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UNIQUENESS FOR ENTROPY SOLUTION TO A CLASS OF NONLINEAR DEGENERATE WEIGHTED ELLIPTIC P(.)-LAPLACIAN PROBLEM WITH L1-DATA

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Abstract. In the present paper, we prove the uniqueness of entropy solution to a class of nonlinear degenerate weighted elliptic p(·)-Laplacian problems. These particular problems are delineated by Dirichlet-type boundary conditions and L^1 data. We use a priori estimates in the framework of weighted variable exponent Sobolev spaces theory.

1. Introduction and preliminaries

Let $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ be a open bounded domain with a connected Lipschitz boundary $\partial\Omega$, and $p(x) \in (1, \infty)$ for all $x \in (0, \infty)$. In this paper, we will study the uniqueness of entropy solution for the nonlinear degenerate weighted elliptic p(.)-Laplacian problem with variable exponent. Moreover, the existence of entropy solution for our following problem (1.1) is also established in [1].

$$\begin{cases} -div(\omega|\nabla u - \theta(u)|^{p(u)-2})(\nabla u - \theta(u)) + \omega|u|^{p(u)-2}u = f & in \ \ \Omega, \\ u = 0 & on \ \ \partial\Omega. \end{cases} \tag{1.1}$$

Where the variable exponent $p : \overline{\Omega} \to (0, \infty)$ is a continuous function, ω is a weight function on Ω, (ω is a measurable, a.e. strictly positive function on Ω and satisfying some integrability conditions, see assumptions (H_1) and (H_2) below) and the right-hand side $f \in L^1(\Omega)$. The notion of entropy solutions has been proposed by Bénilan et al. in [7] for the nonlinear elliptic problems. Many of these models like problem (1.1) have already been analyzed for constant exponents of nonlinearity in particular cases. Especially, when $\omega \equiv 1$, the existence and uniqueness of entropy solution for the problem (1.1) are obtained in [5]. In the recent few years, the study of partial differential equations (i.e, PDEs) and variational problems generalised several types of problems coming from various areas of mathematical physics, such as electrorheological fluid dynamics, elastic mechanics, image processing, biophysics, plasma physics and chemical reactions,

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etc. Additionally, degenerate phenomena have surfaced in fields like oceanography, turbulent fluid flows, induction heating, and electrochemical studies, as documented in references [6], [9], and [12]. The problem (1.1) is modelling several natural phenomena, we present for example the following two models (see [12] and [14]) .

- **Model 1. Fluid flow through porous media.** In this particular model, we are guided by an equation denoted as

$$\frac{\partial \theta}{\partial t} - div(|\nabla \varphi(\theta) - K(\theta)e|^{p-2}(\nabla \varphi(\theta) - K(\theta)e)) = 0,$$

  which articulates the behavior of the system. This equation involves the temporal change of θ, representing the volumetric moisture content, and it's influenced by various factors.These factors include $K(\theta)$, which signifies the hydraulic conductivity $\varphi(\theta)$, which denotes the hydrostatic potential, and e a unit vector oriented in the vertical direction. Together, they interact within the equation to describe the intricate dynamics of the system.
- **Model 2. The thermal model.** Understanding the intricacies of induction heating processes involves a thorough examination of heat transfer mechanisms, primarily influenced by three crucial factors: the dissipation of heat due to eddy currents within the workpiece, the diffusion of heat within the material, and the exchange of heat at the interface between the workpiece and the surrounding air.

  The evolution of temperature within the workpiece is described by the classical heat transfer equation:

$$\rho c \frac{\partial T}{\partial t} - div(k \nabla T) = Q_{em},$$

  here, ρ represents the density of the workpiece, c signifies the specific heat, and k denotes the thermal conductivity, all of which are temperature-dependent parameters in this specific context. Moreover, a heat source term, denoted as Q_{em} is introduced due to the presence of eddy currents. It's important to emphasize that the heat capacity of the workpiece can experience significant fluctuations in value as temperature varies, especially during metallurgical phase transitions and in close proximity to the Curie temperature. Conversely, the thermal conductivity exhibits a gradual decrease in value as temperature rises.

