac{k(2t - n) + 2m - n + 2(m - t)Re(\beta)}{1 + k} + \frac{2k(k^{m-t}|c_m| - |c_t|)}{(1 + k)(k^{m-t}|c_m| + |c_t|)} \right\} |P(z)|.$$

Thus we have for $z \in T_1$,

$$\left| \frac{zD_\alpha P(z)}{(\alpha - z)} + \frac{(m - t)\beta P(z)}{1 + k} \right|$$
$$\geq \frac{1}{2} \left\{ \frac{n(|\alpha|^2 - 1)}{|\alpha - z|^2} + \frac{k(2t - n) + 2m - n + 2(m - t)Re(\beta)}{1 + k} + \frac{2k(k^{m-t}|c_m| - |c_t|)}{(1 + k)(k^{m-t}|c_m| + |c_t|)} \right\} |P(z)|.$$

Using these facts in Theorem 3.1 , we have the following ;

Corollary 3.5. *If all the zeros of a polynomial $P(z)$ lie in $T_k \cup D_k^-, k \leq 1$, with t-fold zeros at the origin, then for every complex number $\beta, |\beta| \leq 1$ and for $z \in T_1$,*

$$\left|\frac{zD_\alpha P(z)}{(\alpha - z)} + \frac{(m-t)\beta P(z)}{1+k}\right|$$
$$\geq \frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{|\alpha - z|^2} + \frac{k(2t-n)+2m-n+2(m-t)Re(\beta)}{1+k} + \frac{2k(k^{m-t}|c_m| - |c_t|)}{(1+k)(k^{m-t}|c_m|+|c_t|)}\right\}|P(z)|.$$

Now if we let $|\alpha| \to \infty$, we get the following;

$$\left|zP'(z) + \frac{(m-t)\beta P(z)}{1+k}\right|$$
$$\geq \frac{1}{2}\left\{n + \frac{k(2t-n)+2m-n+2(m-t)Re(\beta)}{1+k} + \frac{2k(k^{m-t}|c_m| - |c_t|)}{(1+k)(k^{m-t}|c_m|+|c_t|)}\right\}|P(z)|.$$

Remark 3.6. By taking $t = 0, \beta = 0, m = n$, in Corollary 3.5, we get Theorem 1.3.

Next, we prove the following result.

Theorem 3.7. *If $r \in \mathcal{R}_n$, $r(z) = \dfrac{P(z)}{w(z)}$, where $P(z) = \sum_{j=0}^{n} a_j z^j$ has all zeros in $T_k \cup D_k^-, k \leq 1$ and $m^* = \inf_{z\in T_k} |r(z)|$, then for every complex number $\beta, |\beta| \leq 1$ and for $z \in T_1$*

$$\left|zr'(z) + \frac{n\beta}{1+k} r(z)\right| \geq$$
$$\frac{1}{2}\left\{|B'(z)| + \frac{n(1-k+2Re(\beta))}{1+k} + \frac{2k(k^n|a_n| - m^*(k^n + \prod_{j=1}^n |\alpha_j| - |a_0|))}{(1+k)(k^n|a_n| + m^*(k^n + \prod_{j=1}^n |\alpha_j|) + |a_0|)}\right\}|r(z)| +$$
$$\frac{1}{2}\left\{|B'(z)| - \frac{2n|\beta|}{1+k} + \frac{n(1-k+2Re(\beta))}{1+k} + \frac{2k(|k^n a_n| - m^*(1 + \prod_{j=1}^n |\alpha_j| - |a_0|))}{(1+k)(|k^n a_n| + m^*(1 + \prod_{j=1}^n |\alpha_j|) + |a_0|)}\right\}\inf_{z\in T_k} |r(z)|.$$

Remark : Again assuming $r(z)$ has a pole of order n at $z = \alpha, |\alpha| > 1$, then $r(z) = \dfrac{P(z)}{(z-\alpha)^n}$, so that

$$r'(z) = \frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}.$$

Various polynomial inequalities can be obtained by a uniform procedure and continuing as done in Corollary 3.5.

