

## LIOUVILLE-GREEN FORMULAS FOR THE FOURTH ORDER MATRIX DIFFERENTIAL OPERATORS

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**ABSTRACT.** In this paper, Liouville-Green analytical approximations for the solutions of the fourth order matrix differential equation are constructed. We have also determined the error bounds for the Liouville-Green approximations using the Gronwall's inequality.

### 1. INTRODUCTION

The study of matrix differential operators is a fast developing part of mathematics due to its various applications in mathematical physics in different fields like plasma physics, hydrodynamics, astrophysics, control theory, mechanics, quantum mechanics and mathematical physics among others [1, 3]. Most advances in matrix differential operators have led to new insights into the behavior of complex systems and the development of efficient algorithms for solving them. Various methods have been applied to solve matrix differential and difference operators. The commonly used methods include; Matrix exponentiation, Variation of parameters, Transform methods, Finite difference, Matrix pencil theory among others. Luis Verde-Star in [11] used operator identities in order to solve linear homogeneous matrix difference and differential equations and he obtained several explicit formulas for the exponential and for the powers of a matrix as an example of the methods. He used divided differences to find solutions of some initial value problems and he showed how the solution of matrix equations is related to polynomial interpolation. A related study was done in [12] using the matrix pencil theory on higher order linear matrix differential equations of regular case whose coefficients are square constant matrices under consistent and non-consistent initial conditions. Formulas for the solutions were obtained and it was proved that the solution is unique for consistent initial conditions and infinite for nonconsistent initial conditions. Most of the above methods are applicable to the systems that can be easily solved analytically or numerically and not appropriate for a large class of linear matrix differential and difference equations especially for the approximation of highly oscillatory solutions. Highly oscillatory solutions are difficult to compute accurately which has given rise to advanced techniques such as

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the Liouville-Green method that can be employed to approximate the solutions. Different studies have been done for second order like in [4, 5, 6, 8] among others. In [5], Cepale and Spigler developed a Liouville-Green (or WKB) asymptotic approximation theory for a class of almost-diagonal or asymptotically diagonal linear second-order matrix difference equations, they also obtained rigorous and explicitly computable bounds for the error terms where the asymptotics were made with respect to both the index and some parameter affecting the equation. Kovac and Klika in [4] attempted a generalization of the scalar Liouville-Green approximation to multicomponent systems. They gave general approximation theorems for systems of ordinary differential equations without turning points. They also discussed the cases of exponential and oscillatory behaviour first before treating the general case. In this paper, the fourth order matrix differential equations with variable matrix coefficients are studied. We have constructed the Liouville-Green approximations for the solutions of these operators with variable matrix coefficients and also derived, by use of the Gronwall's inequalities, the error bounds for these analytical approximations.

The paper has been organized as follows. Section 1 is the Introduction, which gives the mathematical background to the study in this paper and what the study was intended to achieve. Section 2 has Preliminaries, which give an overview of the concepts used in this study. Section 3 includes the main results for this study.

## 2. PRELIMINARIES

**Definition 2.1.** [9, Definition 2.1] A linear differential system

$$X' = A(t)X, \quad t \geq t_0 \quad (2.1)$$

is said to have an ordinary dichotomy if there exist a fundamental solution matrix  $X(t)$ , a projection  $P$ , and a constant  $C > 0$  such that

$$|X(t)PX^{-1}(s)| \leq C \quad \forall t_0 \leq s \leq t \text{ and } |X(t)[I - P]X^{-1}(s)| \leq C \quad \forall t_0 \leq t \leq s$$

It follows that the above system has an ordinary dichotomy with  $P = I$  if and only if it is uniformly stable

**Theorem 2.2.** [9, Theorem 2.2] *For continuous  $d \times d$  matrices  $\Lambda(t)$  and  $R(t)$ , consider the unperturbed system (2.1) and the perturbed system*

$$Y' = [\Lambda(t) + R(t)]Y, \quad t \geq t_0, \text{ where } R \in L^1[t_0, \infty] \quad (2.2)$$

*Assume that (2.1) satisfies an ordinary dichotomy conditions with projection matrix  $P$ . Then there exists a one-to-one and bicontinuous correspondence between the bounded solutions of (2.1) and (2.2) on  $[t_0, \infty]$ . Moreover, the difference between corresponding solutions of (2.1) and (2.2) tends to zero as  $t \rightarrow \infty$  if  $X(t)P \rightarrow 0$  as  $t \rightarrow \infty$ .*

The above implies that each solution of the perturbed system uniquely corresponds to a solution of the unperturbed system, and also that the correspondence is continuous. Moreover, as time  $t \rightarrow \infty$ , the perturbed system converges to the unperturbed system provided certain conditions on the matrix  $R(t)$  are satisfied. Also with the correspondence continuous, small perturbations in the initial

conditions or the system matrices lead to small differences in the solutions over time.

**Definition 2.3.** Let  $\mathcal{B}$  be a Banach space and  $T : \mathcal{B} \rightarrow \mathcal{B}$ .  $T$  is a contraction mapping if for any  $f, g \in \mathcal{B}$ , there exists a  $k$  such that  $\|Tf - Tg\| \leq k\|f - g\|$  with  $0 \leq k < 1$  and  $k$  is called the contraction factor.

The following theorem is the generalization of Banach contraction principle.

**Theorem 2.4.** [2, Theorem 4] *Let  $(\mathcal{X}, \rho)$  be a complete metric space and  $T : \mathcal{X} \rightarrow \mathcal{X}$  be a continuous self-map. If there exists a function  $\phi : [0, \infty) \rightarrow [0, \infty)$  such that  $\lim_{t \rightarrow 0^+} \phi(t) = 0$ ,  $\phi(0) = 0$  and  $\rho(Tx, Ty) \leq \phi(\rho(x, y)) - \phi(\rho(Tx, Ty))$  for all  $x, y \in X$ , then  $T$  has a unique fixed point.*

The proof can be directly got from [2].

*Remark 2.5.* Note that Theorem 2.4 is a generalization of the Banach contraction principle. If  $T : \mathcal{X} \rightarrow \mathcal{X}$  is a Banach contraction, there exists  $k \in [0, 1)$  such that  $\rho(Tx, Ty) \leq k\rho(x, y)$  for all  $x, y \in \mathcal{X}$ .

The theorem is particularly powerful in the context of proving the existence and uniqueness of solutions to certain types of equations and is applied in establishing the existence of the integral solutions for the perturbed systems in the subsequent sections.

**Corollary 2.6.** *Let  $(\mathcal{X}, \rho)$  be a complete metric space and  $T : \mathcal{X} \rightarrow \mathcal{X}$  be a continuous self-map. Suppose that*

$$\frac{\rho(Tx, Ty) (1 + e^{\rho(Tx, Ty)})}{\rho(x, y)e^{\rho(x, y)}} \leq 1$$

for all  $x, y \in \mathcal{X}$  with  $x \neq y$ . Then  $T$  has a unique fixed point.