The plan of our paper is organized as follows. In the first section, we recall the definitions of weighted Lebesgue spaces with variable exponent (i.e $L^{p(.)}(\Omega, \omega)$) and weighted generalized Lebesgue-Sobolev spaces ($W^{1,p(.)}(\Omega, \omega)$) have been introduced in [4] and some of their basic properties. In this section, we provide definitions, notations, and results that will be using throughout the course of this study. In the second section, we prove the uniqueness of an entropy solution of our problem (1.1) such that their existence is also obtained in [1] and to achieve this goal we will use a priori estimates established by Boccardo and Gallouet in

[2] and [3].
Let $\overline{\Omega}$ be a bounded open domain in $\mathbb{R}^N$, we consider the following set

$$C^+(\overline{\Omega}) = \{p : \overline{\Omega} \to \mathbb{R}^+ \text{ continuous such that } 1 < p^- < p^+ < \infty\},$$

where $p^- = \min_{x\in\overline{\Omega}}(p(x))$ and $p^+ = \max_{x\in\overline{\Omega}}(p(x))$.
Let ω be a measurable positive, a.e finite function defined in $\mathbb{R}^N$ and satisfied the following integrability conditions.

(H_1) $\omega \in L^1_{Loc}(\Omega)$ and $\omega^{\frac{-1}{p(x)-1}} \in L^1_{Loc}(\Omega)$.
(H_2) $\omega^{-s(x)} \in L^1_{Loc}(\Omega)$, where $s(x) \in (\frac{N}{p(x)}, \infty) \cap (\frac{1}{p(x)-1}, \infty]$.

Moreover we add the following hypotheses for the function θ and the regularity of f:

(H_3) θ is a continuous function defined from $\mathbb{R}$ to $\mathbb{R}^N$ such that $\theta(0) = 0$ and for all real numbers x, y we have $|\theta(x) - \theta(y)| < \lambda_0|x - y|$ where λ_0 is a positive constant real number such that

$$0 < \lambda_0 < \min\left\{\left(p^-/2\right)^{1/p^-} ; \left(p^-/2\right)^{1/p^+} ; \left(1/2^{p^+-2}C_0\right)^{1/p^{**}}\right\},$$

where $\lambda_0^{p^{**}} = \min\left(\lambda_0^{p^-} ; \lambda_0^{p^+}\right)$ such that C_0 is a positive constant defined in theorem (1.7) .
(H_4) $f \in L^1(\Omega)$.

For $p(.) \in C^+(\overline{\Omega})$, we define the weighted Lebesgue space with variable exponent $L^{p(.)}(\Omega, \omega)$ by

$$L^{p(.)}(\Omega, \omega) = \left\{u : \Omega \to \mathbb{R} : \text{u is measurable and} \int_\Omega |u|^{p(x)}\omega(x)dx < \infty\right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(.)}(\Omega,\omega)} = \inf\left\{\lambda > 0, \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\omega(x)dx \leq 1\right\}.$$

We denote by $L^{p'(.)}(\Omega, \omega)$ the conjugate space of $L^{p(.)}(\Omega, \omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

and where

$$\omega^*(x) = \omega^{1-p'(x)} \quad \text{for all } x \in \Omega.$$

On the space $L^{p(.)}(\Omega, \omega)$, we consider the function $\varrho_{p(.),\omega} : L^{p(.)}(\Omega, \omega) \to \mathbb{R}^+$ defined by

$$\varrho_{p(.),\omega}(u) = \varrho_{L^{p(.)}(\Omega,\omega)}(u) = \int_\Omega |u(x)|^{p(x)}\omega(x)dx.$$

The weighted Sobolev space with variable exponent is defined by

$$W^{1,p(.)}(\Omega, \omega) = \left\{u \in L^{p(.)}(\Omega, \omega) \text{ such that } \nabla u \in L^{p(.)}(\Omega, \omega)\right\}.$$

With the norm

$$\|u\|_{1,p(.),\omega} = \|u\|_{p(.),\omega} + \|\nabla u\|_{p(.),\omega} \quad \text{for all } u \in W^{1,p(.)}(\Omega, \omega).$$

In the following of this paper, the space $W_0^{1,p(.)}(\Omega,\omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega,\omega)$ (i.e $W_0^{1,p(.)}(\Omega,\omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(.)}(\Omega,\omega)}$) with respect the norm $\|.\|_{1,p(.),\omega}$.
Let $p(.), s(.)$ are two element of space $C^+(\overline{\Omega})$ where the function $s(.)$ satisfies the hypothesis (H_2), we define the following function

$$\begin{aligned} p^*(x) &= \frac{Np(x)}{N-p(x)} \quad \text{for} \;\; p(x) < N, \\ p_s(x) &= \frac{p(x)s(x)}{1+s(x)} < p(x), \\ p_s^*(x) &= \begin{cases} \frac{p(x)s(x)}{(1+s(x))N-p(x)s(x)} & \text{if } N > p_s(x), \\ +\infty & \text{if } N \le p_s(x), \end{cases} \end{aligned}$$

for all most all $x \in \Omega$.

