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THE CONSTRUCTION OF A CLASS OF BIVARIATE COPULAS USING THE FINITE ELEMENT METHOD

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ABSTRACT. The definition of a copula function and the study of its properties are both not so easy tasks, as there is no general method to construct them. In this paper, we present a method that allows us to obtain a class of copulas as a solution to a boundary value problem. For this, we will use the finite element method. This method allows to set up, with the help of the principles inherited from the variational formulation or weak formulation, a discrete mathematical algorithm allowing to search for an approximate solution of a partial differential equation (or PDE) on a compact domain with conditions at the edges and/or in the interior of the compact.

1. Introduction and preliminaries

Copulas thus appear to be a natural tool for constructing multivariate distributions via Sklar's theorem when the marginal distributions are sufficiently regular. Sklar's theorem thus provides a canonical representation of a multivariate distribution, via the data of marginal distributions and dependency structure. The definition and properties of copulas are illustrated in [4]. In this paper, we will limit ourselves to two variables (dimension 2) for the sake of clarity and conciseness. In [5], we had already constructed a class of copula using the finite difference method, but in this paper we construct another copula class using the finite element method.

The aim of this paper is to construct a family of copulas C solution of Poisson's equation based on given finite element. Thus, this new class of copula solution of the boundary value problem is given by the following problem :

$$(\mathcal{P}) : \begin{cases} -\Delta C(u,v) = f(u,v) & f \in L^2([0,1]^2) \text{ and } (u,v) \in [0,1]^2 \\ C(u,0) = 0 = C(0,v) & (u,v) \in \Gamma \\ C(u,1) = u \text{ and } C(1,v) = v & (u,v) \in \Gamma \end{cases} \tag{1.1}$$

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where C is a copula and $f : [0,1]^2 \longrightarrow [0,1]$ a function which is not a copula. $\Delta(.) = \sum_{i=1}^{n} \frac{\partial^2(.)}{\partial u_i^2}$. $\Gamma = \Gamma([0,1]^2)$ is the boundary of the domain.

The problem $(\mathcal{P})$ can be written as follows :

$$(\mathcal{P}) : \begin{cases} -\Delta C = f & \text{in } [0,1]^2 \\ & \\ C|_{\Gamma} = g & \text{on } \Gamma \end{cases} \tag{1.2}$$

where

$$g = \begin{cases} u & \text{if } v = 1 \\ & \\ v & \text{if } u = 1 \\ & \\ 0 & \text{else} \end{cases} \tag{1.3}$$

with $f \in L^2([0,1]^2)$ and $g|_{\Gamma} \in L^2(\Gamma)$.

In [4], if $f \equiv 0$ (Laplace's equation) then C is called harmonic copulas and in this case C is the product copula (or independant copula).

The boundary value problem described above can be thought of as a non-homogenous Dirichlet problem. The solution to this problem is not analytically known in general. An approximation is then made to reduce the problem to a finite number of unknowns (discretization process).

In this paper, we use the finite element method to approximate the solution of the problem $(\mathcal{P})$. It is a method that allows to determine an approximate solution on a spatial domain. The method consists of dividing the spatial domain into small elements, also called meshes, and searching for a simplified formulation of the problem on each element, i.e. transforming the system of equations into a system of linear equations. Each system of linear equations can be represented by a matrix. The systems of equations for all the elements are then put together, forming a large matrix; solving this global system gives the approximate solution to the problem. As with many other numerical methods, in addition to the solution algorithm itself, there are issues of discretization quality, namely : the existence of solutions, uniqueness of the solution, stability, convergence and of course the error measure between a discrete solution and a unique solution of the initial problem.

The solution of the problem (1.2) consists in putting the problem in variational form, i.e. defining a Hilbert space V, the bilinear form $a(u,v)$ and the linear form L such that for a copula $C \in H^1([0,1]^2)$, (1.2) is equivalent to

$$a(u,v) = L(v), \quad v \in V \tag{1.4}$$

In this paper we try to find an approximation C_h of C in a subspace V_h of V. For this purpose, we consider the following subdivisions in the directions of u and v :

$$u_0 = 0 < u_1 = h < \ldots < u_i = ih < u_n = 1$$

and

$$v_0 = 0 < v_1 = h < \ldots < v_i = ih < v_n = 1$$

where $h = 1/n,\ n \in \mathbb{N}^*$. The aim is then to characterise the approximation space $V_h \subset V$, to determine a basis of V_h and to derive the stiffness matrix of the approximated problem. We use the Sobolev spaces which are best suited to characterise the space V_h.

