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NONLINEAR MULTIVALUED HOMOGENEOUS ROBIN BOUNDARY $p(u)$-LAPLACIAN PROBLEM

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ABSTRACT. We establish the existence and a partial uniqueness result of solutions for a nonlinear multivalued Robin boundary $p(u)-$ laplacian problem.

1. INTRODUCTION

We consider the following nonlinear elliptic $p(u)-$laplacian problem with Robin boundary condition.

$$P(\beta, f)\ \begin{cases} \beta(u) - \operatorname{div} a(x,u,\nabla u) \ni f & \text{in } \Omega \\ a(x,u,\nabla u).\eta = -|u|^{r(x,u)-2}u & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$, with smooth boundary and β a maximal monotone graph on $\mathbb{R}$ such that $0 \in \beta(0)$, a is a Leray-Lions operator, η is the outer unit normal vector on $\partial\Omega$ and $f \in L^1(\Omega)$. $\operatorname{div} a(x,u,\nabla u)$ is called $p(u)-$Laplacian operator, a prototype case is $\operatorname{div}(|\nabla u|^{p(.,u)-2}.\nabla u)$. The problem $P(\beta, f)$ is adapted into a generalized Leray-Lions framework under the assumptions that.
$a : \Omega \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}^N$ is a Carathéodory function with

$$a(x,z,0) = 0 \text{ for all } z \in \mathbb{R}, \text{ and a.e. } x \in \Omega, \tag{1.1}$$

satisfying the strict monotonicity assumption

$$\big(a(x,z,\xi) - a(x,z,\eta)\big).(\xi - \eta) > 0 \text{ for all } \xi, \eta \in \mathbb{R}^N, \xi \neq \eta, \tag{1.2}$$

as well as the growth and the coercivity assumptions with variable exponent

$$\big|a(x,z,\xi)\big|^{p'(x,z)} \leq C_1\big(|\xi|^{p(x,z)} + \mathcal{M}(x)\big), \tag{1.3}$$

$$a(x,z,\xi).\xi \geq \frac{1}{C_2}|\xi|^{p(x,z)}. \tag{1.4}$$

Here C_1, C_2 are positive constants and $\mathcal{M}$ is a positive function such that $\mathcal{M} \in L^1(\Omega)$.

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$p : \Omega \times \mathbb{R} \to [p_-, p_+]$ is a Carathéodory function, $1 < p_- \le p(x,z) \le p_+ < \infty$ and $p'(x,z) = \dfrac{p(x,z)}{p(x,z)-1}$ is the conjugate exponent of $p(x,z)$, with

$$p_- := ess \inf_{(x,z)\in\overline{\Omega}\times\mathbb{R}} p(x,z) \text{ and } p_+ := ess \sup_{(x,z)\in\overline{\Omega}\times\mathbb{R}} p(x,z).$$

We assume that

$$p_- > N \text{ and } p \text{ is uniformly log-Hölder continuous in } \overline{\Omega}\times[-M,M], \text{ with } M>0. \quad (1.5)$$

In this paper, we use the Robin boundary condition $a(x,u,\nabla u).\eta = -|u|^{r(x,u)-2}u$ where $r : \partial\Omega \times \mathbb{R} \to [r_-, r_+]$ is a Carathéodory function such that $2 < r_- \le r \le r_+ < \infty$. Thus, the function $t \mapsto |t|^{r(x,t)-2}t$ is continuous for a.e. x on $\partial\Omega$. Furthermore, we make the following hypothesis.

$$(H) \;\; t \mapsto |t|^{r(.,t)-2}t \text{ is increasing.}$$

The problem $P(\beta, f)$ can be seen as an extension of the following problem.

$$\begin{cases} b(u) - \text{div}a(x,u,\nabla u) = f & \text{in } \Omega \\ a(x,u,\nabla u).\eta = -|u|^{r(x,u)-2}u & \text{on } \partial\Omega, \end{cases} \quad (1.6)$$

where $b : \mathbb{R} \to \mathbb{R}$ is nondecreasing, normalized by $b(0) = 0$ and $f \in L^1(\Omega)$. Ouaro and Sawadogo (see [16]) established and proved the existence results, the uniqueness and structural stability issues for such $p(x,u)$ variable exponent problem (1.6).
Since β is nonlinear, its appear in the definition of the solution, a bounded Radon diffuse measure, in order to take into account the border of the domain.
An other difficulty is that the Poincaré inequality and the Poincaré-Wirtinger inequality with variable exponent cannot be used. Nevertheless, we use the Poincaré Wirtinger inequality with constant exponent.
We define $\mathcal{M}_b(\Omega)$ as the set of bounded Radon measures in Ω. For the variable exponent $\pi(.)$, where $\pi(.)$ is to be defined later, given $\mu \in \mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W^{1,\pi(.)}(\Omega)$ if $\mu(A) = 0$, for every set A such that $Cap_{\pi(.)}(A,\Omega) = 0$ (see [15]). For $A \subset \Omega$, we denote

$$S_{\pi(.)}(A) = \left\{ u \in W^{1,\pi(.)}(\Omega) : \;\; u \ge 1 \text{ on all open set which contain } A \text{ and } u \ge 0 \text{ in } \Omega \right\}.$$

The $\pi(.)-$Capacity for every subset A with respect to Ω is defined by

$$Cap_{\pi(.)}(A,\Omega) = \inf_{u\in S_{\pi(.)}(A)} \left\{ \int_\Omega \left(|u|^{\pi(.)} + |\nabla u|^{\pi(.)}\right)dx \right\}.$$

The set of bounded Radon diffuse measures in variable exponent setting $\pi(.)$ is denoted by $\mathcal{M}_b^{\pi(.)}(\Omega)$. Here, we use the techniques of Chipot and de Oliveira in [9] to prove that $|\nabla u|^{\pi(.)} \in L^1(\Omega)$, different to those used the Young measure. (cf. [1, 2, 11, 16]).
In this work, we consider the Robin boundary condition which bring some difficulties to treat the term at the boundary. In order to get our main result, we define a new space which will help us to take into account the boundary condition. This

space in the context of variable exponent was for the first time introduced by Ouaro et *al.* (see [17]). Moreover, we adapt the notion of renormalized solutions for the study of our problem. In fact, the concept of renormalized solution was introduced by Diperna and Lions ([10]). Furthermore, the standard Leray-Lions elliptic problem with L^1 source terms are well posed in the framework of renormalized solutions.

The interest of the study of this kind of problem is due to the fact that they can model phenomena which arise in the study of elastic mechanics (see [3]), electrorheological fluid (see [18]) or image restoration (see [8]).

The remaining part of this article is organized as follows: in the next section, we introduce some preliminary results. In the last section, we study the existence and the partial uniqueness of the renormalized solutions for the problem $P(\beta, f)$.

2. Preliminary results

We will use the so-called truncation function

$$T_k(s) := \begin{cases} s & \text{if } |s| \leq k \\ k sign_0(s) & \text{if } |s| > k \end{cases} \quad , \quad \text{where } sign_0(s) := \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0. \end{cases}$$

The truncation function possesses the following properties.

$$T_k(-s) = -T_k(s), |T_k(s)| = \min\{|s|, k\},$$
$$\lim_{k\to\infty} T_k(s) = s \text{ and } \lim_{k\to 0} \frac{1}{k} T_k(s) = sign_0(s).$$

Taking into account the assumptions (1.3), (1.4) and Robin boundary condition, we need to work in the variable exponent Sobolev space $W^{1,\pi(.)}(\Omega)$ and $L^{s(.)}(\partial\Omega)$, with $\pi(.) := p(., u(.))$ and $s(.) := r(., u(.))$. For the sake of completeness, we also recall the definition of variable exponent Lebesgue and Sobolev spaces $L^{\pi(.)}(\Omega)$ and $W^{1,\pi(.)}(\Omega)$. In the sequel, we will use the same notation $L^{\pi(.)}(\Omega)$ for the space $(L^{\pi(.)}(\Omega))^N$ of vector-valued functions.

Definition 2.1. Let $\pi : \Omega \longrightarrow [1, \infty)$ be a measurable function. $L^{\pi(.)}(\Omega)$ is the space of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that the modular

$$\rho_{\pi(.)}(f) := \int_\Omega |f|^{\pi(x)} dx < \infty.$$

If p_+ is finite, this space is equipped with the Luxembourg norm

$$||f||_{L^{\pi(.)}(\Omega)} := \inf \left\{ \lambda > 0; \quad \rho_{\pi(.)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

$W^{1,\pi(.)}(\Omega)$ is the space of all functions $f \in L^{\pi(.)}(\Omega)$ such that the gradient of f (taken in the sense of distributions) belongs to $L^{\pi(.)}(\Omega)$; if p_+ is finite the space $W^{1,\pi(.)}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,\pi(.)}(\Omega)} := ||u||_{L^{\pi(.)}(\Omega)} + ||\nabla u||_{L^{\pi(.)}(\Omega)}.$$

$W_0^{1,\pi(.)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,\pi(.)}(\Omega)$.
When $1 < p_- \leq \pi(.) \leq p_+ < \infty$, all the above spaces are separable and reflexive Banach spaces.

In the present paper, we assume that $\pi(x) = p(x, u(x))$ verify the log-Hölder continuity assumption (2.1) below. We denote $\pi_\varepsilon(x) := p(x, u_\varepsilon(x))$ for all $x \in \Omega$ and $s_\varepsilon(x) := r(x, u_\varepsilon(x))$ for all $x \in \partial\Omega$.

Proposition 2.2. *(See [1], Proposition 2.3)*
For all measurable function $\pi : \Omega \to [p_-, p_+]$, the following properties hold.

i) *$L^{\pi(.)}(\Omega)$ and $W^{1,\pi(.)}(\Omega)$ are separable and reflexive Banach spaces.*

ii) *$L^{\pi'(.)}(\Omega)$ can be identified with the dual space of $L^{\pi(.)}(\Omega)$, and the following Hölder inequality holds.*

$$\forall f \in L^{\pi(.)}(\Omega), g \in L^{\pi'(.)}(\Omega), \quad \left| \int_\Omega fg dx \right| \leq 2||f||_{L^{\pi(.)}(\Omega)}||g||_{L^{\pi'(.)}(\Omega)}.$$

iii) *One has $\rho_{\pi(.)}(f) = 1$ if and only if $||f||_{L^{\pi(.)}(\Omega)} = 1$; further,*

$$\text{if } \rho_{\pi(.)}(f) \leq 1, \text{ then } ||f||^{p_+}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p_-}_{L^{\pi(.)}(\Omega)};$$

$$\text{if } \rho_{\pi(.)}(f) \geq 1, \text{ then } ||f||^{p_-}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p_+}_{L^{\pi(.)}(\Omega)}.$$

In particular, if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^{\pi(.)}(\Omega)$, then $||f_n||_{L^{\pi(.)}(\Omega)}$ tends to zero (resp., to infinity) if and only if $\rho_{\pi(.)}(f_n)$ tends to zero (resp., to infinity), as $n \to \infty$.

For a measurable function $f \in W^{1,\pi(.)}(\Omega)$ we introduce the following notation.

$$\rho_{1,\pi(.)}(f) = \int_\Omega |f|^{\pi(.)} dx + \int_\Omega |\nabla f|^{\pi(.)} dx.$$

Replacing $p(x)$ by $\pi(x)$ in [5]-Proposition 2.2, we get the following result which is fundamental in this work (See [20, 21]).