**Lemma 2.7.** [7, Lemma 1] *Let  $F(t)$  and  $G(t)$  be real valued, non-negative continuous  $d \times d$  matrices on the real interval  $[a, b]$ . If for all  $t \in [a, b]$  we have  $F(t) \leq F(t_0) + \int_a^t G(s)F(s)ds$ . Then*

$$F(t) \leq F(t_0) \exp \left( \int_a^t G(s)ds \right) ds$$

The above lemma is the Gronwall Lemma. It was first proved by Gronwall in [10] and later, the other variants by different authors.

### 3. MAIN RESULTS

In this section, we present the Liouville-green results for the fourth order matrix differential equation with  $d \times d$  coefficient matrices. To get the solutions, the equation is first converted into first order  $2d \times 2d$  system. Their aim is to transform it into the form on which the given theorems can apply. This involves different transformations and diagonalizations. For this case, we do a shearing transformation, then diagonalize the leading matrix of the resulting system. The resulting system is transformed using  $I + Q$  conditioning transformation to get

the L-diagonal form. A system normalization is made on which asymptotic factorization and the Banach fixed point are applied to obtain the solutions. Finally, Theorem 3.3 gives the error bounds and corollary 3.4 presents the sharper estimates for the bounds. The following theorem gives the main result.

**Theorem 3.1.** *Consider a fourth order matrix differential equation given by*

$$Y^{(iv)} = [F(t) + G(t)] Y$$

on  $[a, +\infty]$ , where  $F(t)$  is a diagonal  $d \times d$  continuously differentiable matrix given by  $F(t) = \text{diag} \{\mu_1(t), \dots, \mu_d(t)\}$  and  $G(t)$  is a continuously differentiable  $d \times d$  matrix where  $G(t) \ll F(t)$ . Then there exists, for  $t \geq t_0$  solutions to equation of the form

$$Y_j(t) = -2 [I \pm E_j(t)] F^{\frac{3}{8}}(t) e^{\int_a^t F^{\frac{1}{4}}(\tau) d\tau}, \quad j = 1, 3$$

and

$$Y_l(t) = 2 [I \pm E_l(t)] F^{\frac{5}{8}}(t) e^{-\int_a^t F^{\frac{1}{4}}(\tau) d\tau}, \quad l = 2, 4$$

where

$$E_j(t) = \int_t^\infty \left( I - e^{\int_t^s 2F^{\frac{1}{4}}} \right) R_l(s) \sum_{i=1}^4 u_{ji}(s) ds,$$

$$E_l(t) = \int_t^\infty \left( I - e^{-\int_t^s 2F^{\frac{1}{4}}} \right) R_j(s) \sum_{i=1}^4 u_{li}(s) ds,$$

and  $u = [u_1, u_2, u_3, u_4]^T$ ,

$$R_j(t) = \frac{1}{4} \left\{ F^{-\frac{3}{8}} e^{\int^t F^{\frac{1}{4}}} G F^{-\frac{3}{8}} e^{-\int^t F^{\frac{1}{4}}} \right\} + \frac{1}{8} F^{-1} F' \pm \frac{5}{64} F^{-\frac{9}{4}} F' \mp \frac{1}{16} F^{-\frac{5}{4}} F''$$

or

$$R_j(t) = \frac{1}{4} \left\{ F^{-\frac{3}{8}} e^{\int^t F^{\frac{1}{4}}} G F^{-\frac{3}{8}} e^{-\int^t F^{\frac{1}{4}}} \right\} + \frac{1}{8} F^{-1} F' \pm \frac{5}{32} F^{-\frac{9}{4}} F' \mp \frac{1}{8} F^{-\frac{5}{4}} F'',$$

$$R_l(t) = \frac{1}{4} \left\{ F^{-\frac{3}{8}} e^{-\int^t F^{\frac{1}{4}}} G F^{-\frac{3}{8}} e^{\int^t F^{\frac{1}{4}}} \right\} + \frac{1}{8} F^{-1} F' \pm \frac{5}{64} F^{-\frac{9}{4}} F' \mp \frac{1}{16} F^{-\frac{5}{4}} F''$$

or

$$R_l(t) = \frac{1}{4} \left\{ F^{-\frac{3}{8}} e^{-\int^t F^{\frac{1}{4}}} G F^{-\frac{3}{8}} e^{\int^t F^{\frac{1}{4}}} \right\} + \frac{1}{8} F^{-1} F' \pm \frac{5}{32} F^{-\frac{9}{4}} F' \mp \frac{1}{8} F^{-\frac{5}{4}} F''$$

and  $R_i \in L^1[0, \infty)$ ,  $i = 1, 2, 3, 4$

*Proof.* We begin by re-writing

$$Y^{(iv)} = [F(t) + G(t)] Y \tag{3.1}$$

as a  $4d \times 4d$  system.

To that effect, one lets  $X = [Y, Y', Y'', Y''']^T$  so that  $X' = [Y', Y'', Y''', Y^{iv}]^T$  leading to

$$X' = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ F(t) + G(t) & 0 & 0 & 0 \end{pmatrix} X, \quad \text{where } X = \begin{pmatrix} Y \\ Y' \\ Y'' \\ Y''' \end{pmatrix} \tag{3.2}$$

We then make a shear transformation on the system (3.2) so that the terms on either sides of the diagonal of the leading matrix have the same magnitude. Therefore, we make the substitution  $X = \text{diag} \left\{ I, F^{\frac{1}{4}}, F^{\frac{2}{4}}, F^{\frac{3}{4}} \right\} \hat{X}$  to have;

$$\hat{X} = \text{diag} \left\{ I, F^{\frac{1}{4}}, F^{\frac{2}{4}}, F^{\frac{3}{4}} \right\}^{-1} X \quad (3.3)$$

so that

$$\hat{X}' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{4} F^{\frac{-5}{4}} F' & 0 & 0 \\ 0 & 0 & \frac{-1}{2} F^{\frac{-3}{2}} F' & 0 \\ 0 & 0 & 0 & \frac{-3}{4} F^{\frac{-7}{4}} F' \end{pmatrix} X + \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & F^{\frac{-1}{4}} & 0 & 0 \\ 0 & 0 & F^{\frac{-1}{2}} & 0 \\ 0 & 0 & 0 & F^{\frac{-3}{4}} \end{pmatrix} X' \quad (3.4)$$