4. Proofs of the Theorems

Proof of Theorem 3.1. By hypothesis $r(z) \neq 0$ for $z \in T_1$, all the poles of r(z) lie in D_1^+ and r(z) has t-folded zero at the origin, therefore we have

$$r(z) = \frac{p(z)}{w(z)} = \frac{z^t h(z)}{w(z)}, \tag{4.1}$$

where

$$h(z) = \sum_{j=0}^{m-t} c_{t+j} z^j = c_m \prod_{j=1}^{m-t} (z - z_j). \tag{4.2}$$

Taking logarithm of (4.2) on both sides and differentiating with respect to z , we get

$$\frac{zr'(z)}{r(z)} = t + \frac{zh'(z)}{h(z)} - \frac{zw'(z)}{w(z)}. \tag{4.3}$$

Hence for every complex β with $|\beta| \leq 1$, we can write (4.3) as,

$$\frac{zr'(z)}{r(z)} + \frac{(m-t)\beta}{1+k} = t + \frac{zh'(z)}{h(z)} - \frac{zw'(z)}{w(z)} + \frac{(m-t)\beta}{1+k}.$$

This gives for all $z \in T_1$, with $r(z) \neq 0$,

$$Re\left(\frac{zr'(z)}{r(z)} + \frac{(m-t)\beta}{1+k}\right) = t + Re\left(\frac{zh'(z)}{h(z)}\right) - Re\left(\frac{zw'(z)}{w(z)}\right) + \frac{(m-t)Re(\beta)}{1+k}.$$

Let $z_1, z_2, .., z_{m-t}$ be the zeros of h(z) , therefore $|z_j| \leq k < 1$ and by using Lemma 2.1, this gives for $|\beta| \leq 1$ and for all $z \in T_1$, where $r(z) \neq 0$

$$Re\left(\frac{zr'(z)}{r(z)} + \frac{(m-t)\beta}{1+k}\right) = t + Re\sum_{j=1}^{m-t}\left(\frac{z}{z-z_j}\right) - \left(\frac{n-|B'(z)|}{2}\right) + \frac{(m-t)Re(\beta)}{1+k}.$$

Now for $z = e^{i\theta}, 0 \leq \theta < 2\pi$ and $z_j = r_j e^{i\theta_j}, r_j < 1$, we have

$$\begin{aligned} Re\left(\frac{z}{z-z_j}\right) &= Re\left(\frac{e^{i\theta}}{e^{i\theta} - r_j e^{i\theta_j}}\right) \\ &= \frac{1 - r_j cos(\theta_j - \theta)}{1 + r_j^2 - 2r_j cos(\theta_j - \theta)} \\ &\geq \frac{1}{1+r_j}, \end{aligned}$$

if and only if

$$1 - r_j cos(\theta_j - \theta) + r_j - r_j^2 cos(\theta_j - \theta) \geq 1 + r_j^2 - 2r_j cos(\theta_j - \theta).$$

That is if and only if

$$r_j(1-r_j) \geq r_j(r_j-1)cos(\theta_j-\theta).$$

Since $r_j \neq 0$ and $r_j < 1$, therefore the above inequality is true if and if

$$cos(\theta_j-\theta) \geq -1,$$

Which is true.
Therefore, we conclude for $z \in T_1$

$$Re\left(\frac{z}{z-z_j}\right) \geq \frac{1}{1+|z_j|}.$$

Thus we get for $|\beta| \leq 1$ and for all $z \in T_1$, with $r(z) \neq 0$

$$Re\left(\frac{zr'(z)}{r(z)}+\frac{(m-t)\beta}{1+k}\right) \geq \frac{|B'(z)|}{2}+\frac{(m-t)Re(\beta)}{1+k}+\sum_{j=1}^{m-t}\left(\frac{1}{1+|z_j|}-\frac{1}{1+k}\right)+\frac{(m-t)}{1+k}-\frac{n}{2}.$$

$$=\frac{1}{2}\left[|B'(z)|+2t+\frac{2(m-t)Re(\beta)}{1+k}\right]+\frac{k}{1+k}\sum_{j=1}^{m-t}\left(\frac{k-|z_j|}{k+k|z_j|}\right)+\frac{2(m-t)-n(1+k)}{2(1+k)}.$$

$$\geq \frac{1}{2}\left[|B'(z)|+\frac{2t(1+k)+2(m-t)Re(\beta)}{1+k}\right]+\frac{k}{1+k}\sum_{j=1}^{m-t}\left(\frac{k-|z_j|}{k+|z_j|}\right)+\frac{2m-2t-n-nk)}{2(1+k)}.$$

$$=\frac{1}{2}\left[|B'(z)|+\frac{k(2t-n)+2m-n+2(m-t)Re(\beta)}{1+k}\right]+\frac{k}{1+k}\sum_{j=1}^{m-t}\left(\frac{1-\frac{|z_j|}{k}}{1+\frac{|z_j|}{k}}\right).$$