Proposition 2.3. *If $f \in W^{1,\pi(.)}(\Omega)$, the following properties hold:*

i) $||f||_{W^{1,\pi(.)}(\Omega)} > 1 \Rightarrow ||f||^{p_-}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < ||f||^{p_+}_{W^{1,\pi(.)}(\Omega)}$;

ii) $||f||_{W^{1,\pi(.)}(\Omega)} < 1 \Rightarrow ||f||^{p_+}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < ||f||^{p_-}_{W^{1,\pi(.)}(\Omega)}$;

iii) $||f||_{W^{1,\pi(.)}(\Omega)} < 1(\textit{respectively} = 1; > 1) \Leftrightarrow \rho_{1,\pi(.)}(f) < 1(\textit{respectively} = 1; > 1)$.

The following lemma show that the space $W^{1,\pi(.)}(\Omega)$ is stable by truncation.

Lemma 2.4. *(See,[1]-Lemma 2.9) If $u \in W^{1,\pi(.)}(\Omega)$ then $T_k(u) \in W^{1,\pi(.)}(\Omega)$.*

Now, we give some embedding results.

Proposition 2.5. *(See [1], Proposition 2.4) Assume that $\pi : \Omega \to [p_-, p_+]$ has a representative which can be extended into a continuous function up to the boundary $\partial\Omega$ and satisfying the log-Hölder continuity assumption:*

$$\exists L > 0, \ \forall x, y \in \overline{\Omega}, x \neq y, \quad -\big(\log|x-y|\big)|\pi(x) - \pi(y)| \leq L. \tag{2.1}$$

i) *Then,* $C^\infty(\overline{\Omega})$ *is dense in* $W^{1,\pi(.)}(\Omega)$.

ii) $W^{1,\pi(.)}(\Omega)$ *is embedded into* $L^{\pi^*(.)}(\Omega)$, *where* $\pi^*(.)$ *is the optimal Sobolev embedding exponent defined in (2.2) below. If* q *is a measurable variable exponent such that* $\operatorname{ess}\inf_{x\in\Omega}(\pi^*(.)-q(.))>0$, *then the embedding of* $W^{1,\pi(.)}(\Omega)$ *into* $L^{q(.)}(\Omega)$ *is compact.*

For a given $\pi(.)$, *a function taking values in* $[p_-,p_+]$, $\pi^*(.)$ *denotes the optimal Sobolev embedding defined for any* $x\in\Omega$ *by*

$$\pi^*(x)=\begin{cases}\dfrac{N\pi(x)}{N-\pi(x)} & \text{if } \pi(x)<N\\ \text{any real value} & \text{if } \pi(x)=N\\ \infty & \text{if } \pi(x)>N.\end{cases} \tag{2.2}$$

Put

$$\pi^{\partial}(x):=\begin{cases}\dfrac{(N-1)\pi(x)}{N-\pi(x)} & \text{if } \pi(x)<N\\ \infty & \text{if } \pi(x)\geq N.\end{cases} \tag{2.3}$$

Proposition 2.6. *(See [17]) Let* $\pi(.)\in C(\overline{\Omega})$ *and* $p_->1$. *If* $q(.)\in C(\partial\Omega)$ *satisfies the condition :*

$$1\leq q(x)<\pi^{\partial}(x),\ \ \forall x\in\partial\Omega,$$

then, there is a compact embedding

$$W^{1,\pi(.)}(\Omega)\hookrightarrow L^{q(.)}(\partial\Omega).$$

In particular,

$$W^{1,\pi(.)}(\Omega)\hookrightarrow L^{\pi(.)}(\partial\Omega).$$

For any $u\in W^{1,\pi(.)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense. We will identify at the boundary u and $\tau(u)$.
The following lemma give us the convergence result in the sens of graphs.

Lemma 2.7. *Let* $(\beta_n)_{n\geq1}$ *be a sequence of maximal monotone graph such that* $\beta_n\to\beta$ *in the sens of graphs (i.e for all* $(x,y)\in\beta$ *there exists* $(x_n,y_n)\in\beta_n$ *such that* $x_n\to x$ *and* $y_n\to y$*). We consider* $(z_n)_{n\geq1}$ *and* $(w_n)_{n\geq1}$ *two sequences of* $L^1(\Omega)$, *such that* $w_n\in\beta_n(z_n)$ $\mathcal{L}^N$ *a.e.* Ω. *If* $(w_n)_{n\geq1}$ *is bounded in* $L^1(\Omega)$ *and* $z_n\to z$ *in* $L^1(\Omega)$, *then* $z\in dom(\beta)$ $\mathcal{L}^N$ *a.e.* Ω.

The main tool of the proof of the above lemma is the "biting lemma of Chacon", see [7].

Lemma 2.8. *Let* Ω *be an open bounded subset of* $\mathbb{R}^N$ *and* $(f_n)_{n\geq1}$ *a bounded sequence in* $L^1(\Omega)$. *Then, there exists* $f\in L^1(\Omega)$, *a subsequence* $(f_{n_k})_{n_k\geq1}$ *and a sequence of the mesurable set* $(E_j)_j$, $E_j\subset\Omega$, $\forall j\in\mathbb{N}^*$ *with* $E_{j+1}\subset E_j$ *and* $\lim_{j\to\infty}|E_j|=0$, *such that for any* $j\in\mathbb{N}^*$, $f_{n_k}\rightharpoonup f$ *in* $L^1(\Omega\setminus E_j)$, *as* $n_k\to\infty$.

Proof of the Lemma 2.7. Since the sequence $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$, using the "biting lemma of Chacon" there exists $w \in L^1(\Omega)$ a subsequence $(w_{n_k})_{n_k\geq 1}$, a sequence of mesurable sets $(E_j)_{j\in\mathbb{N}^*}$, $E_j \subset \Omega$, $\forall j \in \mathbb{N}^*$ with $E_{j+1} \subset E_j$ and $\lim_{j\to\infty} |E_j| = 0$ and for all $j \in \mathbb{N}^*$, $w_{n_k} \rightharpoonup w$ in $L^1(\Omega \setminus E_j)$, as $n_k \to \infty$. Since $z_{n_k} \rightharpoonup z$ in $L^1(\Omega)$ and so in $L^1(\Omega \setminus E_j)$, $\forall j \in \mathbb{N}$ and $\beta_{n_k} \to \beta$ in the sense of graph, we have $w \in \beta(z)$ a.e. in $\Omega \setminus E_j$. Thus, $z \in dom(\beta)$ a.e. in $\Omega \setminus E_j$. Finally, we obtain $z \in dom(\beta)$ a.e. in Ω. □

Furthermore, for every $\varepsilon > 0$, we consider the Yosida regularization β_ε of β, given by

$$\beta_\varepsilon = \frac{1}{\varepsilon}\big(I - (I + \varepsilon\beta)^{-1}\big).$$

Thanks to [6], there exists j a non negative, convex and l.s.c function defined on $\mathbb{R}$ such that $\beta = \partial j$.

To regularize β, we consider $j_\varepsilon(s) = \min_{r\in\mathbb{R}}\left\{\frac{1}{2\varepsilon}|s-r|^2 + j(r)\right\}, \forall s \in \mathbb{R}, \forall \varepsilon > 0.$

Using proposition 2.11-[6], one has

$$\begin{cases} dom(\beta) \subset dom(j) \subset \overline{dom(j)} = \overline{dom(\beta)}, \\ j_\varepsilon(s) = \dfrac{\varepsilon}{2}|\beta_\varepsilon(s)|^2 + j\big(J_\varepsilon(s)\big) \text{ where } J_\varepsilon := (I+\varepsilon\beta)^{-1} \\ j_\varepsilon \text{ is convex, Frechet-differentiable function and } \beta_\varepsilon = \partial j_\varepsilon, \\ j_\varepsilon \uparrow j \text{ as } \varepsilon \downarrow 0. \end{cases}$$

In addition, for any $\varepsilon > 0$, β_ε is nondecreasing and Lipschitz continuous function. We define the function f_ε by $f_\varepsilon(x) = T_{\frac{1}{\varepsilon}}(f(x))$ for any $x \in \Omega$. Then $(f_\varepsilon)_{\varepsilon>0}$ is a sequence of bounded function which converges strongly to $f \in L^1(\Omega)$ as $\varepsilon \to 0$ and such that

$$\|T_{\frac{1}{\varepsilon}}(f)\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \ \ \forall \varepsilon > 0.$$

Lemma 2.9. *(See [15]-Lemma 3.1). The Yosida regularisation β_ε is a surjective operator.*

We need the next two lemmas for the sequel.

Lemma 2.10. *(See [13]-Remark 2.12) Let V be a separable reflexive Banach space and A, $M : V \to V'$ be such that.*

(i) *A is a pseudo-monotone operator;*
(ii) *M is a bounded hemicontinuous and monotone operator;*

then $A + M$ is pseudo-monotone.

Lemma 2.11. *(See [19], Corollary 2.2). If an operator $\mathcal{A}$ is of type (M), bounded and coercive from a separable Banach space to its dual, then $\mathcal{A}$ is surjective.*

We write for all measurable function $u : \Omega \to \mathbb{R}$, $\big\{|u| \leq k \ (< k, > k, \geqslant k, = k\big\}$ or $\big[|u| \leq k \ (< k, > k, \geqslant k, = k\big]$ for the set $\big\{x \in \Omega;\ |u(x)| \leq k \ (< k, > k, \geqslant k, = k\big\}$, and $meas(\Omega)$ or $|\Omega|$ denote the measure of the set Ω.
Let us also set

$$int(dom\beta) = (m, \ \ M) \ \text{ with } \ -\infty \leq m \leq 0 \leq M \leq \infty.$$

3. Existence and partial uniqueness of the renormalized solution

We give the following notion of solution of the problem $P(\beta, f)$ due to Igbida et *al.* in [12].

Definition 3.1. A renormalized solution of the problem $P(\beta, f)$ is a couple (u, w) with u a measurable function such that $T_k(u) \in W^{1,\pi(.)}(\Omega)$ for all $k > 0$, $|u|^{s(.)-2}u \in L^1(\partial\Omega)$ and $u \in dom(\beta)$ $\mathcal{L}^N$ *a.e.* Ω, $w \in L^1(\Omega)$ and $w \in \beta(u)$ $\mathcal{L}^N$ *a.e.* Ω; and there exists a measure $\mu \in \mathcal{M}_b^{\pi(.)}(\Omega)$ such that $\mu \perp \mathcal{L}^N$, μ^+ is concentrated on $\big[u = M\big] \cap \big[u \neq \infty\big]$, μ^- is concentrated on $\big[u = m\big] \cap \big[u \neq -\infty\big]$ such that

$$\begin{aligned} \int_\Omega wh(u)\varphi dx + \int_\Omega a(x,u,\nabla u)\nabla\big(h(u)\varphi\big)dx \quad &+ \quad \int_\Omega h(u)\varphi d\mu \\ &+ \quad \int_{\partial\Omega} |u|^{s(.)-2}uh(u)\varphi d\sigma \\ &= \quad \int_\Omega fh(u)\varphi dx, \end{aligned} \tag{3.1}$$

for all $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$, for all $h \in C_c^1(\mathbb{R})$ and

$$\lim_{M\to\infty} \int_{\big[M<|u|<M+1\big]} a(x,u,\nabla u).\nabla u dx = 0. \tag{3.2}$$

All the terms of (3.1) are well defined. Indeed, $h(u)\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$ then, the first integral of the left-hand side and the right-hand side of (3.1) are well defined. The second integral of the left-hand side is well defined thanks to (1.3). The third integral of the left hand side is also well defined, since the measure μ is diffuse. Moreover, as $\varphi \in C(\overline{\Omega})$ ($\varphi \in W^{1,\pi(.)}(\Omega) \subset W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$, for $p_- > N$), then $\varphi \in L^\infty(\partial\Omega)$, so, the fourth integral is well defined.