In the rest of the document, we will note $I = [0,1]$ and $I^2 = I \times I = [0,1] \times [0,1]$.

Definition 1.1 (see [4]).
A copulas is a function $C : I^2 \longrightarrow I$ with the following properties :

(1) For every $u, v \in I$,
 : (1) $C(u,0) = 0 = C(0,v)$
 : (2) $C(u,1) = u$ and $C(1,v) = v$

(2) C is 2-increasing, i.e for every u_1, u_2, v_1, v_2 in I such that $u_1 \leq u_2$ and $v_1 \leq v_2$, we have :
 : (3) $V_H(B) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$
 where B is the rectangle $[u_1,u_2] \times [v_1,v_2]$ and the expression (3) defines the C-volume of B.

According to sklar theorem (see [4], Theorem 2.3.3) with given F, G and H defined as above, there exists a copula C such that for all $x, y \in \overline{\mathbb{R}}$,

$$H(x,y) = C(F(x), G(y)) \tag{1.5}$$

and conversely, for any copula C, the function H defined with (1.5) is a joint distribution function with margins $F(x)$ and $G(y)$.The aim of this paper is to construct a family of copulas C using the concept of finite element method.

As announced in our introduction, we use the sobolev spaces. Sobolev spaces are functional spaces modelled for the most part on Lebesgue spaces and provided with norms combining that of the function in question and that of the weak derivatives up to a certain order. Intuitively, a Sobolev space is a Banach space or a Hilbert space of functions that can be derived sufficiently many times (e.g. a partial differential equation) and provided with a norm that measures both the size and the regularity of the function, which makes them very important and very suitable tools for the study of partial differential equations. Indeed, the solutions of partial differential equations belong more naturally to a Sobolev space than to a space of differentiable functions whose derivatives are understood in a classical sense.

Definition 1.2. Let Ω be an open set of $\mathbb{R}^N$. We call the Sobolev space of order m and note it $W^{m,p}(\Omega)$, the space :

$$W^{m,p}(\Omega) = \{h \in L^p(\Omega); D^\alpha h \in L^p,\ \forall \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N,\ 0 \leq |\alpha| \leq m\} \tag{1.6}$$

where $D^\alpha = \dfrac{\partial^{|\alpha|} h}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ is the partial derivate of h with order α in the distribution sense and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N$.

In the case where $p = 2$, we note $H^m(\Omega)$ the space $W^{m,2}(\Omega)$. The space $H_0^m(\Omega)$ is defined as the adhesion in $H^m(\Omega)$.

The space $W^{m,p}$ is provided with the following norm :

$$\begin{cases} ||h||_{W^{m,p}(\Omega)} = \left(\displaystyle\sum_{0 \leq |\alpha| \leq m} ||D^\alpha h||^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ ||h||_{W^{m,p}(\Omega)} = \displaystyle\max_{0 \leq |\alpha| \leq \infty} ||D^\alpha h||_{L^\infty(\Omega)} & \text{if } p = \infty \end{cases} \tag{1.7}$$

Proposition 1.3. *The space $\left(W^{m,p}(\Omega), ||.||_{W^{m,p}(\Omega)}\right)$ is a Banach space for $1 \leq p \leq +\infty$.*

In this paper, we introduce a new structure for copulas using finite element method, based on the fact that the particular Sobolev space $W^{1,2}(I^2)$. is a Hilbert space. We show that :

$$< f, g >= \int_{I^2} \nabla f . \nabla g \; d\lambda \tag{1.8}$$

defines a scalar product for copulas with corresponding norm :

$$||C|| = \left(\int_{I^2} |\nabla C|^2 \; d\lambda \right)^{1/2} \tag{1.9}$$

where λ denotes the Lebesgue measure.

Proposition 1.4 (See [1] and [3])**.** *Let $g \in L^2(\Gamma)$, $\exists\ h \in H^2(I^2)$ such that :*

$$h = g; \quad a.e \ on \ \Gamma \tag{1.10}$$

and

$$\exists\ M > 0; \ ||h||_{H^1(I^2)} \leq M ||g||_{L^2(\Gamma)} \tag{1.11}$$

2. Main results

In this section, we will show that the problem (1.2) admit al least one solution and we will show the uniqueness of the solution. As announced in our introduction, we must put the problem (1.2) in variational form by defining a Hilbert space V. The space V will be posed as follows :

$$V = \left\{ C^* \in H^1(I^2); \ C^* : I^2 \to I; \ C^*|_\Gamma = g \right\}$$

Theorem 2.1. *Let $C \in H^1(I^2)$ a copula, be a solution of the following problem*

$$- \Delta C(u, v) = f(u, v), \quad (u, v) \in I^2, \quad (\textit{In the weak sense}). \tag{2.1}$$

for $f \in L^2(I^2)$ with boundary conditions :

$$C(u,0) = 0 = C(0,v)$$
$$C(u,1) = u \text{ and } C(1,v) = v \tag{2.2}$$

where $u, v \in I$. Assume (2.1) − (2.2) hold, then there exists at least one solution of problem $(\mathcal{P})$.