Since $\frac{|z_j|}{k} \leq 1$, therefore by using Lemma 2.2, we get for $z \in T_1$,

$$Re\left(\frac{zr'(z)}{r(z)}+\frac{(m-t)\beta}{1+k}\right) \geq$$

$$\frac{1}{2}\left[|B'(z)|+\frac{k(2t-n)+2m-n+2(m-t)Re(\beta)}{1+k}\right]+\frac{k}{1+k}\left(\frac{1-\prod_{j=1}^{m-t}\frac{|z_j|}{k}}{1+\prod_{j=1}^{m-t}\frac{|z_j|}{k}}\right).$$

$$=\frac{1}{2}\left\{|B'(z)|+\frac{k(2t-n)+2m-n+2(m-t)Re(\beta)}{1+k}+\frac{2k(k^{m-t}|c_m|-|c_t|)}{(1+k)(k^{m-t}|c_m|+|c_t|)}\right\}|r(z)|.$$

Also

$$\left|\frac{zr'(z)}{r(z)}+\frac{(m-t)\beta}{1+k}\right| \geq Re\left(\frac{zr'(z)}{r(z)}+\frac{(m-t)\beta}{1+k}\right),$$

therefore, We conclude for $r(z) \neq 0$ and $z \in T_1$.

$$\left|zr'(z)+\frac{(m-t)\beta}{1+k}r(z)\right|$$
$$\geq \frac{1}{2}\left\{|B'(z)|+\frac{k(2t-n)+2m-n+2(m-t)Re(\beta)}{1+k}+\frac{2k(k^{m-t}|c_m|-|c_t|)}{(1+k)(k^{m-t}|c_m|+|c_t|)}\right\}|r(z)|.$$

In case $r(z)=0$ for $z \in T_1$, then this inequality is trivially true.
This completes proof of Theorem 3.1.

Proof of Theorem 3.7: We assume all the zeros of $r(z)$ lie in D_k^- , so that $m^* = \inf_{z\in T_k} |r(z)| > 0$. Hence for some $\lambda, |\lambda| < 1$ we have $|r(z)| > m^*|\lambda|$ for $z \in T_k$. By Rouches theorem the fuction $G(z) = r(z)+\lambda m^*$ has all its zeros in $D_k^-, k \leq 1$. Now

$$\begin{aligned} G(z) &= r(z)+\lambda m^* \\ &= \frac{P(z)}{w(z)}+\lambda m^* \\ &= \frac{P(z)+\lambda m^* w(z)}{w(z)} \\ &= \frac{\phi(z)}{w(z)}, \end{aligned}$$

where

$$\phi(z) = \sum_{j=1}^{n} a_j z^j + \lambda m^* \prod_{j=1}^{n}(z-a_j)$$
$$= (a_n+\lambda m^*)z^n + ... + (a_0 + \prod_{j=1}^{n}\lambda m^* \prod_{j=1}^{n}(-1)^n\alpha_j).$$

Applying Corollary 3.4 to the function $G(z)$, we have for all $z \in T_1$

$$\left|zG'(z)+\frac{n\beta}{1+k}G(z)\right| \geq$$
$$\frac{1}{2}\left\{|B'(z)|+\frac{n(1-k+2Re(\beta))}{1+k}+\right.$$
$$\left.\frac{2k(k^n|a_n+\lambda m^*|-|\lambda m^*\prod_{j=1}^{n}(-1)^n\alpha_j+a_0|)}{(1+k)(k^n|a_n+\lambda m^*|+|\lambda m^*\prod_{j=1}^{n}(-1)^n\alpha_j+a_0|)}\right\}|G(z)|.$$

This implies for $z \in T_1$

$$\left|zr'(z) + \frac{n\beta}{1+k}(r(z) + \lambda m^*)\right| \geq$$
$$\frac{1}{2}\Bigg\{|B'(z)| + \frac{n(1-k+2Re(\beta))}{1+k} +$$
$$\frac{2k(k^n|a_n + \lambda m^*| - |\lambda m^* \prod_{j=1}^{n}(-1)^n\alpha_j + a_0|)}{(1+k)(k^n|a_n + \lambda m^*| + |\lambda m^* \prod_{j=1}^{n}(-1)^n\alpha_j + a_0|)}\Bigg\}|r(z) + \lambda m^*|.$$