Remark 3.2. If $M = \infty$ and $-\infty < m$ (resp. $m = -\infty$ and $M < \infty$) then, $\mu_+ \equiv 0$ (resp. $\mu_- \equiv 0$). Thus, (3.1) holds true with $\mu \equiv \mu_+$ (resp. $\mu \equiv \mu_-$).
If $M = \infty$ and $m = -\infty$ then, the domain of β is equal to $\mathbb{R}$ and the relation (3.1) becomes

$$\int_\Omega a(x,u,\nabla u)\nabla\big(h(u)\varphi\big)dx + \int_\Omega wh(u)\varphi dx + \int_{\partial\Omega} |u|^{s(.)-2}uh(u)\varphi d\sigma = \int_\Omega fh(u)\varphi dx,$$

for all $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$ and for all $h \in C_c^1(\mathbb{R})$.
In the case where the domain of β is bounded, the renormalization with h is not necessary in the Definition 3.1, we can then take $h \equiv 1$.

Now, we are going to prove the following main existence result.

Theorem 3.3. *Assume that* (1.1)-(1.5) *hold and* $r : \partial\Omega \times \mathbb{R} \to [r_-, r_+]$ *is a Carathéodory function such that* $2 < r_- \leq r \leq r_+ < \infty$. *Then, there exists at least one renormalized solution to the problem* $P(\beta, f)$.

Proof. The proof of Theorem 3.3 is divided into two steps.

Step 1. The approximate problem
Let us consider the following problem.

$$P(\beta_\epsilon, f_\varepsilon)\begin{cases}\beta_\varepsilon(u_\varepsilon) - \mathrm{div} a(x, u_\varepsilon, \nabla u_\varepsilon) - \varepsilon \triangle_{p_+} u_\varepsilon + \varepsilon |u_\varepsilon|^{p_+-2}u_\varepsilon = f_\varepsilon & \text{in } \Omega \\ \big[a(x, u_\varepsilon, \nabla u_\varepsilon) + \varepsilon|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon\big].\eta = T_{\frac{1}{\varepsilon}}\big(- |u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\big) & \text{on } \partial\Omega,\end{cases}$$

where

$$-\triangle_{p_+} u_\varepsilon := -\nabla.(|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon).$$

We define the following reflexive space

$$E = W^{1,p_+}(\Omega) \times L^{p_+}(\partial\Omega).$$

Let

$$X_0 = \{(u, v) \in E : \ \ v = \tau(u)\}.$$

We will identify an element $(u, v) \in X_0$ with its representative $u \in W^{1,p_+}(\Omega)$ in this paper.
In this part, we prove that the problem $P(\beta_\epsilon, f_\varepsilon)$ admits at least one weak solution u_ε.

Theorem 3.4. *There exists at least one weak solution u_ε for the problem $P(\beta_\epsilon, f_\varepsilon)$ in the sense that $u_\varepsilon \in X_0$ and for all $v \in X_0$,*

$$\begin{aligned}\int_\Omega \beta_\varepsilon(u_\varepsilon)vdx + \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla vdx \ &+ \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon)vd\sigma \\ &+ \ \varepsilon \int_\Omega \big[|u_\varepsilon|^{p_+-2}u_\varepsilon v + |\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon \nabla v\big]dx \\ &= \int_\Omega f_\varepsilon vdx. \end{aligned} \tag{3.3}$$

Let

$$< A_\varepsilon u, v >=< Au, v > + < G_\varepsilon u, v >,$$

where

$$< Au, v >= \int_\Omega a(x, u, \nabla u).\nabla vdx, \ \ < G_\varepsilon u, v >= \varepsilon \int_\Omega |\nabla u|^{p_+-2}\nabla u\nabla vdx$$

and

$$< B_\varepsilon u, v >= \int_\Omega \beta_\varepsilon(u)vdx + \varepsilon \int_\Omega |u|^{p_+-2}uvdx + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(|u|^{s(.)-2}u)vd\sigma,$$

with $u, v \in X_0$.
Set $C_\varepsilon = A_\varepsilon + B_\varepsilon$.

Proof of Theorem 3.4. The proof of Theorem 3.4 is done in three claims.

Claim 1: C_ε is bounded.
From Hölder type inequality and relation (1.3) with constant exponent p_+, one deduces that A is bounded. Moreover, using the same argument as in [4]-Proof of

Lemma 4.2 and the fact that β_ε is a lipschitz continuous function with $\beta_\varepsilon(0) = 0$ and as $W^{1,p_+}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p_+ > N$, one proves that $G_\varepsilon + B_\varepsilon$ is bounded. Therefore, C_ε is bounded.

Claim 2. C_ε is of type (M).
Let $(u_\delta)_{\delta>0}$ be a sequence in X_0 such that

$$\begin{cases} u_\delta \rightharpoonup u \text{ in } X_0 \\ C_\varepsilon u_\delta \rightharpoonup \chi \text{ in } X_0' \\ \limsup_{\delta\to 0} < C_\varepsilon u_\delta, u_\delta >=< \chi, u > . \end{cases}$$

We will prove that $\chi = C_\varepsilon u$.
By Fatou's Lemma, we deduce that

$$\liminf_{\delta\to 0}\Big(\int_\Omega \beta_\varepsilon(u_\delta)u_\delta dx + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(|u_\delta|^{s(.)-2}u_\delta)u_\delta d\sigma + \varepsilon\int_\Omega |u_\delta|^{p_+}dx\Big)$$
$$\geq \int_\Omega \beta_\varepsilon(u)udx + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}\big(|u|^{s(.)-2}u\big)ud\sigma + \varepsilon\int_\Omega |u|^{p_+}dx.$$

Moreover, thanks to the Lebesgue dominated convergence Theorem and the fact that

$$|u_\delta|^{p_+-2}u_\delta \rightharpoonup |u|^{p_+-2}u \text{ in } L^{p_+'}(\Omega),$$

we obtain

$$\lim_{\delta\to 0}\Big(\int_\Omega \beta_\varepsilon(u_\delta)vdx + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(|u_\delta|^{s(.)-2}u_\delta)vd\sigma + \varepsilon\int_\Omega |u_\delta|^{p_+-2}u_\delta vdx\Big)$$
$$= \int_\Omega \beta_\varepsilon(u)vdx + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}\big(|u|^{s(.)-2}u\big)vd\sigma + \varepsilon\int_\Omega |u|^{p_+-2}uvdx,$$

for any $v \in X_0$. Thus, as δ goes to 0, it follows that

$$B_\varepsilon u_\delta \rightharpoonup \beta_\varepsilon(u) + T_{\frac{1}{\varepsilon}}\big(|u|^{s(.)-2}u\big) + \varepsilon|u|^{p_+-2}u \text{ in } X_0'.$$

So, it follows that

$$A_\varepsilon u_\delta \rightharpoonup \chi - \Big(\beta_\varepsilon(u) + T_{\frac{1}{\varepsilon}}\big(|u|^{s(.)-2}u\big) + \varepsilon|u|^{p_+-2}u\Big) \text{ in } X_0', \text{ as } \delta\to 0.$$

It remains to prove that A_ε is of type (M). For this, we first show that G_ε is bounded, monotone and hemicontinuous.
From Claim 1, G_ε is bounded. Let us prove that G_ε is monotone. For all $u, v \in W^{1,p_+}(\Omega)$, we have

$$< G_\varepsilon u - G_\varepsilon v, u - v >= \varepsilon\int_\Omega \big(|\nabla u|^{p_+-2}\nabla u - |\nabla v|^{p_+-2}\nabla v\big)\big(\nabla u - \nabla v\big)dx \geq 0,$$

since $\xi \mapsto |\xi|^{p_+-2}\xi$ is increasing as $p_+ > 2$.
Moreover, G_ε is hemicontinuous. Indeed, let $g : t \in \mathbb{R} \mapsto g(t) =< G_\varepsilon(u+tv), v >$ and $t, t_0 \in \mathbb{R}$ such that $t \to t_0$. Let us set $w = u + tv \in W^{1,p_+}(\Omega)$ and $w_0 = u + t_0 v \in W^{1,p_+}(\Omega)$. Then,

$$\|w - w_0\|_{W^{1,p_+}(\Omega)} = \|(t-t_0)v\|_{W^{1,p_+}(\Omega)} = |t-t_0|\|v\|_{W^{1,p_+}(\Omega)} \to 0.$$

So $w \to w_0$ in $W^{1,p_+}(\Omega)$, as $t \to t_0$, which implies that $\nabla w \to \nabla w_0$ in $L^{p_+}(\Omega)$ and we infer that

$$|\nabla w|^{p_+-2}\nabla w \to |\nabla w_0|^{p_+-2}\nabla w_0 \text{ in } L^{p'_+}(\Omega) \text{ as } t \to t_0.$$

Therefore,

$$\begin{aligned} |g(t)-g(t_0)| &= |< G_\varepsilon(u+tv), v> - < G_\varepsilon(u+t_0v), v>| \\ &\le \varepsilon \int_\Omega \big||\nabla w|^{p_+-2}\nabla w - |\nabla w_0|^{p_+-2}\nabla w_0\big||\nabla v|dx \\ &\le \||\nabla w|^{p_+-2}\nabla w - |\nabla w_0|^{p_+-2}\nabla w_0\|_{L^{p'_+}(\Omega)}\|\nabla v\|_{L^{p_+}(\Omega)} \to 0. \end{aligned}$$

Then, we deduce that g is continuous, namely the operator G_ε is hemicontinuous. Additionnaly, A is pseudo-monotone (See [16]-proof of Theorem 3.1). Therefore, it follows from Lemma 2.10 that $A_\varepsilon = A + G_\varepsilon$ is pseudo-monotone. So, the operator A_ε is of type (M) (see, [13]-Proposition 2.5) and one immediately has

$$Au + G_\varepsilon u = \chi - \big(\beta_\varepsilon(u) + T_{\frac{1}{\varepsilon}}\big(|u|^{s(.)-2}u\big) + \varepsilon|u|^{p_+-2}u\big).$$

Thus, one obtains $C_\varepsilon u = \chi$.

Claim 3: C_ε is coercive.
Using (1.4) with constant exponent, we get

$$\begin{aligned} < C_\varepsilon u, u > &= \int_\Omega a(x,u,\nabla u).\nabla u dx + \int_\Omega \beta_\varepsilon(u)u dx \\ &+ \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}\big(|u|^{s(.)-2}u\big)u dx + \varepsilon\int_\Omega \big[|u|^{p_+} + |\nabla u|^{p_+}\big]dx \\ &\ge \frac{1}{C_2}\int_\Omega |\nabla u|^{p_+}dx + \varepsilon\int_\Omega |u|^{p_+}dx \\ &\ge C||u||^{p_+}_{W^{1,p_+}(\Omega)}, \text{ where } C = \min\left\{\frac{1}{C_2}, \varepsilon\right\}. \end{aligned}$$

We deduce that

$$\frac{< C_\varepsilon u, u >}{||u||_{W^{1,p_+}(\Omega)}} \to \infty \text{ as } ||u||_{W^{1,p_+}(\Omega)} \to \infty.$$

Hence, C_ε is coercive. Then, according to Lemma 2.11, C_ε is surjective.
Let $f_\varepsilon = T_{\frac{1}{\varepsilon}}(f) \in X'_0$, then there exists at least one solution $u_\varepsilon \in X_0$ of the problem

$$< C_\varepsilon u_\varepsilon, v > = < f_\varepsilon, v >, \text{ for all } v \in X_0.$$

Therefore, u_ε is a weak solution of the problem $P(\beta_\varepsilon, f_\varepsilon)$. □

Step 2. A priori estimates.
This step is divided into several lemmas.