Proposition 1.1 ([7]). *Let $\Omega \subset \mathbb{R}^N$ a open set of $\mathbb{R}^N$, $p(.) \in C^+(\overline{\Omega})$ and let hypothesis (H_1) be satisfied, we have*

$$L^{p(.)}(\Omega,\omega) \hookrightarrow L^1_{Loc}(\Omega)$$

Proposition 1.2 ([7]). *Assume that hypotheses (H_1) and (H_2) hold and $p(.), s(.) \in C^+(\overline{\Omega})$ then we have the continuous embedding*

$$W^{1,p(.)}(\Omega,\omega) \hookrightarrow W^{1,p_s(.)}(\Omega,\omega)$$

Moreover, we have the compact embedding

$$W^{1,p(.)}(\Omega,\omega) \hookrightarrow\hookrightarrow W^{1,r(.)}(\Omega,\omega)$$

provided that $r \in C^+(\overline{\Omega}), 1 \le r < p^*$ for all $x \in \Omega$.

Definition 1.3. Given a constant $k > 0$, we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| < k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

For a function $u = u(x)$ defined on Ω, we define the truncated function $T_k(u)$ as follows, for every $x \in \Omega$, the value of $(T_k u)$ at x is just $T_k(u(x))$.
By Ph. Bénilan, L. Boccardo, T. Gallouet [7] we define also the space

$$\mathcal{T}_0^{1,p(.)}(\Omega,\omega) =$$
$$\{u : \Omega \to \mathbb{R} \;\text{ where } u \text{ is a measurable function }; \; T_k(u) \in W_0^{1,p(.)}(\Omega,\omega); \forall k > 0\}.$$

Lemma 1.4 ([1],[2]). *For $\xi, \eta \in \mathbb{R}^N$ and $1 < p(.) < \infty$, we have*

$$\frac{1}{p(.)}|\xi|^{p(.)} - \frac{1}{p(.)}|\eta|^{p(.)} \le |\xi|^{p(.)-2}\xi.(\xi-\eta),$$

where a dot symbol denote the Euclidean scalar product in $\mathbb{R}^N$.

Lemma 1.5 ([1],[2]). *For $a>0, b>0$ and $1\leq p(.), q<\infty$ we have*

$$(a+b)^{p(.)}\leq 2^{p(.)-1}(a^{p(.)}+b^{p(.)}).$$

$$a^q+b^q\leq (a+b)^q.$$

Lemma 1.6 ([16]). *Let q, q' two reals numbers, $q, q'>1$ such that $\frac{1}{q}+\frac{1}{q'}=1$ and $\xi,\eta\in\mathbb{R}^N$, the following assertion hold true:*

$$|\phi(\xi)-\phi(\eta)|^{q'}\leq C\left\{(\xi-\eta)\left(\phi(\xi)-\phi(\eta)\right)\right\}^{\frac{\beta}{2}}\left\{|\xi|^q+|\eta|^q\right\}^{1-\frac{\beta}{2}},$$

where $\beta=2$ if $1<q\leq 2$, and $\beta=q'$ if $q>2$, such that $\phi(\xi)=|\xi|^{p(\xi)-2}\xi$, and C is a positive constant.

Theorem 1.7 ([17]). *Suppose that $\Omega\subset\mathbb{R}^N$ be a bounded set. Let the exponent $p(.)\in C(\overline{\Omega})$. Then, for all $u\in W_0^{1,p(.)}(\Omega,\omega)$, the inequality*

$$\|u\|_{p(.),\omega}\leq C_0\|\nabla u\|_{p(.)\ ,\omega}\ ,$$

is satisfied where the constant C_0 depends on the exponent $p(.), diam(\Omega)$ and the dimension N.

Remark 1.8. Further down, c_i; $i\in\{1;2;...\}$ is a positive constant and $meas\{B\}$ denotes the measure of the measurable set $B\subset\mathbb{R}^N$.

2. Main results

In this section, we recall the definition of entropy solution using in (2.1) as well as some fundamental lemmas based on a priori estimates inspired by the ideas of L. Boccardo, T. Gallouet in [2] and theorem that will be use to prove the uniqueness of an entropy solution to our problem (1.1).