Using the fact that the function $g(y) = \dfrac{y - |b|}{y + |b|}$, where $y = k^n|a_n + \lambda m^*|$ and $b = \lambda m^* \prod_{j=1}^{n}(-1)^n\alpha_j + a_0$, is a non-decreasing function of y and the triangle inequality on left hand side of the above inequality, we get for $|\lambda| < 1$ and for $z \in T_1$,

$$\left|zr'(z) + \frac{n\beta}{1+k}r(z)\right| + \frac{n|\lambda||\beta|m^*}{1+k} \geq$$
$$\frac{1}{2}\Bigg\{|B'(z)| + \frac{n(1-k+2Re(\beta))}{1+k} +$$
$$\frac{2k(k^n|a_n| - k^n|\lambda m^*| - |\lambda m^* \prod_{j=1}^{n}\alpha_j| - |a_0|)}{(1+k)(k^n|a_n| + k^n|\lambda|m^*| + |\lambda m^* \prod_{j=1}^{n}\alpha_j| + |a_0|)}\Bigg\}(|r(z) + \lambda m^*|).$$

This gives for $z \in T_1$

$$\left|zr'(z) + \frac{n\beta}{1+k}r(z)\right| + \frac{n|\lambda||\beta|m^*}{1+k} \geq$$
$$\frac{1}{2}\Bigg\{|B'(z)| + \frac{n(1-k+2Re(\beta))}{1+k} +$$
$$\frac{2k(k^n|a_n| - k^n|\lambda|m^* - |\lambda|m^* \prod_{j=1}^{n}|\alpha_j| - |a_0|)}{(1+k)(k^n|a_n| + k^n|\lambda|m^* + |\lambda|m^* \prod_{j=1}^{n}|\alpha_j| + |a_0|)}\Bigg\}(|r(z) + \lambda m^*|.$$

Choosing the argument of λ on the right of the above inequality suitably, we get for $z \in T_1$

$$\left|zr'(z)+\frac{n\beta}{1+k}r(z)\right|+\frac{n|\lambda||\beta|m^*}{1+k}\geq$$
$$\frac{1}{2}\Bigg\{|B'(z)|+\frac{n(1-k+2Re(\beta))}{1+k}+$$
$$\frac{2k(k^n|a_n|-k^n|\lambda|m^*-|\lambda|m^*\prod_{j=1}^{n}|\alpha_j|-|a_0|)}{(1+k)(k^n|a_n|+k^n|\lambda|m^*+|\lambda|m^*\prod_{j=1}^{n}|\alpha_j|+|a_0|)}\Bigg\}(|r(z)|+|\lambda|m^*).$$

Letting $|\lambda|\to 1$, we get for $z\in T_1$,

$$\left|zr'(z)+\frac{n\beta}{1+k}r(z)\right|\geq$$
$$\frac{1}{2}\Bigg\{|B'(z)|+\frac{n(1-k+2Re(\beta))}{1+k}+\frac{2k(k^n|a_n|-m^*(k^n+\prod_{j=1}^{n}|\alpha_j|-|a_0|)}{(1+k)(k^n|a_n|+m^*(k^n+\prod_{j=1}^{n}|\alpha_j|)+|a_0|)}\Bigg\}|r(z)|+$$
$$\frac{1}{2}\Bigg\{|B'(z)|-\frac{2n|\beta|}{1+k}+\frac{n(1-k+2Re(\beta))}{1+k}+\frac{2k(k^n|a_n|-m^*(k^n+\prod_{j=1}^{n}|\alpha_j|-|a_0|))}{(1+k)(k^n|a_n|+m^*(k^n+\prod_{j=1}^{n}|\alpha_j|)+|a_0|)}\Bigg\}\inf_{z\in T_k}|r(z)|.$$

This completes the proof of Theorem 3.7.

Remark : Since the function $g(y)=\dfrac{y-|b|}{y+|b|}$ is a non-decreasing function of y, hence Theorem 3.7 is an improvement of the result due to Milovanovic and Mir [7].

□

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[1] Department of Mathematics, Sharda University, Greater Noida UP, India
Email address: ganiebashir@gmail.com

[2] Department of Mathematics, Sharda University, Greater Noida UP, India
Email address: varun0503ind39@gmail.com

[3] Dartment of Mathematics, Central University of Kashmir, Ganderbal, India
Email address: shah1w@yahoo.co.in