Lemma 3.5. *(i) The sequence $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$.*
(ii)

$$\int_{\partial\Omega} \big|T_{\frac{1}{\varepsilon}}\big(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\big)\big|d\sigma \le \|f\|_{L^1(\Omega)}. \tag{3.4}$$

(iii) The sequence $(u_\varepsilon)_{\varepsilon>0}$ *is such that*

$$\|u_\varepsilon\|_{W^{1,p_-}(\Omega)} \leq const\big(p_-, \Omega, f\big). \tag{3.5}$$

Proof. Taking $v = T_k(u_\varepsilon)$ in (3.3) and the fact that all the terms of the left hand side in the resulting equality are nonnegative, one deduces that

$$\int_\Omega \beta_\varepsilon(u_\varepsilon) T_k(u_\varepsilon) dx \leq \int_\Omega f_\varepsilon T_k(u_\varepsilon) dx \leq k\|f\|_{L^1(\Omega)}$$

and

$$\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}\big(|u_\varepsilon|^{s_\varepsilon(.)-2} u_\varepsilon\big) T_k(u_\varepsilon) d\sigma \leq \int_\Omega f_\varepsilon T_k(u_\varepsilon) dx \leq k\|f\|_{L^1(\Omega)}.$$

Dividing the above inequalities by k and letting k goes to 0, it follows that

$$\int_\Omega \big|\beta_\varepsilon(u_\varepsilon)\big| dx \leq \|f\|_{L^1(\Omega)}$$

and

$$\int_{\partial\Omega} \Big|T_{\frac{1}{\varepsilon}}\big(|u_\varepsilon|^{s_\varepsilon(.)-2} u_\varepsilon\big)\Big| d\sigma \leq \|f\|_{L^1(\Omega)}.$$

Hence, (i) and (ii) are proved.
(iii) Moreover, β is a Lipschitz, continuous nondecreasing and surjective function, then $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$. So, there exists a positive constant C_3 such that

$$\int_\Omega |u_\varepsilon| dx \leq C_3. \tag{3.6}$$

Hence,

$$\tilde{u}_\varepsilon := \frac{1}{meas(\Omega)} \int_\Omega u_\varepsilon dx \leq const(\Omega).$$

Furthermore, by using Poincaré-Wirtinger inequality, one has

$$\int_\Omega |u_\varepsilon - \tilde{u}_\varepsilon|^{p_-} dx \leq const(p_-, \Omega) \int_\Omega |\nabla u_\varepsilon|^{p_-} dx.$$

Therefore,

$$\int_\Omega |u_\varepsilon|^{p_-} dx \leq const(p_-, \Omega) \int_\Omega |\nabla u_\varepsilon|^{p_-} dx + \left| \int_\Omega \tilde{u_\varepsilon} dx \right|^{p_-}.$$

Thus,

$$\int_\Omega |u_\varepsilon|^{p_-} dx \leq C_4 \int_\Omega |\nabla u_\varepsilon|^{p_-} dx + C_5, \tag{3.7}$$

where $C_4 = cons(p_-, \Omega)$ and $C_5 = const(p_-, C_3, \Omega)$ are two positive constants.
Moreover, since $W^{1,p_-}(\Omega) \hookrightarrow L^\infty(\Omega)$, there exists a positive constant C_6 such that

$$\|u_\varepsilon\|^{p_-}_{L^\infty(\Omega)} \leq C_6 \|u_\varepsilon\|^{p_-}_{W^{1,p_-}(\Omega)}. \tag{3.8}$$

Using (1.4) on $a(x,u_\varepsilon,\nabla u_\varepsilon)$ and Theorem 3.4, the sequence u_ε satisfies

$$\begin{aligned}\int_\Omega \beta_\varepsilon(u_\varepsilon)u_\varepsilon dx+\frac{1}{C_2}\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx &+ \varepsilon\Big(\int_\Omega |\nabla u_\varepsilon|^{p_+}dx+\int_\Omega |u_\varepsilon|^{p_+}dx\Big)\\ &+ \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}\left(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\right)u_\varepsilon d\sigma\\ &\leq \int_\Omega f_\varepsilon u_\varepsilon dx. \qquad (3.9)\end{aligned}$$

Applying Young inequality on the right hand side of (3.9) and using (3.8), it follows

$$\begin{aligned}\int_\Omega f_\varepsilon u_\varepsilon dx &\leq \int_\Omega |f||u_\varepsilon|dx\\ &\leq \|f\|_{L^1(\Omega)}\|u_\varepsilon\|_{L^\infty(\Omega)}\\ &= \left(\frac{2C_2C_6(C_4+1)}{p_-}\right)^{\frac{1}{p_-}}\|f\|_{L^1(\Omega)}.\left(\frac{p_-}{2C_2C_6(C_4+1)}\right)^{\frac{1}{p_-}}\|u_\varepsilon\|_{L^\infty(\Omega)}\\ &\leq \frac{1}{p'_-}\left(\frac{2C_2C_6(C_4+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\|f\|^{p'_-}_{L^1(\Omega)}+\frac{1}{p_-}\frac{p_-}{2C_2C_6(C_4+1)}\|u_\varepsilon\|^{p_-}_{L^\infty(\Omega)}\\ &\leq \frac{1}{p'_-}\left(\frac{2C_2C_6(C_4+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\|f\|^{p'_-}_{L^1(\Omega)}+\frac{1}{2C_2(C_4+1)}\|u_\varepsilon\|^{p_-}_{W^{1,p_-}(\Omega)}.\end{aligned} \qquad (3.10)$$

Moreover, one has

$$\int_\Omega |\nabla u_\varepsilon|^{p_-}dx \leq meas(\Omega)+\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx, \qquad (3.11)$$

since $p_- < \pi_\varepsilon(.)$. Combining (3.7) and (3.11), we get

$$\int_\Omega |u_\varepsilon|^{p_-}dx \leq C_4 meas(\Omega)+C_4\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx+C_5.$$

We infer from the above inequality and (3.11) that

$$\begin{aligned}\|u_\varepsilon\|^{p_-}_{W^{1,p_-}(\Omega)} &= \int_\Omega \left[|u_\varepsilon|^{p_-}+|\nabla u_\varepsilon|^{p_-}\right]dx\\ &\leq (C_4+1)meas(\Omega)+C_5+(C_4+1)\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx. \qquad (3.12)\end{aligned}$$

Furthermore, using (3.10) and (3.12), we get

$$\begin{aligned}\int_\Omega f_\varepsilon u_\varepsilon dx &\leq \frac{1}{p'_-}\left(\frac{2C_2C_6(C_4+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\|f\|^{p'_-}_{L^1(\Omega)}+\frac{meas(\Omega)}{2C_2}+\frac{C_5}{2C_2(C_4+1)}\\ &+ \frac{1}{2C_2}\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx. \qquad (3.13)\end{aligned}$$

Combining (3.9) and (3.13), one has

$$\int_\Omega \beta_\varepsilon(u_\varepsilon)u_\varepsilon dx + \frac{1}{2C_2}\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx + \varepsilon||u_\varepsilon||^{p_+}_{W^{1,p_+}(\Omega)} + \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}\big(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\big)u_\varepsilon d\sigma \leq C_7, \tag{3.14}$$

where

$$C_7 = \frac{1}{p'_-}\left(\frac{2C_2C_6(C_4+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\|f\|^{p'_-}_{L^1(\Omega)} + \frac{meas(\Omega)}{2C_2} + \frac{C_5}{2C_2(C_4+1)}.$$

Thus, we deduce from (3.14) that

$$\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx \leq C_8. \tag{3.15}$$

Now, using (3.12) and the last inequality, we infer

$$\|u_\varepsilon\|_{W^{1,p_-}(\Omega)} \leq const\big(p_-, \Omega, f\big)$$

and (iii) follows. □

Lemma 3.6. *$(u_\varepsilon)_{\varepsilon>0}$ converges a.e. to v on $\partial\Omega$.*

Proof. $(u_\varepsilon)_{\varepsilon>0}$ being uniformly bounded in $W^{1,p_-}(\Omega)$, then, up to extraction of a subsequence still denoted $(u_\varepsilon)_{\varepsilon>0}$, it converges a.e. in Ω (and also weakly in $W^{1,p_-}(\Omega)$) to a limit u.
We know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$. Note that, $W^{1,p_-}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ because $p_- > 1$. Therefore, $u_\varepsilon \to u$ in $L^1(\partial\Omega)$ and a.e. on $\partial\Omega$, as $\varepsilon \to 0$. Thus, $v = u_{|\partial\Omega}$ has definite meaning. □

Remark 3.7. From (3.5) and the compact embedding $W^{1,p_-}(\Omega) \hookrightarrow L^{p_-}(\Omega)$, we get for a subsequence still labelled with ε and a function u, that

$$u_\varepsilon \rightharpoonup u \text{ in } W^{1,p_-}(\Omega), \text{ as } \varepsilon \to 0; \tag{3.16}$$
$$\nabla u_\varepsilon \rightharpoonup \nabla u \text{ in } L^{p_-}(\Omega), \text{ as } \varepsilon \to 0; \tag{3.17}$$
$$u_\varepsilon \to u \text{ in } L^{p_-}(\Omega), \text{ as } \varepsilon \to 0; \tag{3.18}$$
$$u_\varepsilon \to u \text{ a.e. in } \Omega, \text{ as } \varepsilon \to 0. \tag{3.19}$$

To establish that $|\nabla u|^{\pi(.)} \in L^1(\Omega)$, we need the following result due to Chipot and de Oliveira in [9].

Lemma 3.8. *(See[9], Lemma 3.1)*
Assume that

$$1 < p_- \leq \pi_\varepsilon(x) \leq p_+ < \infty, \forall\varepsilon > 0, \quad a.e.\ x \in \Omega; \tag{3.20}$$
$$\pi_\varepsilon(.) \to \pi(.) \ a.e.\ in\ \Omega, \ as\ \varepsilon \to 0; \tag{3.21}$$
$$\nabla u_\varepsilon \rightharpoonup \nabla u \ in\ L^1(\Omega), \ as\ \varepsilon \to 0; \tag{3.22}$$
$$\||\nabla u_\varepsilon|^{\pi_\varepsilon(x)}\|_{L^1(\Omega)} \leq C; \tag{3.23}$$

with C a positive constant not depending on ε. Then, $|\nabla u| \in L^{\pi(.)}(\Omega)$ and

$$\int_\Omega |\nabla u|^{\pi(x)}dx \leq \liminf_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(x)}dx. \tag{3.24}$$

Lemma 3.9. *Assume that $(u_\varepsilon)_{\varepsilon>0}$ converges a.e. in Ω to a function u, as $\varepsilon \to 0$, then $\pi_\varepsilon(.) \to \pi(.)$ a.e. in Ω as $\varepsilon \to 0$. Furthermore, $u \in W^{1,\pi(.)}(\Omega)$.*

Proof. It should be noticed that, due to (1.5), $p(.,.)$ is locally uniformly log-Hölder continuous. Then, $p(.,.)$ is locally uniformly continuous.
Moreover, $1 < p_- \leq \pi_\varepsilon(x) \leq p_+ < \infty$, $\forall \varepsilon > 0$, a.e. $x \in \Omega$ and $u_\varepsilon \in W^{1,p_-}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$ for $p_- > N$. So,

$$p(x, u_\varepsilon(x)) \to p(x, u(x)) \text{ a.e. in } \Omega \Longleftrightarrow \pi_\varepsilon(.) \to \pi(.) \text{ a.e. in } \Omega, \text{ as } \varepsilon \to 0. \quad (3.25)$$

Now, using (3.15), (3.17), (3.25) and according to Lemma 3.8, we get $|\nabla u| \in L^{\pi(.)}(\Omega)$. We also have $u \in W^{1,p_-}(\Omega) \subset L^\infty(\Omega) \subset L^{\pi(.)}(\Omega)$. Hence, $u \in W^{1,\pi(.)}(\Omega)$. □

Lemma 3.10. *(i) $|u|^{s(.)-2}u \in L^1(\partial\Omega)$.*
(ii) $(\nabla u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$.
(iii) $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$.
(iv) $u \in dom(\beta)$ $\mathcal{L}^N-$a.e. in Ω and $T_k(u) \in W^{1,\pi(.)}(\Omega)$.