Definition 2.1. A function $u\in\mathcal{T}_0^{1,p(.)}(\Omega,\omega)$ is an entropy solution of degenerate elliptic problem (1.1) if and only if

$$\int_\Omega \omega\Phi(\nabla u-\theta(u))\nabla T_k(u-\varphi)+\int_\Omega \omega|u|^{p(u)-2}uT_k(u-\varphi)\leq\int_\Omega fT_k(u-\varphi) \quad (2.1)$$

for all $k>0$, $\varphi\in W_0^{1,p(.)}(\Omega,\omega)\cap L^\infty(\Omega)$ and $\Phi(\xi)=|\xi|^{p(\xi)-2}\xi$, $\forall\xi\in\mathbb{R}^N$.

Theorem 2.2. *Let hypotheses $(H_1),(H_2),(H_3)$ and (H_4) be satisfied, if the problem (1.1) has a entropy solution, then this solution is unique.*

To prove the uniqueness of entropy solution for problem (1.1)(i.e the proof of Theorem (2.2)) is divided into several steps.

Step 1: Some technical Lemmas.

Lemma 2.3. *Let hypotheses $(H_1),(H_2),(H_3),(H_4)$ be satisfied and u an entropy solution for problem (1.1), then*

$$\lim_{h\to+\infty} meas\{|u|>h\}=0.$$

Proof. Let u an entropy solution for our problem (1.1) and h a positive real number be large enough, we get

$$\begin{aligned}
\min_{x\in\Omega}\left(|\omega(x)|\right) h^{p^-} meas\{|u|>h\} &\leq \int_\Omega |\omega| h^{p(u)}dx,\\
&\leq \int_\Omega \omega(|u|^{p(u)-2}u)T_h(u)dx,\\
&\leq \int_\Omega \omega\Phi(\nabla u-\theta(u))\nabla T_h(u)dx\\
&+ \int_\Omega \omega(|u|^{p(u)-2}u)T_h(u)dx,\\
&\leq \int_\Omega fT_h(u)dx,\\
&\leq h\|f\|_1.
\end{aligned}$$

This implies that

$$meas\{|u>h|\} \leq \frac{c_1}{h^{p^--1}}.$$

Where

$$c_1=\frac{\|f\|_1}{\min\limits_{x\in\Omega}\left(|\omega(x)|\right)}.$$

Then

$$\lim_{h\to+\infty} meas\{|u|>h\}=0.$$

□

Lemma 2.4. *Let hypotheses $(H_1),(H_2),(H_3)$ and (H_4) be satisfied, if u is an entropy solution of problem (1.1), then*

1- $\lim\limits_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\{h<|u|<h+k\}}\omega|\nabla u-\theta(u)|^{p(u)-2}(\nabla u-\theta(u))\nabla u dx=0.$

2- $\lim\limits_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\{h<|u|<h+k\}}\omega|\nabla u-\theta(u)|^{p(u)}dx=0.$

3- $\lim\limits_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\{h<|u|<h+k\}}\omega|\nabla u|^{p(u)}dx=0.$

Proof. **Firstly.** Let u an entropy solution for the problem (1.1), k and h be two real numbers such that $1<k<h$.
Taking $\varphi=T_h(u)$ is a test function in inequality (2.1), we get

$$\begin{aligned}
\int_\Omega \omega\Phi(\nabla u-\theta(u))\nabla T_k(u-T_h(u))dx+\int_\Omega \omega|u|^{p(u)-2}uT_k(u-T_h(u))dx \\
\leq \int_\Omega fT_k(u-T_h(u))dx.
\end{aligned}\tag{2.2}$$

Remark that:

$$sign(u)\chi_{\{|u|>h\}}=sign(T_k(u-hsign(u))\chi_{\{|u|>h\}}.$$

Then

$$\int_\Omega \omega |u|^{p(u)-2} u T_k(u - T_h(u)))dx \geq 0. \tag{2.3}$$

We make the following change only for the reason of simplifcation.