Hence,

$$\hat{X}' = \left\{ \begin{pmatrix} 0 & F^{\frac{1}{4}} & 0 & 0 \\ 0 & 0 & F^{\frac{1}{4}} & 0 \\ 0 & 0 & 0 & F^{\frac{1}{4}} \\ F^{\frac{1}{4}} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{4} F^{-1} F' & 0 & 0 \\ 0 & 0 & \frac{-1}{2} F^{-1} F' & 0 \\ F^{\frac{-3}{4}} G & 0 & 0 & \frac{-3}{4} F^{-1} F' \end{pmatrix} \right\} \hat{X} \quad (3.5)$$

To diagonalize the leading matrix in (3.5), we let

$$\hat{X} = \begin{bmatrix} -I & I & -I & I \\ I & I & -I & -I \\ -I & I & I & -I \\ I & I & I & I \end{bmatrix} \tilde{X} = S \tilde{X} \quad (3.6)$$

and on further simplifications, this gives

$$\tilde{X}' = [\Lambda(t) + V_1(t) + V_2(t)] \tilde{X} \quad (3.7)$$

$$\text{where } \Lambda(t) = \begin{bmatrix} F^{\frac{1}{4}} & 0 & 0 & 0 \\ 0 & -F^{\frac{1}{4}} & 0 & 0 \\ 0 & 0 & F^{\frac{1}{4}} & 0 \\ 0 & 0 & 0 & -F^{\frac{1}{4}} \end{bmatrix}, \quad V_1 = \frac{F^{-1} F'}{8} \begin{bmatrix} 3I & I & 0 & 2I \\ I & 3I & 2I & 0 \\ 2I & 0 & I & 3I \\ 0 & 2I & 3I & I \end{bmatrix} \text{ and}$$

$$V_2(t) = \frac{F^{\frac{-3}{4}} G}{4} \begin{bmatrix} I & -I & I & -I \\ I & -I & I & -I \\ I & -I & I & -I \\ -I & I & -I & I \end{bmatrix}$$

To obtain the required L-diagonal form, one needs to diagonalize  $\Lambda(t) + V_1(t)$ . Thus, do an elementary row operation on  $V_1(t)$  to make it symmetric and let  $\tilde{X} = [I + \bar{Q}(t)]Z$  where  $\bar{Q}(t)$  is a  $4d \times 4d$  such that  $Z = [I + \bar{Q}(t)]^{-1} \tilde{X}$  and

$\|Q(t)\| = o\|I\| = o(1)$ . We choose  $\bar{Q}$  such that

$$[I + \bar{Q}] = \begin{pmatrix} I + Q & -Q & Q & -Q \\ Q & I - Q & Q & -Q \\ -Q & Q & I - Q & Q \\ -Q & Q & -Q & I + Q \end{pmatrix}$$

to have  $[I + \bar{Q}]^{-1} = [I - \bar{Q}]$ . So

$$\begin{aligned} Z &= [I + \bar{Q}]^{-1} \tilde{X} = [I - \bar{Q}] \tilde{X} \quad \text{and} \\ Z' &= -\bar{Q}' \tilde{X} + [I - \bar{Q}] \tilde{X}' \end{aligned}$$

With

$$\bar{Q}'\bar{Q} = \bar{Q}^2 = \bar{Q}V_1 = V_2\bar{Q} = \bar{Q}V_2 = \bar{Q}\Lambda\bar{Q} = \bar{Q}V_1\bar{Q} = \bar{Q}V_2\bar{Q} = \bar{0},$$

$$Z' = [-\bar{Q}' + \Lambda + V_1 + V_2 - \bar{Q}\Lambda + \Lambda\bar{Q} + V_1\bar{Q}] Z$$

in which the value of  $Q$  is determined by setting  $\Lambda\bar{Q} - \bar{Q}\Lambda + V_1 - \text{diag}\{V_1\} = 0$ . Hence, we can re-write the above as

$$Z' = [(\Lambda + \text{diag}\{V_1\}) + (\Lambda\bar{Q} - \bar{Q}\Lambda + V_1 - \text{diag}\{V_1\}) + V_2 + V_1\bar{Q} - \bar{Q}'] Z$$

and we therefore get

$$Z' = [\text{Diag}\{\Lambda_k, k = 1, 2, 3, 4\} + (\Lambda\bar{Q} - \bar{Q}\Lambda + V_1 - \text{diag}\{V_1\}) + V_2 + (V_1\bar{Q} - \bar{Q}')] Z$$

where

$$\Lambda_1 = \Lambda_3 = F^{\frac{1}{4}} + \frac{3}{8}F^{-1}F' \quad \text{and} \quad \Lambda_2 = \Lambda_4 = -F^{\frac{1}{4}} + \frac{3}{8}F^{-1}F',$$

$$\Lambda\bar{Q} - \bar{Q}\Lambda + V_1 - \text{diag}\{V_1\}$$

$$= \begin{pmatrix} F^{\frac{1}{4}}Q - QF^{\frac{1}{4}} & -F^{\frac{1}{4}}Q - QF^{\frac{1}{4}} + \frac{F^{-1}F'}{8} & -F^{\frac{1}{4}}Q - QF^{\frac{1}{4}} & -F^{\frac{1}{4}}Q - QF^{\frac{1}{4}} + \frac{F^{-1}F'}{4} \\ -F^{\frac{1}{4}}Q - QF^{\frac{1}{4}} + \frac{F^{-1}F'}{8} & F^{\frac{1}{4}}Q - QF^{\frac{1}{4}} & -F^{\frac{1}{4}}Q - QF^{\frac{1}{4}} + \frac{F^{-1}F'}{4} & F^{\frac{1}{4}}Q - QF^{\frac{1}{4}} \\ -F^{\frac{1}{4}}Q + QF^{\frac{1}{4}} & F^{\frac{1}{4}}Q + QF^{\frac{1}{4}} + \frac{F^{-1}F'}{4} & -F^{\frac{1}{4}}Q + QF^{\frac{1}{4}} & F^{\frac{1}{4}}Q + QF^{\frac{1}{4}} + \frac{F^{-1}F'}{8} \\ F^{\frac{1}{4}}Q + QF^{\frac{1}{4}} + \frac{F^{-1}F'}{4} & -F^{\frac{1}{4}}Q + QF^{\frac{1}{4}} & F^{\frac{1}{4}}Q + QF^{\frac{1}{4}} + \frac{F^{-1}F'}{8} & -F^{\frac{1}{4}}Q + QF^{\frac{1}{4}} \end{pmatrix}$$

and

$$V_2 + V_1\bar{Q} - \bar{Q}' = \frac{1}{4}F^{\frac{-3}{4}}G \begin{pmatrix} I & -I & I & -I \\ I & -I & I & -I \\ I & -I & I & -I \\ I & -I & I & -I \end{pmatrix} + \left\{ \frac{F^{-1}F'}{8} - Q' \right\} \begin{pmatrix} I & -I & I & -I \\ I & -I & I & -I \\ -I & I & -I & I \\ -I & I & -I & I \end{pmatrix}$$

which with some elementary row operations on  $V_2$  becomes

$$V_2 + V_1\bar{Q} - \bar{Q}' = \left\{ \frac{1}{4}F^{\frac{-3}{4}}G + \frac{1}{8}F^{-1}F' - Q' \right\} \begin{pmatrix} I & -I & I & -I \\ I & -I & I & -I \\ -I & I & -I & I \\ -I & I & -I & I \end{pmatrix}.$$