Proof. (i) From Fatou's Lemma and (3.4), we obtain

$$\int_{\partial\Omega} \left||u|^{s(.)-2}u\right| d\sigma \leq \liminf_{\varepsilon\to 0} \int_{\partial\Omega} \left|T_{\frac{1}{\varepsilon}}\left(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\right)\right| d\sigma \leq \|f\|_{L^1(\Omega)}.$$

(ii) Using (3.5) and the fact that $p_- > 1$, we get

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon| dx &\leq \int_\Omega \left(1 + |\nabla u_\varepsilon|^{p_-}\right) dx \\ &\leq meas(\Omega) + \|u_\varepsilon\|^{p_-}_{W^{1,p_-}(\Omega)} \\ &\leq C, \end{aligned}$$

with C a positive constant depending on $meas(\Omega)$ and p_-. Thus, as Ω is bounded, (ii) follows.
(iii) follows from (3.6).
(iv) Since $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$ and $u_\varepsilon \to u$ in $L^1(\Omega)$, as $\varepsilon \to 0$, then $u \in dom(\beta)$ $\mathcal{L}^N-$a.e. in Ω, thanks to Lemma 2.7. Furthermore, as $u \in W^{1,\pi(.)}(\Omega)$, then $T_k(u) \in W^{1,\pi(.)}(\Omega)$, thanks to Lemma 2.4. □

Lemma 3.11. *For all $\delta > 0$, for all $\sigma > 0$, there exists $\eta(\sigma) > 0$ such that*

$$\forall \varepsilon \leq \eta(\sigma), \;\; meas(\{|\nabla u_\varepsilon - \nabla u| \geq \delta\}) \leq \sigma.$$

Proof. Notice that for all $k > 0$ and $\beta > 0$, one has

$$\begin{aligned} & \{|\nabla u_\varepsilon - \nabla u| \geq \delta\} \\ \subset\; & \{|u_\varepsilon| \geq k\} \cup \{|u| \geq k\} \cup \{|\nabla u_\varepsilon| \geq k\} \cup \{|\nabla u| \geq k\} \cup \{|u_\varepsilon - u| \geq \beta\} \cup \\ & \{|\nabla u_\varepsilon - \nabla u| \geq \delta,\ |u_\varepsilon| \leq k,\ |u| \leq k,\ |\nabla u_\varepsilon| \leq k,\ |\nabla u| \leq k, |u_\varepsilon - u| \leq \beta\}. \end{aligned}$$

We will denoted $E_1,\ E_2,\ ...,E_6$ respectively the six sets of the right hand side of above relation. Since $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$, we obtain

$$\int_\Omega |u_\varepsilon|dx \geq \int_{E_1} |u_\varepsilon|dx \geq k meas(E_1).$$

So,

$$meas(E_1) \leq \frac{1}{k}\int_\Omega |u_\varepsilon|dx \leq \frac{C_3}{k} < \sigma.$$

Moreover, $(\nabla u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$ and $u \in W^{1,p_-}(\Omega) \subset W^{1,1}(\Omega)$. Then, there exists $k_0 > 0$ such that $meas(E_2) < \sigma$, $meas(E_3) < \sigma$ and $meas(E_4) < \sigma$, for every $k > k_0$.
Furthermore, $u_\varepsilon \to u$ a.e. in Ω, as $\varepsilon \to 0$. So, there exists $\eta_1 > 0$ such that for all $\varepsilon \leq \eta_1$, $meas(E_5) < \sigma$. It remains to prove that $meas(E_6)$ is bounded. Using (1.2), one has

$$\big[a(x,s,\xi_1) - a(x,s,\xi_2)\big]\big(\xi_1 - \xi_2\big) > 0 \text{ for all } \xi_1 \neq \xi_2.$$

Notice that,

$$\big\{(s,\xi_1,\xi_2) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N;\ |s| \leq k,\ |\xi_1| \leq k,\ |\xi_2| \leq k,\ |\xi_1 - \xi_2| \geq \delta\big\}$$

is a compact set and $a(x,.,.)$ is continuous in $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$ a.e. $x \in \Omega$.
So, $\big[a(x,s,\xi_1)-a(x,s,\xi_2)\big]\big(\xi_1-\xi_2\big)$ has a minimun denoted $\gamma(x)$ with $\gamma(x) > 0$ a.e. $x \in \Omega$, on this compact set. Moreover, the absolute continuity of the Lebesgue integral guarantees that, there exists $\sigma' > 0$ such that

$$\int_{E_6} \gamma(x)dx \leq \sigma' \Rightarrow meas(E_6) \leq \sigma.$$

Now, our aims is to prove that

$$\int_{E_6} \gamma(x)dx \leq \sigma'.$$

One has

$$\begin{aligned}
\int_{E_6} \gamma(x)dx &\leq \int_{E_6} \big[a(x,u_\varepsilon,\nabla u_\varepsilon) - a(x,u_\varepsilon,\nabla T_k(u))\big]\big(\nabla u_\varepsilon - \nabla T_k(u)\big)dx\\
&= \int_\Omega \big[a(x,u_\varepsilon,\nabla u_\varepsilon) - a(x,u_\varepsilon,\nabla T_k(u))\big]\big(\nabla u_\varepsilon - \nabla T_k(u)\big)\chi_{\{|u_\varepsilon - T_k(u)|\leq\beta\}}dx\\
&= \int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla T_\beta\big(u_\varepsilon - T_k(u)\big)dx\\
&- \int_\Omega a(x,u_\varepsilon,\nabla T_k(u))\nabla T_\beta\big(u_\varepsilon - T_k(u)\big)dx,
\end{aligned} \tag{3.26}$$

since on E_6, $|u_\varepsilon - T_k(u)| = |u_\varepsilon - u| \leq \beta$ and
$\nabla T_\beta\big(u_\varepsilon - T_k(u)\big) = \big(\nabla u_\varepsilon - \nabla T_k(u)\big)\chi_{\{|u_\varepsilon - T_k(u)|\leq\beta\}}$. For the second integral of the

right hand side of (3.26), we have

$$\begin{aligned}&\int_\Omega a(x,u_\varepsilon,\nabla T_k(u))\nabla T_\beta\big(u_\varepsilon - T_k(u)\big)dx\\&=\int_\Omega a(x,T_{k+\beta}(u_\varepsilon),\nabla T_k(u))\nabla T_\beta\big(T_{k+\beta}(u_\varepsilon)-T_k(u)\big)dx,\end{aligned}\tag{3.27}$$

since, if $|u_\varepsilon|\geq k+\beta$ then $|u_\varepsilon - T_k(u)|\geq\beta$ and $\nabla T_\beta\big(T_{k+\beta}(u_\varepsilon)-T_k(u)\big)=0$.
Taking $v=T_h(u_\varepsilon)\in W^{1,p_+}(\Omega)\cap L^\infty(\Omega)$ in (3.3) with $h>0$, we get

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla T_h(u_\varepsilon)dx\leq h\|f\|_{L^1(\Omega)}.$$

In fact, with the choice $v=T_h(u_\varepsilon)$, the other terms of the left hand side of (3.3) are nonnegative.
Using (1.4), we have

$$\begin{aligned}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla T_h(u_\varepsilon)dx &= \int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\chi_{\{|u_\varepsilon|\leq h\}}dx\\ &\geq \frac{1}{C_2}\int_\Omega|\nabla u_\varepsilon|^{p_+}\chi_{\{|u_\varepsilon|\leq h\}}dx=\frac{1}{C_2}\int_\Omega|\nabla T_h(u_\varepsilon)|^{p_+}dx,\end{aligned}$$

which implies that $\|\nabla T_h(u_\varepsilon)\|_{L^{p_+}(\Omega)}\leq C$. Furthermore, from Lemma 3.10-(iii), $(T_h(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$. So, $(T_h(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^{p_+}(\Omega)$, thanks to Poincaré Wirtinger inequality. Therefore, $\big(T_h(u_\varepsilon)\big)_{\varepsilon>0}$ is uniformly bounded in $W^{1,p_+}(\Omega)$ for all $h>0$.
Let us set $h=k+\beta$. Then, for a subsequence still labelled with ε, we deduce that

$$T_{k+\beta}(u_\varepsilon)\rightharpoonup T_{k+\beta}(u)\ \text{ in }\ W^{1,p_+}(\Omega),\ \text{ as }\ \varepsilon\to 0\tag{3.28}$$

$$T_{k+\beta}(u_\varepsilon)\to T_{k+\beta}(u)\ \text{ in }\ L^{p_+}(\Omega),\ \text{ as }\ \varepsilon\to 0,\tag{3.29}$$

$$T_{k+\beta}(u_\varepsilon)\to T_{k+\beta}(u)\ \text{ a.e. in }\ \Omega,\ \text{ as }\ \varepsilon\to 0.\tag{3.30}$$

Thus,

$$a(x,T_{k+\beta}(u_\varepsilon),\nabla T_k(u))\to a(x,T_{k+\beta}(u),\nabla T_k(u))\ \text{ a.e. in }\ \Omega,\ \text{ as }\ \varepsilon\to 0.$$

Moreover, $|a(x,T_{k+\beta}(u_\varepsilon),\nabla T_k(u))|^{(p_+)'}$ is equi-integrable on Ω. Therefore, thanks to Vitali's Theorem,

$$a(x,T_{k+\beta}(u_\varepsilon),\nabla T_k(u))\to a(x,T_{k+\beta}(u),\nabla T_k(u))\ \text{ in }\ L^{(p_+)'}(\Omega),\ \text{ as }\ \varepsilon\to 0.$$

Since

$$\chi_{\{|u_\varepsilon-T_k(u)|\leq\beta\}}\to\chi_{\{|u-T_k(u)|\leq\beta\}}\ \text{ a.e. in }\ \Omega,\ \text{ as }\ \varepsilon\to 0,$$

and it is bounded, then, using Lebesgue dominated convergence theorem,

$$\begin{aligned}a(x,T_{k+\beta}(u_\varepsilon),\nabla T_k(u))\chi_{\{|u_\varepsilon-T_k(u)|\leq\beta\}}\ &\to\ a(x,T_{k+\beta}(u),\nabla T_k(u))\chi_{\{|u-T_k(u)|\leq\beta\}}\\ &\qquad\text{in }\ L^{(p_+)'}(\Omega),\ \text{ as }\ \varepsilon\to 0.\end{aligned}\tag{3.31}$$