$$\mathcal{I}_{h,k} = \int_{\Omega_k^h} \omega\Phi(\nabla u - \theta(u))\nabla T_k(u - T_h(u))dx,$$

and

$$\Omega_k^h = \{h \leq |u| < h + k\}.$$

Now, using (2.2), (2.3), lemma (2.3) and Lebesgue's Dominated convergence theorem, we get

$$\begin{aligned}
\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{I}_{h,k} &= \lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\Omega_k^h} \omega\Phi(\nabla u - \theta(u))\nabla T_k(u - T_h(u))dx, \qquad (2.4)\\
&\leq \lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\Omega_k^h} \omega\Phi(\nabla u - \theta(u))\nabla T_k(u - T_h(u))dx\\
&+ \lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\Omega_k^h} \omega|u|^{p(u)-2}uT_k(u - T_h(u)))dx,\\
&\leq \lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\Omega_k^h} fT_k(u - T_h(u))dx,\\
&\leq \int_\Omega \lim_{h\to+\infty} |f|\chi_{\{|u|>h\}}dx,\\
&\leq \lim_{h\to+\infty}\int_\Omega |f|\chi_{\{|u|>h\}}dx.
\end{aligned}$$

By lemma (2.3), we conclude that

$$\lim_{h\to+\infty}\int_\Omega |f|\chi_{\{|u|>h\}}dx = 0. \tag{2.5}$$

This implies that

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\{h<|u|<h+k\}} \omega|\nabla u - \theta(u)|^{p(u)-2}(\nabla u - \theta(u))\nabla u dx = 0. \tag{2.6}$$

Secondly. Now, using (2.2), lemma (1.4) and (H_3), we get that

$$\begin{aligned}\frac{1}{p^+}\int_{\Omega_k^h}\omega|\nabla u-\theta(u)|^{p(u)}dx &\leq \int_{\Omega_k^h}\omega|\nabla u-\theta(u)|^{p(u)-2}(\nabla u-\theta(u))\nabla u dx\\ &+ \frac{1}{p^-}\int_{\Omega_k^h}\omega|\theta(u)|^{p(u)}dx, \qquad (2.7)\\ &\leq \int_{\Omega_k^h}\omega|\nabla u-\theta(u)|^{p(u)-2}(\nabla u-\theta(u))\nabla u dx\\ &+ \frac{\lambda_1^{p^*}}{p^-}\int_{\Omega_k^h}\omega|u|^{p(u)}dx,\\ &\leq \int_{\Omega_k^h}\omega\Phi(\nabla u-\theta(u))\nabla T_k(u-T_h(u))dx,\\ &+ \int_{\Omega_k^h}\omega|u|^{p(u)}dx+\left(\frac{\lambda_1^{p^*}}{p^-}-1\right)\int_{\Omega_k^h}\omega|u|^{p(u)}dx\\ &\leq \int_{\Omega_k^h}fT_k(u-T_h(u))dx.\end{aligned}$$

Where

$$\lambda_1^{p^*}=\max\left(\lambda_0^{p^-};\lambda_0^{p^+}\right).$$

Then

$$\begin{aligned}\lim_{\substack{k\to 0\\ h\to+\infty}}\frac{1}{k}\int_{\Omega_k^h}\omega|\nabla u-\theta(u)|^{p(u)}dx &\leq p^+\left(\lim_{\substack{k\to 0\\ h\to+\infty}}\frac{1}{k}\int_{\Omega_k^h}fT_k(u-T_h(u))dx\right),\\ &\leq p^+\left(\lim_{h\to+\infty}\int_{\Omega}|f|\chi_{\{|u|>h\}}dx\right) \qquad (2.8)\end{aligned}$$

Now, using (2.5) and (2.8), we deduce that

$$\lim_{\substack{k\to 0\\ h\to+\infty}}\frac{1}{k}\int_{\{h<|u|<h+k\}}\omega|\nabla u-\theta(u)|^{p(u)}dx=0 \tag{2.9}$$

Thirdly. The same approach is employed here to demonstrate coercivity object by using (2.2), lemma (1.4) lemma (1.5) and (H_3) in [1], then we obtain

$$\begin{aligned}\frac{1}{p^+}\frac{1}{2^{p^+-1}}\int_{\Omega_k^h}\omega|\nabla u|^{p(u)}dx &\leq \int_{\Omega_k^h}\omega|\nabla u-\theta(u)|^{p(u)-2}(\nabla u-\theta(u))\nabla u dx\\ &+ \left(\frac{2\lambda_1^{p^*}}{p^-}-1\right)\int_{\Omega_k^h}\omega|u|^{p(u)}dx+\int_{\Omega_k^h}\omega|u|^{p(u)}dx\\ &\leq \int_{\Omega_k^h}fT_k(u-T_h(u))dx,\end{aligned}$$