Setting  $\Lambda\bar{Q} - \bar{Q}\Lambda + V_1 - \text{diag}\{V_1\} = 0$ ,  $Q = \pm\frac{1}{16}F^{-\frac{5}{4}}F'$  or  $Q = \pm\frac{1}{8}F^{-\frac{5}{4}}F'$  leads to the L-diagonal form given by

$$Z' = \left[ \begin{pmatrix} \Lambda_1(t) & 0 & 0 & 0 \\ 0 & \Lambda_2(t) & 0 & 0 \\ 0 & 0 & \Lambda_3(t) & 0 \\ 0 & 0 & 0 & \Lambda_4(t) \end{pmatrix} + R(t) \begin{pmatrix} I & -I & I & -I \\ I & -I & I & -I \\ -I & I & -I & I \\ -I & I & -I & I \end{pmatrix} \right] Z$$

where  $R(t) = \frac{1}{4}F^{-\frac{3}{4}}G + \frac{1}{8}F^{-1}F' - Q'$ . For simplicity, one can therefore write

$$Z' = [\tilde{\Lambda}(t) + \tilde{R}(t)] Z \quad (3.8)$$

Also substituting  $Q$  in  $R(t)$ , we have

$$R_k(t) = \frac{1}{4}F^{-\frac{3}{4}}G + \frac{1}{8}F^{-1}F' \pm \frac{5}{64}F^{-\frac{9}{4}}F' \mp \frac{1}{16}F^{-\frac{5}{4}}F''$$

or

$$R_k(t) = \frac{1}{4}F^{-\frac{3}{4}}G + \frac{1}{8}F^{-1}F' \pm \frac{5}{32}F^{-\frac{9}{4}}F' \mp \frac{1}{8}F^{-\frac{5}{4}}F''$$

where  $k = 1, 3$  or  $k = 2, 4$  since  $\Lambda_1 = \Lambda_3$  and  $\Lambda_2 = \Lambda_4$

Consider the system (3.8) as a perturbation of the diagonal system  $Z' = \tilde{\Lambda}(t)Z$ . It is clear that the diagonal(unperturbed) system does not necessarily satisfy the Levinson's dichotomy conditions. Therefore, for one to determine the existence of solutions corresponding to  $\Lambda_k(t)$  one uses the fact that  $Z_k = e^{\int^t \Lambda_k(s) d(s)}$  to make in (3.8) a normalization of the form

$$Z(t) = \text{diag} \left\{ e^{\int^t \Lambda_k(s) d(s)}, e^{\int^t \Lambda_k(s) d(s)}, e^{\int^t \Lambda_k(s) d(s)}, e^{\int^t \Lambda_k(s) d(s)} \right\} U(t) = DU(t)$$

with  $k = 1, 2, 3, 4$ . This leads to

$$U_k = \begin{pmatrix} e^{-\int^t \Lambda_k(s) d(s)} & & & \\ & e^{-\int^t \Lambda_k(s) d(s)} & & \\ & & e^{-\int^t \Lambda_k(s) d(s)} & \\ & & & e^{-\int^t \Lambda_k(s) d(s)} \end{pmatrix} Z = AZ \quad (3.9)$$

and therefore we have

$$U'_k = A'Z + AZ' = [A'Z + A\tilde{\Lambda}D + A\tilde{R}D] U_k$$

which yields the following systems.

For  $j = 1, 3$  and  $l = 2, 4$ ,

$$U'_j = \left[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2F^{\frac{1}{4}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2F^{\frac{1}{4}} \end{pmatrix} + R_l(t) \begin{pmatrix} I & -I & I & -I \\ I & -I & I & -I \\ -I & I & -I & I \\ -I & I & -I & I \end{pmatrix} \right] U_j \quad (3.10)$$

where  $R_l(t) = e^{-\int^t \Lambda_j(s) d(s)} R(t) e^{\int^t \Lambda_j(s) d(s)}$  and

$$U'_i = \left[ \begin{pmatrix} 2F^{\frac{1}{4}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2F^{\frac{1}{4}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + R_j(t) \begin{pmatrix} I & -I & I & -I \\ I & -I & I & -I \\ -I & I & -I & I \\ -I & I & -I & I \end{pmatrix} \right] U_l \quad (3.11)$$

where  $R_j(t) = e^{-\int^t \Lambda_i(s) d(s)} R(t) e^{\int^t \Lambda_i(s) d(s)}$

With the above,  $R_k$  can be further simplified as

$$R_j(t) = \frac{1}{4} \left\{ F^{-\frac{3}{8}} e^{\int^t F^{\frac{1}{4}}} G F^{-\frac{3}{8}} e^{-\int^t F^{\frac{1}{4}}} \right\} + \frac{1}{8} F^{-1} F' \pm \frac{5}{64} F^{-\frac{9}{4}} F' \mp \frac{1}{16} F^{-\frac{5}{4}} F'' \quad (3.12)$$

or

$$R_j(t) = \frac{1}{4} \left\{ F^{-\frac{3}{8}} e^{\int^t F^{\frac{1}{4}}} G F^{-\frac{3}{8}} e^{-\int^t F^{\frac{1}{4}}} \right\} + \frac{1}{8} F^{-1} F' \pm \frac{5}{32} F^{-\frac{9}{4}} F' \mp \frac{1}{8} F^{-\frac{5}{4}} F'' \quad (3.13)$$

and also

$$R_l(t) = \frac{1}{4} \left\{ F^{-\frac{3}{8}} e^{-\int^t F^{\frac{1}{4}}} G F^{-\frac{3}{8}} e^{\int^t F^{\frac{1}{4}}} \right\} + \frac{1}{8} F^{-1} F' \pm \frac{5}{64} F^{-\frac{9}{4}} F' \mp \frac{1}{16} F^{-\frac{5}{4}} F'' \quad (3.14)$$

or

$$R_l(t) = \frac{1}{4} \left\{ F^{-\frac{3}{8}} e^{-\int^t F^{\frac{1}{4}}} G F^{-\frac{3}{8}} e^{\int^t F^{\frac{1}{4}}} \right\} + \frac{1}{8} F^{-1} F' \pm \frac{5}{32} F^{-\frac{9}{4}} F' \mp \frac{1}{8} F^{-\frac{5}{4}} F'' \quad (3.15)$$

since  $\Lambda_1 = \Lambda_3$  and  $\Lambda_2 = \Lambda_4$

Next is to prove the existence of solutions for the systems above. By Theorem 2.2, it follows that

$$U'_k = \left[ \tilde{\Lambda}_k(t) + \tilde{R}_k(t) \right] U_k \quad (3.16)$$

have for every  $t \in [t_0, \infty]$ ,  $4d \times d$  matrix solutions  $\tilde{u}(t)$  in the form of integral solutions.