Moreover,

$$\nabla T_{k+\beta}(u_\varepsilon)-\nabla T_k(u)\rightharpoonup\nabla T_{k+\beta}(u)-\nabla T_k(u)\ \text{ in }\ L^{p_+}(\Omega),\ \text{ as }\ \varepsilon\to 0,\tag{3.32}$$

thanks to (3.28). Thus, combining (3.31) and (3.32), one obtains

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,T_{k+\beta}(u_\varepsilon),\nabla T_k(u))\nabla T_\beta\big(T_{k+\beta}(u_\varepsilon)-T_k(u)\big)dx$$
$$=\int_\Omega a(x,T_{k+\beta}(u),\nabla T_k(u))\nabla T_\beta\big(T_{k+\beta}(u)-T_k(u)\big)dx. \tag{3.33}$$

Since $\nabla T_\beta\big(T_{k+\beta}(u)-T_k(u)\big)\to 0$ a.e. in Ω, as $\beta\to 0$, then, for $\beta<1$, $\nabla T_\beta\big(T_{k+\beta}(u)-T_k(u)\big)\le \nabla T_1\big(T_{k+1}(u)-T_k(u)\big)\in L^{p_+}(\Omega)$ and $a(x,T_{k+\beta}(u),\nabla T_k(u))\in L^{(p_+)'}(\Omega)$. So,

$$\lim_{\beta\to 0}\int_\Omega a(x,T_{k+\beta}(u),\nabla T_k(u))\nabla T_\beta\big(T_{k+\beta}(u)-T_k(u)\big)dx=0. \tag{3.34}$$

For the first integral of the right hand side of (3.26), choosing $v=T_\beta\big(u_\varepsilon-T_k(u)\big)\in W^{1,p_+}(\Omega)\cap L^\infty(\Omega)$ in (3.3) and using Lemma 3.5-(i) and (ii), we get

$$\begin{aligned}
&\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla T_\beta\big(u_\varepsilon-T_k(u)\big)dx+\int_\Omega \varepsilon|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon.\nabla T_\beta\big(u_\varepsilon-T_k(u)\big)dx\\
+&\int_\Omega \varepsilon|u_\varepsilon|^{p_+-2}u_\varepsilon.T_\beta\big(u_\varepsilon-T_k(u)\big)dx\\
=&\int_\Omega fT_\beta\big(u_\varepsilon-T_k(u)\big)dx-\int_\Omega \beta_\varepsilon(u_\varepsilon)T_\beta\big(u_\varepsilon-T_k(u)\big)dx\\
-&\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}\left(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\right)T_\beta\big(u_\varepsilon-T_k(u)\big)d\sigma\\
\le\ & 3\beta\|f\|_{L^1(\Omega)}:=C\beta.
\end{aligned}\tag{3.35}$$

The relation (3.35) is equivalent to

$$\begin{aligned}
&\int_\Omega a(x,T_{k+\beta}(u_\varepsilon),\nabla T_{k+\beta}(u_\varepsilon))\nabla T_\beta\big(T_{k+\beta}(u_\varepsilon)-T_k(u)\big)dx\\
\le\ &-\int_\Omega \varepsilon|\nabla T_{k+\beta}(u_\varepsilon)|^{p_+-2}\nabla T_{k+\beta}(u_\varepsilon)\nabla T_\beta\big(T_{k+\beta}(u_\varepsilon)-T_k(u)\big)dx\\
-&\int_\Omega \varepsilon|u_\varepsilon|^{p_+-2}u_\varepsilon T_\beta\big(u_\varepsilon-T_k(u)\big)dx+C\beta.
\end{aligned}\tag{3.36}$$

Moreover,

$$\lim_{\varepsilon\to 0}\varepsilon\left[\int_\Omega |\nabla T_{k+\beta}(u_\varepsilon)|^{p_+-2}\nabla T_{k+\beta}(u_\varepsilon).\nabla T_\beta\big(T_{k+\beta}(u_\varepsilon)-T_k(u)\big)+|u_\varepsilon|^{p_+-2}u_\varepsilon T_\beta\big(u_\varepsilon-T_k(u)\big)\right]dx=0,$$

thanks to (3.14) and the fact that $\big(T_{k+\beta}(u_\varepsilon)\big)_{\varepsilon>0}$ is uniformly bounded in $W^{1,p_+}(\Omega)$.

Let us fix $\beta<\dfrac{\sigma'}{3C}$. Then, there exists $\eta_2>0$ such that $\forall\varepsilon\le\eta_2$,

$$\begin{aligned}
&\left|\varepsilon\int_\Omega \left[|\nabla T_{k+\beta}(u_\varepsilon)|^{p_+-2}\nabla T_{k+\beta}(u_\varepsilon).\nabla T_\beta\big(T_{k+\beta}(u_\varepsilon)-T_k(u)\big)+|u_\varepsilon|^{p_+-2}u_\varepsilon T_\beta\big(u_\varepsilon-T_k(u)\big)\right]dx\right|\\
&\le\ \frac{\sigma'}{3}
\end{aligned}\tag{3.37}$$

and

$$\int_\Omega a(x, T_{k+\beta}(u), \nabla T_k(u))\nabla T_\beta\big(T_{k+\beta}(u) - T_k(u)\big)dx \leq \frac{\sigma'}{6}, \quad (3.38)$$

thanks to (3.34).
Thus, choosing $\eta_3 > 0$ such that $\forall \varepsilon \leq \eta_3$,

$$\Big|\int_\Omega a(x, T_{k+\beta}(u_\varepsilon), \nabla T_k(u))\nabla T_\beta\big(T_{k+\beta}(u_\varepsilon) - T_k(u)\big)dx - \int_\Omega a(x, T_{k+\beta}(u), \nabla T_k(u))\nabla T_\beta\big(T_{k+\beta}(u) - T_k(u)\big)dx\Big| \leq \frac{\sigma'}{6}. \quad (3.39)$$

Now, combining (3.38) and (3.39), we obtain

$$\begin{aligned} &\Big|\int_\Omega a(x, u_\varepsilon, \nabla T_k(u))\nabla T_\beta\big(u_\varepsilon - T_k(u)\big)dx\Big| \\ = &\Big|\int_\Omega a(x, T_{k+\beta}(u_\varepsilon), \nabla T_k(u))\nabla T_\beta\big(T_{k+\beta}(u_\varepsilon) - T_k(u)\big)dx\Big| \\ \leq &\frac{\sigma'}{3}. \end{aligned} \quad (3.40)$$

Finally, it follows from (3.36), (3.37) and (3.40) that

$$\int_{E_6} \gamma(x)dx \leq \frac{C\sigma'}{3C} + \frac{\sigma'}{3} + \frac{\sigma'}{3} = \sigma'.$$

Therefore, an appropriate choice of σ' implies that $meas(E_6) \leq \sigma$. So, $\forall \varepsilon \leq \min(\eta_1, \eta_2, \eta_3)$, $meas\big(\{|\nabla u_\varepsilon - \nabla u| \geq \delta\}\big) \leq 6\sigma$ and $\big(\nabla u_\varepsilon\big)_{\varepsilon>0}$ converges to ∇u in measure, as $\varepsilon \to 0$. Then, up to a subsequence still labelled with ε, $\big(\nabla u_\varepsilon\big)_{\varepsilon>0}$ converges to ∇u a.e. in Ω, as $\varepsilon \to 0$. □

Lemma 3.12. *For all $\varphi \in C^\infty(\overline{\Omega})$ and $h \in C_c^1(\mathbb{R})$,*

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi)dx \to \int_\Omega a(x, u, \nabla u).\nabla(h(u)\varphi)dx, \quad \text{as } \varepsilon \to 0. \quad (3.41)$$

Proof. From (3.19) and Lemma 3.11, we have

$$a(x, u_\varepsilon, \nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi) \to a(x, u, \nabla u).\nabla(h(u)\varphi) \text{ a.e. in } \Omega, \text{ as } \varepsilon \to 0. \quad (3.42)$$

It remains to prove that $\big(a(x, u_\varepsilon, \nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi)\big)_{\varepsilon>0}$ is equi-integrable in order to use Vitali's convergence theorem to get the convergence in $L^1(\Omega)$.
Let $E \subset \Omega$, it follows from Young inequality that

$$\begin{aligned} &\int_E \big|a(x, u_\varepsilon, \nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi)\big|dx \\ \leq &\int_E \frac{\big|a(x, u_\varepsilon, \nabla u_\varepsilon)\big|^{\pi'_\varepsilon(.)}}{\pi'_\varepsilon(.)}dx + \int_E \frac{\big|\nabla(h(u_\varepsilon)\varphi)\big|^{\pi_\varepsilon(.)}}{\pi_\varepsilon(.)}dx \\ \leq &\int_E \big|a(x, u_\varepsilon, \nabla u_\varepsilon)\big|^{\pi'_\varepsilon(.)}dx + \int_E \big|\nabla(h(u_\varepsilon)\varphi)\big|^{\pi_\varepsilon(.)}dx. \end{aligned} \quad (3.43)$$

Using (1.3) for the first term of the right hand side of (3.43), we have

$$\begin{aligned}\int_E \big|a(x,u_\varepsilon,\nabla u_\varepsilon)\big|^{\pi'_\varepsilon(.)}dx &\le \int_E C_1\big(\mathcal{M}(x)+|\nabla u_\varepsilon\big|^{\pi_\varepsilon(.)}\big)dx\\ &= C_1\int_E \mathcal{M}(x)dx + C_1\int_E |\nabla u_\varepsilon\big|^{\pi_\varepsilon(.)}dx.\end{aligned}$$

So, it follows that $\int_E \big|a(x,u_\varepsilon,\nabla u_\varepsilon)\big|^{\pi'_\varepsilon(.)}dx$ is small for $meas(E)$ small enough, since $\mathcal{M}\in L^1(\Omega)$ and $|\nabla u_\varepsilon\big|^{\pi_\varepsilon(.)}\in L^1(\Omega)$.
For the second term of the right hand side of (3.43), we first recall that

$$\nabla(h(u_\varepsilon)\varphi) = h'(u_\varepsilon)\varphi\nabla u_\varepsilon + h(u_\varepsilon)\nabla\varphi.$$

Since $h\in C^1_c(\mathbb{R})$ and $\varphi\in L^\infty(\Omega)$, then $|h'(u_\varepsilon)\varphi|\le K_1$ and $|h(u_\varepsilon)|\le K_2$, where K_1 and K_2 are some positive constants. It follows that

$$|\nabla(h(u_\varepsilon)\varphi)| \le K_1|\nabla u_\varepsilon| + K_2|\nabla\varphi|.$$

We recall that

$$\frac{1}{2^p}(a+b)^p \leqslant \frac{1}{2}(a^p+b^p),$$

for all $a,b>0$ and $p>1$. Thus, for all set $E\subset\Omega$,

$$\begin{aligned}\int_E |\nabla(h(u_\varepsilon)\varphi)|^{\pi_\varepsilon(.)}dx &\le \int_E 2^{\pi_\varepsilon(.)-1}\big[(K_1|\nabla u_\varepsilon|)^{\pi_\varepsilon(.)} + (K_2|\nabla\varphi|)^{\pi_\varepsilon(.)}\big]dx\\ &\le \int_E 2^{p_+-1}\big[(K_1|\nabla u_\varepsilon|)^{\pi_\varepsilon(.)} + 1 + (K_2|\nabla\varphi|)^{p_+}\big)\big]dx.\end{aligned} \tag{3.44}$$