Now, using Lebesgue's Dominated convergence theorem, we conclude that

$$\begin{aligned}\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\Omega_k^h} \omega|\nabla u|^{p(u)}dx &\leq p^+2^{p^+-1}\left(\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\Omega_k^h} fT_k(u-T_h(u))dx\right),\\ &\leq p^+2^{p^+-1}\left(\lim_{h\to+\infty}\int_\Omega |f|\chi_{\{|u|>h\}}dx\right).\end{aligned}$$

Finally, using (2.5) we obtain that

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\{h<|u|<h+k\}} \omega|\nabla u|^{p(u)}dx = 0. \tag{2.10}$$

Step 2: Uniqueness.
Now, let u and v are two entropy solutions of our degenerate weighted elliptic problem (1.1) and let h,k two positive real numbers such that $1<k<h$. In inequality (2.5), we take for the solution u, $\varphi = T_h(v)$ and for the solution v, we take $\varphi = T_h(u)$ as a test function. Using Lebesgue's Dominated Convergence Theorem, pass to limit when $k\to 0$ and $h\to+\infty$, we get that

$$\|g(u)-g(v)\|_{L^1(\Omega,\omega)} + \lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}(h,k) \leq 0, \tag{2.11}$$

where

$$g(u) = |u|^{p(u)-2}u \ \text{ for all } \ u\in W_0^{1,p(.)}(\Omega,\omega).$$

And

$$\mathcal{L}(h,k) = \int_\Omega \omega\{\Phi(\nabla u-\theta(u))\nabla T_k(u-T_h(v)) + \Phi(\nabla v-\theta(v))\nabla T_k(v-T_h(u))\}dx.$$

Now, we will prove that

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}(h,k)\geq 0.$$

For that, we consider the following decomposition

$$\begin{gathered}\Omega_h^1 = \{|u|\leq h, |v|\leq h\}; \Omega_h^2 = \{|u|\leq h, |v|>h\},\\ \Omega_h^3 = \{|u|>h, |v|\leq h\}; \Omega_h^4 = \{|u|>h, |v|>h\},\end{gathered}$$

and $\mathcal{L}_i(h,k)$ is the integral on Ω_h^i for $i=1,...,4$, defined by where

$$\mathcal{L}_i(h,k) = \int_{\Omega_h^i} \omega\left(\Phi(\nabla u-\theta(u))\nabla T_k(w_{u,v}^h) + \Phi(\nabla v-\theta(v))\nabla T_k(\overline{w}_{u,v}^h)\right)dx.$$

Where

$$w_{u,v}^h = u - T_h(v), \overline{w}_{u,v}^h = v - T_h(u).$$

Firstly, we pose

$$\mathcal{L}_1(h,k) = \mathcal{L}_1^1(h,k) + \mathcal{L}_1^2(h,k),$$

where

$$\begin{aligned}
\mathcal{L}_1^1(h,k) &= \int_{\Omega^1_{k,h}} \omega\{\Phi(\nabla u-\theta(u))-\Phi(\nabla v-\theta(v))\}\psi_\theta(u,v)dx,\\
\mathcal{L}_1^2(h,k) &= \int_{\Omega^1_{k,h}} \omega\{\Phi(\nabla u-\theta(u))-\Phi(\nabla v-\theta(v))\}\Psi_\theta(u,v)dx,
\end{aligned}$$

and

$$\Omega^1_{k,h}=\{|u-v|\le k; |u|\le h; |v|\le h\},$$

$$\begin{aligned}
\psi_\theta(u,v) &= (\nabla u-\theta(u))-(\nabla v-\theta(v)),\\
\Psi_\theta(u,v) &= \theta(u)-\theta(v).
\end{aligned}$$

To show that

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_1(k,h)=0,$$

we consider two cases according to the value of p^+.

Case 1: $1<p^+\le 2$.

Let $\varepsilon>0$, using Young's inequality with $\varepsilon>0$ and (H_3) we get

$$\begin{aligned}
\lim_{k\to 0}\frac{1}{k}\mathcal{L}_1^2(h,k) &\le \lim_{k\to 0}\frac{\varepsilon}{kp'^+}\int_{\Omega^1_{k,h}}\omega|\Phi(\nabla u-\theta(u))-\Phi(\nabla v-\theta(v))|^{p'^+}dx\\
&+ \lim_{k\to 0}\frac{1}{\varepsilon kp^+}\int_{\Omega^1_{k,h}}\omega|\theta(u)-\theta(v)|^{p^+}dx, \qquad (2.12)\\
&\le \lim_{k\to 0}\left(\varepsilon\frac{c_2}{k}\mathcal{L}_1^1(k,h)+\frac{(k\lambda_0)^{p^+}}{\varepsilon}\right),\\
&\le \lim_{k\to 0}\varepsilon\frac{c_2}{k}\mathcal{L}_1^1(k,h),
\end{aligned}$$

where

$$c_2=C/p'^+.$$

Remark 2.5. Using lemma (1.6) we deduce that c_2 is a positive constant.