For  $k = 1$ , the solution is given by

$$u_1(t) = \begin{pmatrix} I_d \\ 0 \\ I_d \\ 0 \end{pmatrix} + \int_{t_0}^t \begin{pmatrix} I_d & & & \\ & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} & & \\ & & I_d & \\ & & & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \end{pmatrix} \tilde{R}_l(s) u_1(s) ds \quad (3.17)$$

To prove that, we make a normalization of the form

$$w_1(t) = u_1(t) - \begin{pmatrix} I_d \\ 0 \\ I_d \\ 0 \end{pmatrix} \quad (3.18)$$

in (3.17) above, and by Theorem 2.2, showing that  $w_1(t) = (T w_1)(t)$  has a solution  $w(t)$  for which  $\lim_{n \rightarrow \infty} w(t) = 0_{4d \times d}$  is equivalent to proving the existence of a



solution of the integral form above for (3.17).

First, we show that  $w_1(t) = (Tw_1)(t)$ . That is,

$$\begin{aligned}
 w_1(t) &= \int_{t_0}^t \begin{pmatrix} I_d & & & \\ & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} & & \\ & & I_d & \\ & & & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \end{pmatrix} \tilde{R}_l(s) u_1(s) ds \\
 &= \int_{t_0}^t \begin{pmatrix} I_d & & & \\ & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} & & \\ & & I_d & \\ & & & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \end{pmatrix} \tilde{R}_l(s) \left[ w_1(s) + \begin{pmatrix} I_d \\ 0 \\ I_d \\ 0 \end{pmatrix} \right] ds \\
 &= \int_{t_0}^t \begin{pmatrix} I_d & & & \\ & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} & & \\ & & I_d & \\ & & & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \end{pmatrix} \tilde{R}_l(s) w_1(s) ds + \\
 &\quad \int_{t_0}^t \begin{pmatrix} I_d & & & \\ & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} & & \\ & & I_d & \\ & & & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \end{pmatrix} \begin{pmatrix} 2R_l(s) \\ 2R_l(s) \\ -2R_l(s) \\ -2R_l(s) \end{pmatrix} ds = (Tw_1)(t)
 \end{aligned}$$

where

$$\tilde{R}_l(s) = \begin{pmatrix} R_l(s) & -R_l(s) & R_l(s) & -R_l(s) \\ R_l(s) & -R_l(s) & R_l(s) & -R_l(s) \\ -R_l(s) & R_l(s) & -R_l(s) & R_l(s) \\ -R_l(s) & R_l(s) & -R_l(s) & R_l(s) \end{pmatrix}$$

Now to prove the existence of the solution to the integral equation (3.17) on  $[t_0, \infty)$ , one chooses for any  $m \times n$  matrix  $A$  a submultiplicative norm given by

$$|A| = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$$

which is the maximum of the absolute row sum.

Consider a function  $\phi_1 : [t_0, \infty) \rightarrow \mathbb{R}^+$  defined by

$$\phi_1(t) = \int_t^\infty |2R_l(s)| ds$$

and assume without loss of generality that  $\phi_1(t) > 0$  for all  $t \geq t_0$ , that is, assume that the perturbation term  $R_1(t)$  is non vanishing for all  $t \geq t_0$ .

Now define the set

$$\mathcal{B}_1 = \left\{ w(t) \in \mathbb{C}^{4d \times d} : \|w_1\|_1 = \sup_{t \geq t_0} \frac{|w_1(t)|}{e^{4\phi_1(t)} - 1} < \infty \right\},$$

then  $\mathcal{B}_1$  is a Banach space and implies  $w_1(t) \rightarrow 0$  as  $t \rightarrow \infty$ . One also defines

$$\mathcal{W}_1 = \{w_1(t) \in \mathcal{B}_1 : \|w_1\|_1 \leq 1\},$$

therefore  $\mathcal{W}_1$  is a closed subset of  $\mathcal{B}_1$ . With this, one now applies the Banach fixed point theorem principle in determining the existence of solutions (see section 2). To do that, begin with the following lemma.

**Lemma 3.2.** *The operator  $T$  for which  $w_1(t) = (Tw_1)(t)$  above maps  $\mathcal{W}_1$  into  $\mathcal{W}_1$  and it is a contraction.*

*Proof.* Given  $w_1(t) \in \mathcal{W}_1$ , it can be clearly seen that

$$(Tw_1)(t) \leq \int_{t_0}^t \left| \begin{pmatrix} I_d & & \\ & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} & \\ & & I_d \\ & & & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \end{pmatrix} \tilde{R}_l(s) w_1(s) \right| ds +$$

$$\int_{t_0}^t \left| \begin{pmatrix} I_d & & \\ & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} & \\ & & I_d \\ & & & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \end{pmatrix} \begin{pmatrix} 2R_l(s) \\ 2R_l(s) \\ -2R_{1l}(s) \\ -2R_l(s) \end{pmatrix} \right| ds$$

and

$$(Tw_1)(t) \leq 4 \int_{t_0}^t |R_l(s)| (e^{4\phi_1(s)} - 1) ds + 2 \int_{t_0}^t |R_l(s)| ds$$

$$= \phi_1(t) + 4 \int_{t_0}^t |R_l(s)| e^{4\phi_1(s)} ds - 4 \int_{t_0}^t |R_l(s)| ds$$

$$= e^{4\phi_1(t)} - 1 - \phi_1(t) < e^{4\phi_1(t)} - 1$$

and it follows that

$$\frac{(Tw_1)(t)}{e^{4\phi_1(t)} - 1} < 1.$$

Taking the supremum over all  $t \geq t_0$  establishes that  $\|Tw_1\| \leq 1$  and hence  $T$  maps  $\mathcal{W}_1$  into  $\mathcal{W}_1$ .