From (3.15) and the density argument between $C^\infty(\overline{\Omega})$ and $W^{1,p_+}(\Omega)$, it follows that $(K_1|\nabla u_\varepsilon|)^{\pi_\varepsilon(.)}\in L^1(\Omega)$ and $(K_2|\nabla\varphi|)^{p_+}\in L^1(\Omega)$. So, the left hand side of (3.44) is small for $meas(E)$ small enough. Therefore, $\int_E \big|a(x,u_\varepsilon,\nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi)\big|dx$ is small for $meas(E)$ small enough. Hence, $\big(a(x,u_\varepsilon,\nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi)\big)_{\varepsilon>0}$ is equi-integrable. Finally, (3.41) follows from Vitali's convergence theorem. □

Lemma 3.13. *The problem $P(\beta,f)$ admits at least one renormalized solution.*

Proof. Let us consider $\varphi\in C^\infty(\overline{\Omega})$, $h\in C^1_c(\mathbb{R})$ and $h(u_\varepsilon)\varphi$ as a test fonction in (3.3), one has

$$\begin{aligned}&\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi)dx\\ +\ &\varepsilon\int_\Omega \big[|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon.\nabla(h(u_\varepsilon)\varphi) + |u_\varepsilon|^{p_+-2}u_\varepsilon h(u_\varepsilon)\varphi\big]dx\\ +\ &\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}\big(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\big)h(u_\varepsilon)\varphi d\sigma + \int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)\varphi dx\\ =\ &\int_\Omega fh(u_\varepsilon)\varphi dx.\end{aligned} \tag{3.45}$$

Since $\big(\beta_\varepsilon(u_\varepsilon)\big)_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$. Then, up to a subsequence still labelled with ε, there exists $z \in \mathcal{M}_b(\Omega)$, such that

$$\beta_\varepsilon(u_\varepsilon) \rightharpoonup^* z \text{ in } \mathcal{M}_b(\Omega), \text{ as } \varepsilon \to 0. \tag{3.46}$$

We consider the following decomposition.

$$\int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)\varphi dx = \int_\Omega \beta_\varepsilon(u_\varepsilon)\big[h(u_\varepsilon) - h(u)\big]\varphi dx + \int_\Omega \beta_\varepsilon(u_\varepsilon)h(u)\varphi dx. \tag{3.47}$$

The sequence $\Big(\beta_\varepsilon(u_\varepsilon)\big[h(u_\varepsilon) - h(u)\big]\varphi\Big)_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$ and it converges to 0 a.e. in Ω, as $\varepsilon \to 0$. Therefore, thanks to Vitali's Theorem, the first term of the right hand side of (3.47) converges to 0, as $\varepsilon \to 0$. For the second term of the right hand side of (3.47), one has

$$\lim_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(u_\varepsilon)h(u)\varphi dx = \int_\Omega h(u)\varphi dz,$$

due to (3.46), the fact that $h(u)\varphi \in C_c(\Omega)$ and the density argument between $C_c(\Omega)$ and $C_0(\Omega)$. Thus, passing to the limit in (3.47), as $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)\varphi dx = \int_\Omega h(u)\varphi dz. \tag{3.48}$$

For the right hand side of (3.45), we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega fh(u_\varepsilon)\varphi dx = \int_\Omega fh(u)\varphi dx, \tag{3.49}$$

thanks to the Lebesgue dominated convergence theorem. For the first term of the left hand side of (3.45), using Lemma 3.12, one gets.

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi)dx = \int_\Omega a(x,u,\nabla u).\nabla(h(u)\varphi)dx. \tag{3.50}$$

For the second term of the left hand side of (3.45), we have

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega \big[|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon.\nabla(h(u_\varepsilon)\varphi) + |u_\varepsilon|^{p_+-2}u_\varepsilon(h(u_\varepsilon)\varphi)\big]dx = 0. \tag{3.51}$$

For the third term of the left hand side of (3.45), we obtain

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}\big(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\big)h(u_\varepsilon)\varphi d\sigma = \int_{\partial\Omega}|u|^{s(.)-2}uh(u)\varphi d\sigma. \tag{3.52}$$

Indeed, $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $W^{1,p_-}(\Omega)$ and $W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $p_- > N$. Then, $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\partial\Omega)$. Moreover, $t \mapsto |t|^{r(.,t)-2}t$ is continuous a.e on $\partial\Omega$. Therefore, $\big(|u_\varepsilon|^{r(.,u_\varepsilon)-2}u_\varepsilon\big)_{\varepsilon>0} := \big(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\big)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\partial\Omega)$ and

$$\big|T_{\frac{1}{\varepsilon}}\big(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\big)\big| \leq \big||u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\big|.$$

So, $\big(T_{\frac{1}{\varepsilon}}(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon)\big)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\partial\Omega)$. Thus, there exists a positive constant C such that

$$\big|T_{\frac{1}{\varepsilon}}\big(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\big)h(u_\varepsilon)\varphi\big| \leq C|\varphi| \text{ a.e. on } \partial\Omega.$$

Furthermore, $T_{\frac{1}{\varepsilon}}\left(|u_\varepsilon|^{s_\varepsilon(.)-2}u_\varepsilon\right)h(u_\varepsilon)\varphi \to |u|^{s(.)-2}uh(u)\varphi$ a.e. on $\partial\Omega$ and (3.52) follows from Lebesgue's dominated convergence theorem. Then, using (3.48), (3.49), (3.50), (3.51) and (3.52), we infer from (3.45) that

$$\begin{aligned}\int_\Omega h(u)\varphi dz &= \int_\Omega fh(u)\varphi dx - \int_\Omega a(x,u,\nabla u).\nabla(h(u)\varphi)dx \\ &- \int_{\partial\Omega} |u|^{s(.)-2}uh(u)\varphi d\sigma,\end{aligned} \tag{3.53}$$

for all $\varphi \in C^\infty(\overline{\Omega})$.

Remark 3.14. Since $u \in W^{1,p_-}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$ for $p_- > N$ and $p(.,.)$ is locally uniformly log-Hölder continuous, then $p(.,u(.)) := \pi(.)$ verifies (1.5). Therefore, $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(.)}(\Omega)$. Thus, the relation (3.53) holds true with $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$.

In other words,

$$\begin{aligned}\int_\Omega h(u)\varphi dz &= \int_\Omega fh(u)\varphi dx - \int_\Omega a(x,u,\nabla u).\nabla(h(u)\varphi)dx \\ &- \int_{\partial\Omega} |u|^{s(.)-2}uh(u)\varphi d\sigma,\end{aligned} \tag{3.54}$$

for all $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$, which implies that $z \in \mathcal{M}_b^{\pi(.)}(\Omega)$.
Now, we give a Radon Nikodym decomposition result of the measure z.

Lemma 3.15. *The Radon Nikodym decomposition of the measure z given by (3.54) with respect to $\mathcal{L}^N$,*

$$z = w\mathcal{L}^N + \mu, \ \textit{with } \mu \perp \mathcal{L}^N,$$

satisfies the following properties.

$$\begin{cases} w \in \beta(u)\mathcal{L}^N - \ a.e. \ in \ \Omega, \ w \in L^1(\Omega), \ \mu \in \mathcal{M}_b^{\pi(.)}(\Omega), \\ \mu^+ \ is \ concentrated \ on \ \left[u = M\right] \cap \left[u \neq \infty\right] and \\ \mu^- \ is \ concentrated \ on \ \left[u = m\right] \cap \left[u \neq -\infty\right]. \end{cases}$$

Proof. For the proof of the Lemma 3.15, we use the arguments of [12]-Lemma 3.2. Let $(z_\varepsilon)_{\varepsilon>0}$ be a subsequence of $\left(\beta_\varepsilon(u_\varepsilon)\right)_{\varepsilon>0}$ such that $z_\varepsilon \rightharpoonup^* z$ in $\mathcal{M}_b(\Omega)$.
Since, for any $\varepsilon > 0$, $z_\varepsilon \in \partial j_\varepsilon(u_\varepsilon)$, we have

$$j(t) \geq j_\varepsilon(t) \geq j_\varepsilon(u_\varepsilon) + (t-u_\varepsilon)z_\varepsilon \ \mathcal{L}^N - a.e. \text{ in } \Omega, \ \forall t \in \mathbb{R}.$$

Then, for any $h \in C_c^1(\mathbb{R})$, $h \geq 0$ and $\psi \geq 0$, we have

$$\psi h(u_\varepsilon)j(t) \geq \psi h(u_\varepsilon)j_\varepsilon(u_\varepsilon) + (t-u_\varepsilon)\psi h(u_\varepsilon)z_\varepsilon \ \mathcal{L}^N - a.e. \text{ in } \Omega, \ \forall t \in \mathbb{R}.$$

Moreover, for any $0 < \varepsilon < \tilde{\varepsilon}$, we have

$$\psi h(u_\varepsilon)j(t) \geq \psi h(u_\varepsilon)j_{\tilde{\varepsilon}}(u_\varepsilon) + (t-u_\varepsilon)\psi h(u_\varepsilon)z_\varepsilon \ \mathcal{L}^N - a.e. \text{ in } \Omega, \ \forall t \in \mathbb{R},$$

and integrating over Ω, we get

$$\int_{\Omega} \psi h(u_\varepsilon) j(t) dx \geq \int_{\Omega} \psi h(u_\varepsilon) j_{\tilde{\varepsilon}}(u_\varepsilon) dx + \int_{\Omega} (t - u_\varepsilon) \psi h(u_\varepsilon) z_\varepsilon dx.$$

As $\varepsilon \to 0$, using Fatou's Lemma, we obtain

$$\int_{\Omega} \psi h(u) j(t) dx \geq \int_{\Omega} \psi h(u) j_{\tilde{\varepsilon}}(u) dx + \liminf_{\varepsilon \to 0} \int_{\Omega} (t - u_\varepsilon) \psi h(u_\varepsilon) z_\varepsilon dx.$$

Now, for any $\psi \in C_c^1(\Omega)$ and $t \in \mathbb{R}$, let $\tilde{h}(r) = (t - r)h(r)$, we have

$$\lim_{\varepsilon \to 0} \int_{\Omega} (t - u_\varepsilon) \psi h(u_\varepsilon) z_\varepsilon dx = \lim_{\varepsilon \to 0} \int_{\Omega} \tilde{h}(u_\varepsilon) \psi z_\varepsilon dx = \int_{\Omega} (t - u) h(u) \psi dz.$$

So,

$$\int_{\Omega} \psi h(u) j(t) dx \geq \int_{\Omega} \psi h(u) j_{\tilde{\varepsilon}}(u) dx + \int_{\Omega} (t - u) h(u) \psi dz.$$

As $\tilde{\varepsilon} \to 0$, using Fatou's Lemma, one has

$$\int_{\Omega} \psi h(u) j(t) dx \geq \int_{\Omega} \psi h(u) j(u) dx + \int_{\Omega} (t - u) h(u) \psi dz.$$

From the last inequality, we infer

$$h(u) j(t) \geq h(u) j(u) + (t - u) h(u) z \text{ in } \mathcal{M}_b(\Omega), \ \ \forall t \in \mathbb{R}. \tag{3.55}$$