Proof. The idea is obviously to return the problem (1.2) to the case of a homogeneous Dirichlet condition by carrying out a space bearing (see proposition 1.4). Then, we pose : $\varphi = C - h,\ h \in H^2(I^2)$ and $\varphi : I^2 \to I$ a function. we have : $\varphi|_\Gamma = C|_\Gamma - h|_\Gamma = g - g = 0$. we get :

$$-\Delta\varphi = -\Delta C + \Delta h = f + \Delta h = F \tag{2.3}$$

The problem (1.2) can be write as follow :

$$(\mathcal{P}') : \begin{cases} -\Delta\varphi = F & \text{in } I^2 \\ \\ \varphi|_\Gamma = 0 & \text{on } \Gamma \end{cases} \tag{2.4}$$

It is a Poisson problem with homogeneous boundary conditions. Then, $\exists\ !\ \varphi \in H_0^1(I^2)$ solution of :

$$\int_{I^2} \nabla\varphi.\nabla\psi \ d\lambda = \int_{I^2} F\psi \ d\lambda, \quad \psi \in H_0^1(I^2). \tag{2.5}$$

φ depends on h and more $\varphi = C - h \Rightarrow C = \varphi + h$. Then C exists as long as φ and h exist. With this we have the existence of $C \in H^1(I^2)$ solution of the variational problem [(2.4)] associated to the initial problem $(\mathcal{P})$ □

Theorem 2.2. *Assume (2.1) − (2.2) hold. Then the solution of problem $(\mathcal{P})$ is unique.*

Proof. Let's consider

$$V = \left\{C^* \in H^1(I^2);\ C^* : I^2 \to I;\ C^*|_\Gamma = g\right\}$$

If $C \in V$ and $C^* \in V \Rightarrow (C^* - C)|_\Gamma = 0$, then we can write :

$$-\Delta C = f \Rightarrow -\int_{I^2} \Delta C(C^* - C)\ d\lambda = \int_{I^2} f(C^* - C)\ d\lambda \tag{2.6}$$

(2.6), using Green formula, is equivalent to :

$$\int_{I^2} \nabla C.\nabla(C^* - C)\ d\lambda - \int_\Gamma \frac{\partial C}{\partial n}(C^* - C)\ d\lambda = \int_{I^2} f(C^* - C)\ d\lambda \tag{2.7}$$

we obtain :

$$\int_{I^2} \nabla C.\nabla(C^* - C)\ d\lambda = \int_{I^2} f(C^* - C)\ d\lambda \tag{2.8}$$

because $(C^* - C)|_\Gamma = 0$. we deduce that C is solution of the following variational problem :

$$(\mathcal{VP}) : \begin{cases} a(C, C^* - C) = L(C^* - C) & \forall\ C^* \in I^2 \\ \\ C \in V \end{cases} \tag{2.9}$$

where $a(.,.)$ is a bilinear form and $L(.)$ is a linear function.

Let's consider C_1 and C_2 two copulas solutions of variational problem (2.9), so we can write :

$$(\mathcal{R}) : \begin{cases} a(C_1, C^* - C_1) = L(C^* - C_1) & \forall\ C^* \in I^2 \\ \\ a(C_2, C^* - C_2) = L(C^* - C_2) & \forall\ C^* \in I^2 \end{cases} \tag{2.10}$$

we have :

$$(\mathcal{R}') : \begin{cases} \text{if } C^* = C_2 \Rightarrow a(C_1, C_2 - C_1) = L(C_2 - C_1) \\ \\ \text{if } C^* = C_1 \Rightarrow a(C_2, C_1 - C_2) = L(C_1 - C_2) \end{cases} \tag{2.11}$$