Now, to show that  $T$  is a contraction, let  $w_1$  and  $\tilde{w}_1$  be elements in  $\mathcal{W}_1$ . Then  $w_1 - \tilde{w}_1 \in \mathcal{W}_1$  and it follows that for  $t \geq t_0$ ,

$$|(Tw_1)(t) - (T\tilde{w}_1)(t)| = \int_{t_0}^t \left| \begin{pmatrix} I_d & & \\ & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} & \\ & & I_d \\ & & & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \end{pmatrix} \tilde{R}_l(s) \right| |w_1(s) - \tilde{w}_1(s)| ds$$

$$\leq \int_{t_0}^t (|4R_l(s)| [e^{4\phi_1(s)} - 1]) \|w_1 - \tilde{w}_1\| ds$$

$$= \|w_1 - \tilde{w}_1\| (e^{4\phi_1(t)} - 1 - 2\phi_1(t))$$

and so

$$\begin{aligned} \frac{|(Tw_1)(t) - (T\tilde{w}_1)(t)|}{e^{4\phi_1(t)} - 1} &\leq \|w_1 - \tilde{w}_1\| \left(1 - \frac{2\phi_1(t)}{e^{4\phi_1(t)} - 1}\right) \\ &\leq \|w_1 - \tilde{w}_1\| \left(1 - \frac{2\phi_1(t_0)}{e^{4\phi_1(t_0)} - 1}\right) \end{aligned}$$

since  $\phi_1(t)$  is a non-increasing function on  $[t_0, \infty)$  and also the fact that  $1 - \frac{2y}{e^{4y} - 1}$  is strictly increasing for positive values of  $y$ .

Now if one lets

$$k_1 = 1 - \frac{2\phi_1(t_0)}{e^{4\phi_1(t_0)} - 1},$$

then  $k_1 \in (0, 1)$  and taking the supremum over over all  $t \geq t_0$  yields

$$|T(w_1 - \tilde{w}_1)| \leq k_1 \|w_1 - \tilde{w}_1\|.$$

It follows from definition (2.3) that  $T$  is a contraction.  $\square$

Finally, since  $T$  is a contraction, the Banach fixed point principle now implies that there exists for  $t \geq t_0$  a unique function  $w_1(t) \in \mathcal{W}_1$  for which  $w_1(t) = (Tw_1)(t)$ . And by equation (3.18), for  $k = 1$ ,  $U'_1 = [\tilde{\Lambda}_1(t) + \tilde{R}_1(t)] U_1$  has a solution  $u_1(t)$  of the form (3.17) for all  $t \geq t_0$ . Following similar arguments with  $\Lambda_1 = \Lambda_3$ ,  $U'_3 = [\tilde{\Lambda}_3(t) + \tilde{R}_3(t)] U_3$  has for all  $t \geq t_0$  a solution of the form

$$u_3(t) = \begin{pmatrix} I_d \\ 0 \\ I_d \\ 0 \end{pmatrix} + \int_{t_0}^t \begin{pmatrix} I_d & & & \\ & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} & & \\ & & I_d & \\ & & & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \end{pmatrix} \tilde{R}_l(s) u_3(s) ds \quad (3.19)$$

Finally the solutions  $Y_j$  corresponding to  $j = 1, 3$  can be obtained through asymptotic factorization of the following form

$$Y_j(t) = [BS(I + Q)A] u_j(t), \quad j = 1, 3 \quad (3.20)$$

where  $A$  and  $S$  are defined as before and  $B = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & F^{\frac{1}{4}} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & F^{\frac{1}{4}} \end{pmatrix}$ .

On simplification, this becomes

$$Y_j(t) = \begin{pmatrix} -e^{\int^t \Lambda_j(s) d(s)} & e^{\int^t \Lambda_j(s) d(s)} & -e^{\int^t \Lambda_j(s) d(s)} & e^{\int^t \Lambda_j(s) d(s)} \\ M_1 e^{\int^t \Lambda_j(s) d(s)} & M_2 e^{\int^t \Lambda_j(s) d(s)} & M_3 e^{\int^t \Lambda_j(s) d(s)} & M_4 e^{\int^t \Lambda_j(s) d(s)} \\ -e^{\int^t \Lambda_j(s) d(s)} & e^{\int^t \Lambda_j(s) d(s)} & e^{\int^t \Lambda_j(s) d(s)} & -e^{\int^t \Lambda_j(s) d(s)} \\ F^{\frac{1}{4}} e^{\int^t \Lambda_j(s) d(s)} & F^{\frac{1}{4}} e^{\int^t \Lambda_j(s) d(s)} & F^{\frac{1}{4}} e^{\int^t \Lambda_j(s) d(s)} & F^{\frac{1}{4}} e^{\int^t \Lambda_j(s) d(s)} \end{pmatrix} u_j(t)$$

where  $u_j(t)$  is defined as in (3.17) and (3.19) for  $j = 1$  and  $j = 3$  respectively,  $M_1 = F^{\frac{1}{4}}(I + 4Q)$ ,  $M_2 = F^{\frac{1}{4}}(I - 4Q)$ ,  $M_3 = F^{\frac{1}{4}}(-I + 4Q)$ ,  $M_4 = F^{\frac{1}{4}}(-I - 4Q)$  and  $Y_j = [Y_j, Y'_j, Y''_j, Y'''_j]^T$  where  $T$  denotes the usual transpose. Further

simplification of the above and equating the corresponding components gives the solution as

$$Y_j(t) = -2 [I \pm E_j(t)] F^{\frac{3}{8}}(t) e^{\int_a^t F^{\frac{1}{4}}(\tau) d\tau}, \quad j = 1, 3. \tag{3.21}$$

where  $E_j(t) = \int_{t_0}^t \left( I - e^{\int_{t_0}^s 2F^{\frac{1}{4}}} \right) R_l(s) \sum_{i=1}^4 u_{ji}(s) ds$  given that  $u = [u_1, u_2, u_3, u_4]^T$

Doing a similar analysis, one can also prove that for  $k = 2$ ,  $U'_2 = [\tilde{\Lambda}_2(t) + \tilde{R}_j(t)] U_2$  has a solution  $u_2(t)$  of the integral form

$$u_2(t) = \begin{pmatrix} 0 \\ I_d \\ 0 \\ I_d \end{pmatrix} - \int_t^\infty \begin{pmatrix} e^{-\int_t^s 2F^{\frac{1}{4}} ds} & & & \\ & I_d & & \\ & & e^{-\int_t^s 2F^{\frac{1}{4}} ds} & \\ & & & I_d \end{pmatrix} \tilde{R}_j(s) u_2(s) ds \tag{3.22}$$

and a similar case with  $\Lambda_2 = \Lambda_4$  gives that  $U'_4 = [\tilde{\Lambda}_4(t) + \tilde{R}_4(t)] U_4$  has for all  $t \geq t_0$  an inetgral form solution given by

$$u_4(t) = \begin{pmatrix} 0 \\ I_d \\ 0 \\ I_d \end{pmatrix} - \int_t^\infty \begin{pmatrix} e^{-\int_t^s 2F^{\frac{1}{4}} ds} & & & \\ & I_d & & \\ & & e^{-\int_t^s 2F^{\frac{1}{4}} ds} & \\ & & & I_d \end{pmatrix} \tilde{R}_j(s) u_4(s) ds \tag{3.23}$$