If $\lim\limits_{k\to 0}\frac{1}{k}\mathcal{L}_1^1(k,h)=0$.

Then

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_1(h,k)=0.$$

If $0<\lim\limits_{k\to 0}\frac{1}{k}\mathcal{L}_1^1(k,h)<\infty$.

We pose in (2.12)

$$\varepsilon=1/\left(h\lim_{k\to 0}\frac{1}{k}\mathcal{L}_1^1(h,k)\right).$$

This implies that

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_2(h,k)=0.$$

Then

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_1(h,k) \geq 0.$$

If $\lim\limits_{k\to 0} \frac{1}{k}\mathcal{L}_1^1(k,h) = +\infty.$

Using hypothesis (H_3) and lemma (2.4), we find that

$$\begin{aligned}
\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_1^2(k,h) &\leq \lim_{\substack{k\to 0\\ h\to+\infty}} (\lambda_0 k)\left(\frac{1}{k}\int_{\Omega^1_{k,h}} \omega|\Phi(\nabla u-\theta(u))-\Phi(\nabla v-\theta(v))|dx\right),\\
&\leq \lim_{\substack{k\to 0\\ h\to+\infty}} (\lambda_0 k)\left(\frac{1}{k}\int_{\Omega^1_{k,h}} \omega|\nabla u-\theta(u)|^{p(u)}dx\right) \qquad (2.13)\\
&+ \lim_{\substack{k\to 0\\ h\to+\infty}} (\lambda_0 k)\left(\frac{1}{k}\int_{\Omega^1_{k,h}} \omega|\nabla v-\theta(v)|^{p(v)}dx\right).
\end{aligned}$$

Now, using (2.13), we conclude that

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_1^2(k,h) = 0 \qquad (2.14)$$

This implies that

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_1(k,h) = +\infty. \qquad (2.15)$$

Case 2: $p^+ > 2$.

Let p'^+ the conjugate of p^+ (i.e: $\frac{1}{p^+}+\frac{1}{p'^+} = 1$).

Then $1 < p'^+ \leq 2$. Now, using Young's inequality with $\varepsilon > 0$ and (H_3) we get

$$\begin{aligned}
\lim_{k\to 0} \frac{1}{k}\mathcal{L}_1^2(h,k) &\leq \lim_{k\to 0} \frac{\varepsilon}{kp^+}\int_{\Omega^1_{k,h}} \omega|\Phi(\nabla u-\theta(u))-\Phi(\nabla v-\theta(v))|^{p^+}dx\\
&+ \lim_{k\to 0} \frac{1}{\varepsilon k p'^+}\int_{\Omega^1_{k,h}} \omega|\theta(u)-\theta(v)|^{p'^+}dx, \qquad (2.16)\\
&\leq \lim_{k\to 0}\left(\varepsilon\frac{c_3}{k}\mathcal{L}_1^1(k,h) + \frac{(k\lambda_0)^{p'^+}}{\varepsilon}\right),\\
&\leq \lim_{k\to 0} \varepsilon\frac{c_3}{k}\mathcal{L}_1^1(k,h),
\end{aligned}$$

where

$$c_3 = C/p^+.$$

Remark 2.6. Using lemma (1.6), we deduce that c_3 is a positive constant.

By similarity method in case 1 , we have

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_1(k,h) \geq 0. \qquad (2.17)$$

Secondly, we pose

$$\mathcal{L}_2(k,h) = \mathcal{L}_2^1(k,h) + \mathcal{L}_2^2(k,h),$$

where

$$\begin{aligned} \mathcal{L}_2^1(k,h) &= \int_{\Omega_h^2} \omega(\Phi(\nabla v - \theta(v))\nabla T_k(v-u)dx, \\ &= \int_{\Omega_{k,h}^{2,1}} \omega(\Phi(\nabla v - \theta(v))\nabla v dx - \int_{\Omega_{k,h}^{2,1}} \omega(\Phi(\nabla v - \theta(v))\nabla u dx, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \mathcal{L}_2^2(k,h) &= \int_{\Omega_h^2} \omega(\Phi(\nabla u - \theta(u))\nabla T_k(u - hsign(v))dx, \\ &= \int_{\Omega_{k,h}^{2,2}} \omega(\Phi(\nabla u - \theta(u))\nabla u dx, \end{aligned} \tag{2.19}$$