By summing (2.11), we get

$$a(C_1, C_2 - C_1) + a(C_2, C_1 - C_2) = 0 \tag{2.12}$$

then

$$a(C_1 - C_2, C_2 - C_1) = 0. \tag{2.13}$$

But $C_1 - C_2 \in H_0^1(I^2)$ and a is coercive on $H_0^1(I^2)$, so we can write :

$$\exists\ \alpha > 0;\ ||C_1 - C_2||^2_{H^1(I^2)} \leq a(C_1 - C_2, C_2 - C_1) = 0 \tag{2.14}$$

finally

$$C_1 = C_2 \quad a.e \quad \text{in } I^2 \tag{2.15}$$

So, C is unique □

Theorem 2.3 (A priori estimate.)**.** *Assume* (2.1)−(2.2) *hold, then* $\exists\,!$ *copula* $C \in H^1(I^2)$ *such that the weak solution associate to the problem* (1.2) *verifies the following relation :*

$$\exists\ \alpha > 0,\ ||C||_{H^1(I^2)} \leq \alpha \left[||f||_{L^2(I^2)} + ||g||_{L^2(\Gamma)}\right] \tag{2.16}$$

Proof. Let's pose $\varphi = C - h \Rightarrow \Delta\varphi = \Delta C - \Delta h \Rightarrow -\Delta C = -\Delta\varphi - \Delta h = f$. So, we can write :

$$-\int_{I^2} \Delta\varphi.\psi\ d\lambda = \int_{I^2} f\psi\ d\lambda + \int_{I^2} \Delta h.\psi\ d\lambda \tag{2.17}$$

By using Green theorem, we have :

$$\int_{I^2} \nabla\varphi.\nabla\psi \; d\lambda - \int_{\Gamma} \frac{\partial \varphi}{\partial n}\psi \; d\lambda = \int_{I^2} f\psi \; d\lambda - \int_{I^2} \nabla h.\nabla\psi \; d\lambda + \int_{I^2} \frac{\partial h}{\partial n}\psi \; d\lambda \quad (2.18)$$

but $\varphi|_{\Gamma} = 0 \Rightarrow \varphi \in H_0^1(I^2)$. We obtain the following variational problem :

$$\begin{cases} \text{Find } \varphi \in H_0^1(I^2) \\ \\ \displaystyle\int_{I^2} \nabla\varphi.\nabla\psi \; d\lambda = \int_{I^2} f\psi \; d\lambda - \int_{I^2} \nabla h.\nabla\psi \; d\lambda \end{cases} \quad (2.19)$$

If $\psi = \varphi$, we have :

$$\int_{I^2} |\nabla\varphi|^2 \; d\lambda = \int_{I^2} f\varphi \; d\lambda - \int_{I^2} \nabla h.\nabla\varphi \; d\lambda \; ; \quad \varphi \in H_0^1(I^2) \quad (2.20)$$

if $\varphi \in H_0^1(I^2)$, we can use the Poincaré inégality given by :

$$\exists \; C_p > 0; \; ||\varphi||^2_{L^2(I^2)} \leq C_p^2 ||\nabla\varphi||^2_{L^2(I^2)} \Rightarrow ||\varphi||^2_{H^1(I^2)} \leq (1 + C_p^2)||\nabla\varphi||^2_{L^2(I^2)}$$

Then, $\exists \; \beta > 0$, with $\left(\beta = \dfrac{1}{1 + C_p^2}\right)$, such that :

$$\begin{aligned} \beta||\varphi||^2_{H^1(I^2)} &\leq ||\nabla\varphi||^2_{L^2(I^2)} = \int_{I^2} f\varphi \; d\lambda - \int_{I^2} \nabla h.\nabla\varphi \; d\lambda \\ &\leq \left| \int_{I^2} f\varphi \; d\lambda - \int_{I^2} \nabla h.\nabla\varphi \; d\lambda \right| \\ &\leq \int_{I^2} |f\varphi| \; d\lambda + \int_{I^2} |\nabla h.\nabla\varphi| \; d\lambda \\ &\leq ||f||_{L^2(I^2)}.||\varphi||_{L^2(I^2)} + ||\nabla h||_{L^2(I^2)}.||\nabla\varphi||_{L^2(I^2)} \\ &\leq \left[||f||_{L^2(I^2)} + ||\nabla h||_{L^2(I^2)}\right] \times ||\varphi||_{H^1(I^2)} \end{aligned} \quad (2.21)$$

then , (2.21) give us :

$$||\varphi||_{H^1(I^2)} \leq \frac{1}{\beta}\left[||f||_{L^2(I^2)} + ||\nabla h||_{L^2(I^2)}\right] \quad (2.22)$$

But, $C = \varphi + h \Rightarrow ||C||_{H^1(I^2)} = ||\varphi + h||_{H^1(I^2)} \leq ||\varphi||_{H^1(I^2)} + ||h||_{H^1(I^2)}$
we obtain :

$$||C||_{H^1(I^2)} \leq \frac{1}{\beta}[||f||_{L^2(I^2)} + ||\nabla h||_{L^2(I^2)}] + ||h||_{H^1(I^2)} \quad (2.23)$$