Asymptotic factorization of the above also yields the solutions  $Y_l$ ,  $l = 2, 4$  corresponding  $\Lambda_l$ . Hence one has

$$Y_l(t) = [BS(I + Q)A] u_l(t), \quad l = 2, 4 \tag{3.24}$$

where  $A$  and  $S$  are defined as before,  $u_l(t)$  defined as in (3.22) and (3.23), and  $B = \text{diag} \left\{ F^{\frac{1}{4}}, I, F^{\frac{1}{4}}, I \right\}$ . This, on working it out and further simplification gives the solutions as

$$Y_l(t) = 2 [I \pm E_l(t)] F^{\frac{5}{8}}(t) e^{-\int_a^t F^{\frac{1}{4}}(\tau) d\tau}, \quad l = 2, 4. \tag{3.25}$$

with  $E_l(t) = \int_t^\infty \left( I - e^{-\int_t^s 2F^{\frac{1}{4}}} \right) R_j(s) \sum_{i=1}^4 u_{li}(s) ds$  given that  $u = [u_1, u_2, u_3, u_4]^T$  □

**Theorem 3.3.** *Let  $Y_j(t)$  and  $Y_l(t)$  be solutions to the fourth order differential equation*

$$Y^{(iv)} = [F(t) + G(t)] Y$$

on  $[a, +\infty]$ , where  $F(t)$  and  $G(t)$  are continuously differentiable  $d \times d$  matrices with  $F(t)$  diagonal and  $G(t)$  smaller than  $F(t)$ . Under given conditions in theorem (3.1), suppose  $E_j(t)$  and  $E_l(t)$  are the respective error terms to the solutions, then;

$$|E_j(t)|, \left| \frac{E'_j(t)}{F^{\frac{1}{4}}(t)} \right| \leq e^{2\phi_j(t)} - 1 \quad \text{and} \quad |E_l(t)|, \left| \frac{E'_l(t)}{F^{\frac{1}{4}}(t)} \right| \leq e^{2\phi_l(t)} - 1$$

**Corollary 3.4.** *Suppose that  $F(t) \geq 0$ ,  $\mu_i(t) \geq 0$  for all  $1 \leq i \leq d$ , then for all  $t \geq t_0$  the error estimates above become*

$$|E_j(t)| \leq \exp \left[ \int_{t_0}^t 2K_j(\tau) |R_l(\tau)| d\tau \right] ds - 1, \quad \left| \frac{E'_j(t)}{F^{\frac{1}{4}}(t)} \right| \leq 2 \left[ e^{\int_{t_0}^t 2|R_l(u)| du} - 1 \right]$$

and

$$|E_l(t)| \leq \exp \left[ \int_{t_0}^t 2K_l(\tau) |R_j(\tau)| d\tau \right] ds - 1, \quad \left| \frac{E'_l(t)}{F^{\frac{1}{4}}(t)} \right| \leq 2 \left[ e^{\int_{t_0}^t 2|R_j(u)| du} - 1 \right]$$

To determine the upper bounds for  $E_j(t)$ ,  $E'_j(t)$ ,  $E_l(t)$  and  $E'_l(t)$ , we shall use the Gronwalls inequality to determine the estimates for  $\sum_{i=1}^4 u_{ji}(s)$  and  $\sum_{i=1}^4 u_{li}(s)$ ,  $j = 1, 3$  and  $l = 2, 4$ . To that effect, we have the following;

*Proof.* To obtain bounds for  $E_j(t)$  and  $E'_j(t)$ , consider the solution  $u_j(t)$ ,  $j = 1, 3$  as defined in (3.17) and (3.19). One can write these as

$$u_j(t) = \begin{pmatrix} I_d \\ 0 \\ I_d \\ 0 \end{pmatrix} + \int_{t_0}^t \begin{pmatrix} I_d & & & \\ & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} & & \\ & & I_d & \\ & & & e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \end{pmatrix} \tilde{R}_l(s) u_j(s) ds \quad (3.26)$$

Expanding it and adding the components gives

$$\sum_{i=1}^4 u_{ji}(t) = 2I + \int_{t_0}^t \left( 2I + 2e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \right) R_l(s) \sum_{i=1}^4 u_{ji}(s) ds$$

and

$$\left| \sum_{i=1}^4 u_{ji}(t) \right| \leq 2 + \int_{t_0}^t \left| \left( 2I + 2e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \right) \right| |R_l(s)| \left| \sum_{i=1}^4 u_{ji}(s) \right| ds \quad (3.27)$$

Suppose that  $F(t) \neq 0$  and one chooses branches of the fourth root function  $F^{\frac{1}{4}}(t)$  for which  $Re \left( F^{\frac{1}{4}}(t) \right) = Re \left( \sqrt[4]{\mu_i(t)} \right) \geq 0$ . Then  $F^{\frac{1}{4}}(t)$  is well defined for real  $t$  and  $\int_t^s 2F^{\frac{1}{4}}(\tau) d\tau$  is real and so we have that for all  $s \geq t$ ,

$$\left| 2I + 2e^{\int_t^s 2F^{\frac{1}{4}} ds} \right| = \max_{1 \leq i \leq d} \left| 2I + 2e^{\int_t^s 2\sqrt[4]{\mu_i(\tau)} d\tau} \right| \leq 4.$$

Hence applying the usual Gronwall's inequality in (3.27) gives

$$\left| \sum_{i=1}^4 u_{ji}(t) \right| \leq 2 \exp \left[ \int_{t_0}^t 4 |R_l(s)| ds \right] \quad (3.28)$$

Now, from  $E_j(t) = \int_{t_0}^t \left( I - e^{\int_{t_0}^s 2F^{\frac{1}{4}} ds} \right) R_l(s) \sum_{i=1}^4 u_{ji}(s) ds$ , substituting (3.28) in  $E_j(t)$  yields

$$|E_j(t)| \leq \int_{t_0}^t 4 |R_l(s)| \exp \left[ \int_{s_0}^s 4 |R_l(\tau)| d\tau \right] ds.$$