and

$$\Omega_{k,h}^{2,1} = \{|u| \le h; |v| > h; |v-u| \le k\},$$

$$\Omega_{k,h}^{2,2} = \{|u| \le h; |v| > h; |u - hsign(v)| \le k\},$$

Remark 2.7. it is important to note that the following statement hold

$$\Omega_{k,h}^{2,1} \subset \Omega_k^h. \tag{2.20}$$

Now, using (2.20), we get

$$\begin{aligned} -\int_{\Omega_{k,h}^{2,1}} \omega|\Phi(\nabla v - \theta(v))\nabla u dx &= -\int_{\Omega_{k,h}^{2,1}} \omega(\Phi(\nabla v - \theta(v))\nabla v dx \\ &- \int_{\Omega_{k,h}^{2,1}} \omega(\Phi(\nabla v - \theta(v))\nabla (u-v) dx, \\ &\ge -\int_{\Omega_k^h} \omega|\Phi(\nabla v - \theta(v))\nabla v| dx \\ &- \int_{\Omega_k^h} \omega|\Phi(\nabla v - \theta(v))\nabla (u-v)| dx, \\ &\ge -k\left(\frac{1}{k}\int_{\Omega_k^h} \omega|(\nabla v - \theta(v)|^{p(u)-1}|\nabla v| dx\right) \\ &- k^2 C_0 \left(\frac{1}{k}\int_{\Omega_k^h} \omega|\nabla v - \theta(v)|^{p(u)-1} dx\right), \end{aligned} \tag{2.21}$$

Now, using (2.21) and lemma (2.4) , we obtain that

$$-\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\int_{\Omega_{k,h}^{2,1}} \omega|\Phi(\nabla v - \theta(v))\nabla u dx \ge 0 \tag{2.22}$$

Applying the same methodology employed to illustrate coerciveness, we obtain

$$\begin{aligned}\int_{\Omega^{2,1}_{k,h}} \omega(\Phi(\nabla v-\theta(v))\nabla v dx &\geq \frac{1}{p^+}\left(\frac{1}{2^{p^+-1}}-2C_0\lambda_0^{p^{**}}\right)\int_{\Omega^{2,1}_{k,h}} \omega|\nabla v|^{p(v)}dx,\\ &\geq c_4\left(\frac{1}{k}\int_{\Omega^{2,1}_{k,h}} \omega|\nabla v|^{p(v)}dx\right), \qquad (2.23)\end{aligned}$$

where

$$c_4 = k/p^+\left(1/2^{p^+-1}-2C_0\lambda_0^{p^{**}}\right).$$

Remark 2.8. Using hypothesis (H_3), we deduce that c_4 is a positive constant.

In the same way, we have

$$\int_{\Omega^{2,2}_{k,h}} \omega(\Phi(\nabla u-\theta(u))\nabla u dx \geq c_5\left(\frac{1}{k}\int_{\Omega^{2,1}_{k,h}} \omega|\nabla u|^{p(u)}dx\right). \qquad (2.24)$$

Where c_5 is a positive constant.

Remark 2.9. The constant c_5 depends on lemma (1.4), lemma (1.5) and hypothesis (H_3).

Now, using (2.22) - (2.24) and lemma (2.3), we conclude that

$$\begin{aligned}\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_2^1(k,h) &\geq 0,\\ \lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_2^2(k,h) &\geq 0.\end{aligned}$$

This implies that

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}_2(k,h) \geq 0. \qquad (2.25)$$

Finally, in the same manner, we obtain that

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\left(\mathcal{L}_3(k,h)+\mathcal{L}_4(k,h)\right) \geq 0. \qquad (2.26)$$

Hence

$$\lim_{\substack{k\to 0\\ h\to+\infty}} \frac{1}{k}\mathcal{L}(k,h) \geq 0. \qquad (2.27)$$

Therefore, inequality (2.11) becomes

$$\|g(u)-g(v)\|_{L^1(\Omega,\omega)} \leq 0 \qquad (2.28)$$

This implies that

$$u = v \quad \text{a.e} \quad \text{in} \quad \Omega.$$

As a result, we ascertain the uniqueness of the entropy solution for our problem (1.1).

Remark 2.10. We have clearly demonstrate the existence of entropy solution for our problem (1.1) in [1].

□

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