Hence;

$$\begin{aligned} ||C||_{H^1(I^2)} &\leq \frac{1}{\beta}[|f||_{L^2(I^2)} + ||h||_{H^1(I^2)}] + [|f||_{L^2(I^2)} + ||h||_{H^1(I^2)}] \\ &= \left(1+\frac{1}{\beta}\right)\left[|f||_{L^2(I^2)} + ||h||_{H^1(I^2)}\right] \end{aligned} \tag{2.24}$$

the relation (2.24) can be write :

$$\exists\ M>0;\ ||h||_{H^1(I^2)} \leq M||g||_\Gamma \Longrightarrow ||C||_{H^1(I^2)} \leq (1+\frac{1}{\beta})[||f||_{L^2(I^2)} + M||g||_\Gamma]$$

if we take $\alpha = \max\left[\left(1+\frac{1}{\beta}\right); M\left(1+\frac{1}{\beta}\right)\right]$ we obtain the result □

The theorem below gives us a characterisation of the solution of problem $(\mathcal{P})$ by the finite element method. This method, which is used in this article to construct the copulas, proposes to set up, on the basis of weak formulations, a discretisation algorithm allowing to search for an approximate solution of a partial derivative problem. The reliability of a result depends on the reliability of the whole chain of approximations performed since the modelling of the problem. The following theorem follows:

Theorem 2.4 (Approximation of Copula via finite element)**.** *Let C, a copula, be the exact solution of the problem (1.2). Let be $n \in \mathbb{N}$, we pose $h = \frac{1}{n}$ and C_h is the desired approximate copulas of C using triangular finite element. Assume (1.10) and (2.8) hold then C_h verify the following linear system equation :*

$$(VP_h) : \begin{cases} \textit{Find} \quad {}^t\mathcal{C}_h = [C_h^1, C_h^2, \dots, C_h^{K_h}] \\ \\ \mathbb{A}.\mathcal{C}_h = b \end{cases} \tag{2.25}$$

where $\mathbb{A}$ is a invertible matrix and $\dim V_h = K_h, \quad \forall\ C_h \in V_h \subset V$.

Proof. Let's introduce a space $V_h \subset V$ such that $\dim V_h = K_h < +\infty$ then,

$\Longrightarrow \exists\ (\xi_i)_{1,\dots,Kh},$ base of the space V_h. By using the variational problem (2.8) we can write the following approximate variational problem :

$$(VP_h) : \begin{cases} \text{Find} \quad C_h \in V_h \\ \\ \displaystyle\int_{I^2} \nabla C_h.\nabla(C_h^* - C_h)\ d\lambda = \int_{I^2} f(C_h^* - C_h)\ d\lambda,\ \forall\ v_h \in V_h \end{cases} \tag{2.26}$$

we obtain :

$$(VP_h) : \begin{cases} C_h^* - C_h = \xi_i, \ i = 1, \dots, K_h \\ \\ C_h(u,v) = \displaystyle\sum_{j=1}^{K_h} C_h^j.\xi_j(u,v), \qquad u, v \in V_h \subset V \\ \\ \displaystyle\int_{I^2} \nabla \left(\sum_{j=1}^{K_h} C_h^j \xi_j \right) .\nabla \xi_i \ d\lambda = \int_{I^2} f \xi_i \ d\lambda, \ \forall \ i = 1, \dots, K_h \end{cases} \tag{2.27}$$

the system of K_h - equations can be written as follow :

$$\sum_{j=1}^{K_h} \left(\int_{I^2} \nabla \xi_i . \nabla \xi_j \ d\lambda \right) C_h^j = \int_{I^2} f \xi_i \ d\lambda \tag{2.28}$$

we pose :

$$\mathbb{A}_{ij} = \int_{I^2} \nabla \xi_i . \nabla \xi_j \ d\lambda \tag{2.29}$$

and

$$b_i = \int_{I^2} f \xi_i \ d\lambda \tag{2.30}$$

then

$$\sum_{j=1}^{K_h} \mathbb{A}_{ij} C_h^j = b_i \tag{2.31}$$

if we introduce ${}^t\mathcal{C}_h = [C_h^1, C_h^2, \dots, C_h^{K_h}]$, we get $\mathbb{A}.\mathcal{C}_h = b$. Finally we obtain the approximate variational problem $\square$

3. Conclusion

In this paper, we constructed a new class of copula by the finite element method. This paper has allowed us to highlight the different stages of the resolution with the finite element method. But this method of solving does not guarantee an exact solution, nevertheless it offers a way to get closer to it.

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