Since  $|\sum_{i=1}^4 u_{ji}(t)| \leq 2e^{2\phi_j(t)}$ , one finally gets that

$$|E_j(t)| \leq e^{2\phi_j(t)} - 1 \quad (3.29)$$

Similarly, to determine the estimate for  $E'_j(t)$ , one uses the fact that

$$\frac{d}{dt} \left( e^{\int_t^s 2F^{\frac{1}{4}}(\tau) d\tau} \right) = 2F^{\frac{1}{4}}(t) e^{\int_t^s 2F^{\frac{1}{4}}(\tau) d\tau}$$

to get

$$E'_j(t) = 2F^{\frac{1}{4}}(t) \int_{t_0}^t \left( e^{\int_t^s 2F^{\frac{1}{4}}(\tau) d\tau} \right) R_l(s) \sum_{i=1}^4 u_{ji}(s) ds,$$

$$\left| F^{-\frac{1}{4}}(t) E'_j(t) \right| \leq 2 \int_{t_0}^t \left| \left( e^{\int_t^s 2F^{\frac{1}{4}}(\tau) d\tau} \right) \right| |R_l(s)| \left| \sum_{i=1}^4 u_{ji}(s) \right| ds.$$

Hence by (3.28), the above becomes

$$\left| F^{-\frac{1}{4}}(t) E'_j(t) \right| \leq 2 \int_{t_0}^t 2 |R_l(s)| e^{2\phi_j(s)} ds.$$

Therefore

$$\left| \frac{E'_j(t)}{F^{\frac{1}{4}}(t)} \right| \leq e^{2\phi_j(t)} - 1 \quad (3.30)$$

Now, suppose that  $\mu_i(t) \geq 0$  for all  $1 \leq i \leq d$ , then for all  $s \geq t$ ,

$$\left| 2I + 2e^{\int_{t_0}^s 2F^{\frac{1}{4}}(\tau) d\tau} \right| = \max_{1 \leq i \leq d} 2 \left\{ I + e^{\int_{t_0}^s 2\sqrt[4]{\mu_i}(\tau) d\tau} \right\} \leq \max_{1 \leq i \leq d} 2 \left\{ I + e^{\int_{t_0}^s 2\sqrt[4]{\mu_i(\tau)} d\tau} \right\} = 2K_j(s)$$

Therefore equation (3.27) becomes

$$\left| \sum_{i=1}^4 u_{ji}(t) \right| \leq 2 \exp \left[ \int_{t_0}^t 2K_j(s) |R_l(s)| ds \right] \quad (3.31)$$

and substituting (3.31) in  $E_j(t)$  further gives

$$|E_j(t)| \leq \int_{t_0}^t 2K_j(s) |R_l(s)| \exp \left[ \int_{t_0}^t 2K_j(\tau) |R_l(\tau)| d\tau \right] ds.$$

With  $\eta(t) = \int_{t_0}^t r(u) du$ , one has  $\int_{t_0}^t r(s) e^{\eta(s)} ds = e^{\eta(t)} - 1$  and this leads us to  $|E_j(t)| \leq \exp \left[ \int_{t_0}^t 2K_j(\tau) |R_l(\tau)| d\tau \right] ds - 1$ . Hence

$$|E_j(t)| \leq \exp \left[ \int_{t_0}^t 2K_j(\tau) |R_l(\tau)| d\tau \right] ds - 1 \quad (3.32)$$

Similarly from

$$E'_j(t) = 2F^{\frac{1}{4}}(t) \int_{t_0}^t \left( e^{\int_t^s 2F^{\frac{1}{4}}(\tau) d\tau} \right) R_l(s) \sum_{i=1}^4 u_{ji}(s) ds,$$

one can write

$$\left| F^{-\frac{1}{4}}(t) E'_j(t) \right| \leq 4 \int_{t_0}^t \left| \left( e^{\int_t^s 2F^{\frac{1}{4}}(\tau) d\tau} \right) \right| |R_l(s)| \exp \left[ \int_{t_0}^t 2K_j(u) |R_l(u)| du \right] ds$$

which on simplification reduces to

$$\left| \frac{E'_j(t)}{F^{\frac{1}{4}}(t)} \right| \leq 2 \left[ e^{\int_{t_0}^t 2|R_l(u)| du} - 1 \right] \tag{3.33}$$

Similarly to get bounds for  $E_l(t)$  and  $E'_l(t)$ , we consider the integral solution  $u_l(t), l = 2, 4$  as defined in (3.22) and (3.23). These can generally be written as

$$u_l(t) = \begin{pmatrix} 0 \\ I_d \\ 0 \\ I_d \end{pmatrix} - \int_t^\infty \begin{pmatrix} e^{-\int_t^s 2F^{\frac{1}{4}} ds} & & & \\ & I_d & & \\ & & e^{-\int_t^s 2F^{\frac{1}{4}} ds} & \\ & & & I_d \end{pmatrix} \tilde{R}_j(s) u_l(s) ds \tag{3.34}$$

Following a similar analysis like the one for 3.29, one gets

$$\sum_{i=1}^4 u_{li}(t) = 2I - \int_t^\infty \left( 2I - 2e^{-\int_t^s 2F^{\frac{1}{4}}} \right) R_j(s) \sum_{i=1}^4 u_{li}(s) ds,$$

$$\left| \sum_{i=1}^4 u_{li}(t) \right| \leq 2 + \int_t^\infty \left| \left( 2I + 2e^{-\int_t^s 2F^{\frac{1}{4}}} \right) \right| |R_j(s)| \left| \sum_{i=1}^4 u_{li}(s) \right| ds \tag{3.35}$$

Application of the usual Gronwall's inequality on (3.35) and further simplifying leads to

$$|E_l(t)| \leq e^{2\phi_l(t)} - 1 \tag{3.36}$$

Also from

$$E'_l(t) = -2F^{\frac{1}{4}}(t) \int_t^\infty \left( e^{-\int_t^s 2F^{\frac{1}{4}}(\tau) d\tau} \right) R_j(s) \sum_{i=1}^4 u_{li}(s) ds,$$

a similar analysis yields

$$\left| \frac{E'_l(t)}{F^{\frac{1}{4}}(t)} \right| \leq e^{2\phi_l(t)} - 1 \tag{3.37}$$

Suppose that  $\mu_i(t) \geq 0$  for all  $1 \leq i \leq d$ , then for all  $s \geq t$ , then

$$\left| 2I - 2e^{\int_t^s -2F^{\frac{1}{4}}} \right| = \max_{1 \leq i \leq d} \left\{ 2I - 2e^{\int_t^s -2\sqrt[4]{\mu_i}} \right\} \leq \max_{1 \leq i \leq d} \left\{ 2I - 2e^{\int_{t_0}^s -2\sqrt[4]{\mu_i(\tau)} d\tau} \right\} = K_l(s)$$

and following the same procedure leads to

$$|E_l(t)| \leq \exp \left[ \int_{t_0}^t 2K_l(\tau) |R_j(\tau)| d\tau \right] ds - 1 \tag{3.38}$$

and also

$$\left| \frac{E'_l(t)}{F^{\frac{1}{4}}(t)} \right| \leq 2 \left[ e^{\int_{t_0}^t 2|R_j(u)| du} - 1 \right] \tag{3.39}$$

□

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