Using the Radon Nikodym decomposition of z, we have $z = w\mathcal{L}^N + \mu$ with $\mu \perp \mathcal{L}^N$, $w \in L^1(\Omega)$, then comparing the regular part and singular part of (3.55), for any $h \in C_c^1(\mathbb{R})$, we obtain

$$h(u) j(t) \geq h(u) j(u) + (t - u) h(u) w \ \ \mathcal{L}^N(\Omega) - \text{ a.e. in } \Omega, \ \ \forall t \in \mathbb{R} \tag{3.56}$$

and

$$(t - u) h(u) \mu \leq 0 \text{ in } \mathcal{M}_b(\Omega), \ \ \forall t \in \overline{dom(j)}. \tag{3.57}$$

From (3.56) we get

$$j(t) \geq j(u) + (t - u) w \ \ \mathcal{L}^N(\Omega) - \text{ a.e. in } \Omega, \ \ \forall t \in \mathbb{R}.$$

So, $w \in \partial j(u)$ $\mathcal{L}^N(\Omega)-$ a.e. in Ω. Relation (3.57) implies that for any $t \in \overline{dom(j)}$,

$$\mu \geq 0 \ \text{ in } [u \in (t, \infty) \cap supp(h)] \tag{3.58}$$

and

$$\mu \leq 0 \ \text{ in } [u \in (-\infty, t) \cap supp(h)]. \tag{3.59}$$

In particular, this implies that

$$\mu([m < u < M]) = 0.$$

Furthermore, if $m \neq -\infty$ (resp. $M \neq \infty$) then, (3.58) (resp. (3.59)) implies that μ^+ is concentrated on $\big[u = M\big] \cap \big[u \neq \infty\big]$(resp. μ^- is concentrated on $\big[u = m\big] \cap \big[u \neq -\infty\big]$). By construction of the measure z, one sees that

$$\mu([u = \pm\infty]) = 0.$$

□

Furthermore, using the Radon Nikodym decomposition of measure z, the first term of (3.54) becomes

$$\int_\Omega h(u)\varphi dz = \int_\Omega h(u)w\varphi dx + \int_\Omega h(u)\varphi d\mu. \tag{3.60}$$

Combining (3.54) and (3.60), we infer

$$\begin{aligned}\int_\Omega h(u)w\varphi dx + \int_\Omega a(x,u,\nabla u).\nabla(h(u)\varphi)dx \;&-\; \int_{\partial\Omega} |u|^{s(.)-2}uh(u)\varphi d\sigma \\ &+\; \int_\Omega h(u)\varphi d\mu = \int_\Omega fh(u)\varphi dx,\end{aligned} \tag{3.61}$$

for all $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$.
It remains to prove that

$$\lim_{M\to\infty} \int_{\left[M<|u|<M+1\right]} a(x,u,\nabla u).\nabla u dx = 0, \tag{3.62}$$

to end the proof of Theorem 3.3.

Since $|\nabla u| \in L^{\pi(.)}(\Omega)$, then using (1.3) with variable exponent $\pi(.)$, $a(x,u,\nabla u) \in L^{\pi'(.)}(\Omega)$ and it follows from Hölder type inequality with variable exponent, that $a(x,u,\nabla u).\nabla u \in L^1(\Omega)$.
Moreover, $u \in W^{1,p_-}(\Omega) \subset W^{1,1}(\Omega)$, which implies that $\int_\Omega |u|dx < \infty$. Therefore,

$$\int_\Omega |u|dx \geq \int_{\left[|u|\geq M\right]} |u|dx \geq M meas(\left[|u| \geq M\right]).$$

Then,

$$meas(\left[|u| \geq M\right]) \leq \frac{1}{M}\int_\Omega |u|dx \leq \frac{C}{M},$$

for any $M > 0$ and C a positive constant not depending of M. So,

$$meas(\left[|u| \geq M\right]) \to 0, \text{ as } M \to \infty.$$

Now, one has

$$\begin{aligned}\int_{\left[M<|u|<M+1\right]} a(x,u,\nabla u).\nabla u dx \;&\leq\; \int_{\left[|u|>M\right]} a(x,u,\nabla u).\nabla u dx \\ &\leq\; \int_{\left[|u|\geq M\right]} a(x,u,\nabla u).\nabla u dx.\end{aligned} \tag{3.63}$$

Since $a(x,u,\nabla u).\nabla u \in L^1(\Omega)$ and $meas(\left[|u| \geq M\right]) \to 0$, as $M \to \infty$, one obtains

$$\lim_{M\to\infty} \int_{\left[|u|\geq M\right]} a(x,u,\nabla u).\nabla u dx = 0.$$

Hence,

$$\lim_{M\to\infty}\int_{\left[M<|u|<M+1\right]} a(x,u,\nabla u).\nabla u dx = 0.$$

Finally, using Lemma 3.10, Lemma 3.15, (3.61) and (3.62), one deduces that $(u,\ w)$ is a renormalized solution of problem $P(\beta,\ f)$. □

This conclude the proof of the existence result. □

To prove the uniqueness result, we make the following hypotheses on the function a, namely the local Lipschitz continuity with respect to z.
For all bounded subset K of $\mathbb{R}\times\mathbb{R}^N$, there exists a constant $C(K)$ such that

$$\begin{aligned}&\text{a.e. } x\in\Omega,\ \text{ for all } (z,\eta),(\tilde{z},\eta)\in K,\\ &|a(x,z,\eta)-a(x,\tilde{z},\eta)|\leq C(K)|z-\tilde{z}|.\end{aligned} \tag{3.64}$$

Remark 3.16. Let (u,w) be a renormalized solution of the problem $P(\beta,f)$, then $u\in C(\overline{\Omega})$, since $u\in W^{1,p_-}(\Omega)$ and $p_->N$. Moreover, if u is a Lipschitz continuous function, then $u\in W^{1,\infty}(\Omega)$. Indeed, Ω is an open bounded domain with smooth boundary $\partial\Omega$, so, the space of Lipschitz functions $C^{0,1}(\overline{\Omega})$ and $W^{1,\infty}(\Omega)$ are homeomorphic and they can be identified. The uniqueness in the sense of Theorem 3.3 seems difficult to prove. Therefore, our partial uniqueness result reduces to the case where the domain of β is bounded.

Theorem 3.17. *Assume (1.1)-(1.5), (H) and (3.64), with $\mathcal{M}$ in (1.3) which can be taken constant. Assume that $-\infty<m\leq 0\leq M<\infty$. Moreover, assume that $f\in L^1(\Omega)$ such that the problem $P(\beta,f)$ has a solution (u,w) in the sense that u is a measurable and Lipschitz continuous function such that $T_k(u)\in W^{1,\pi(.)}(\Omega)$ for all $k>0$, $|u|^{s(.)-2}u\in L^1(\partial\Omega)$ and $u\in dom(\beta)\,\mathcal{L}^N$ a.e. Ω, $w\in L^1(\Omega)$ and $w\in\beta(u)\,\mathcal{L}^N$ a.e. Ω; and there exists a measure $\mu\in\mathcal{M}_b^{\pi(.)}(\Omega)$ such that $\mu\perp\mathcal{L}^N$, μ^+ is concentrated on $\left[u=M\right]$, μ^- is concentrated on $\left[u=m\right]$ such that*

$$\int_\Omega w\varphi dx+\int_\Omega a(x,u,\nabla u)\nabla\varphi dx+\int_\Omega\varphi d\mu+\int_{\partial\Omega}|u|^{s(.)-2}u\varphi d\sigma=\int_\Omega f\varphi dx, \tag{3.65}$$

for all $\varphi\in W^{1,\pi(.)}(\Omega)\cap L^\infty(\Omega)$.
Then, any other solution $(\tilde{u},\tilde{w})$ of the problem $P(\beta,f)$ in the sense of equality (3.65) associated to $\tilde{\mu}\in\mathcal{M}_b^{\pi(.)}(\Omega)$, partially coincides with (u,w), which is

$$\begin{cases} u=\tilde{u} & a.e.\ \ on\ \partial\Omega,\\ w=\tilde{w} & a.e.\ \ in\ \Omega\\ \ and & \\ \mu=\tilde{\mu}. & \end{cases}$$

First of all, to complete the proof of the existence, we justify the relation (3.65). We consider the function h_k where k is a positive constant such that

$$\begin{cases} h_k \in C_c^1(\mathbb{R}), \;\; h_k(s) \geq 0, \; \forall s \in \mathbb{R}, \\ \\ h_k(s) = 1 \text{ if } |s| < k \text{ and } h_k(s) = 0 \text{ if } |s| \geq k. \end{cases}$$

Since the domain of β is bounded and the equality (3.61) holds for any $h \in C_c^1(\mathbb{R})$, we take $h(s) = h_k(s) = 1$ for all $s \in \big[m,\, M\big] \subsetneq \big[-k,\, k\big] = supp(h_k)$, which implies

$$\int_\Omega w\varphi dx + \int_\Omega a(x,u,\nabla u).\nabla\varphi dx + \int_\Omega \varphi d\mu + \int_{\partial\Omega} |u|^{s(.)-2}u\varphi d\sigma = \int_\Omega f\varphi dx, \tag{3.66}$$

for all $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$.

Proof. Now, we prove the uniqueness result. For more details, see [2]-Proof of Theorem 2.8.

Let (u_1, w_1) be a solution of the problem $P(\beta, f)$, with u_1 a Lipschitz continuous function and (u_2, w_2) be an other solution of the problem $P(\beta, f)$ in the sense of the equality (3.65).

Let $\phi := \dfrac{1}{k}T_k(u_1 - u_2)$, then ϕ is an admissible test function in the formulations for both (u_1, w_1) and (u_2, w_2). So, with this test function, one has

$$\begin{aligned} \int_\Omega w_1 \frac{1}{k}T_k(u_1-u_2)dx \;\; &+ \;\; \frac{1}{k}\int_\Omega a(x,u_1,\nabla u_1).\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx \\ &+ \;\; \int_\Omega \frac{1}{k}T_k(u_1-u_2)d\mu_1 + \int_{\partial\Omega} |u_1|^{r(.,u_1)-2}u_1\frac{1}{k}T_k(u_1-u_2)d\sigma \\ &= \;\; \int_\Omega f\frac{1}{k}T_k(u_1-u_2)dx \end{aligned} \tag{3.67}$$

and

$$\begin{aligned} \int_\Omega w_2 \frac{1}{k}T_k(u_1-u_2)dx \;\; &+ \;\; \frac{1}{k}\int_\Omega a(x,u_2,\nabla u_2).\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx \\ &+ \;\; \int_\Omega \frac{1}{k}T_k(u_1-u_2)d\mu_2 + \int_{\partial\Omega} |u_2|^{r(.,u_2)-2}u_2\frac{1}{k}T_k(u_1-u_2)d\sigma \\ &= \;\; \int_\Omega f\frac{1}{k}T_k(u_1-u_2)dx. \end{aligned} \tag{3.68}$$

We substract (3.67) and (3.68) to get

$$\begin{aligned} &\frac{1}{k}\int_\Omega \big(a(x,u_1,\nabla u_1) - a(x,u_2,\nabla u_2)\big).\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx \\ + \;\; &\int_\Omega \big(w_1 - w_2\big)\frac{1}{k}T_k(u_1-u_2)dx + \int_\Omega \frac{1}{k}T_k(u_1-u_2)(d\mu_1 - d\mu_2) \\ + \;\; &\int_{\partial\Omega} \frac{1}{k}T_k(u_1-u_2)(|u_1|^{r(.,u_1)-2}u_1 - |u_2|^{r(.,u_2)-2}u_2)d\sigma = 0. \end{aligned} \tag{3.69}$$

Let's denote by I the first term of the left hand side of (3.69). We know that

$$\begin{aligned}(a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_2))\nabla(u_1-u_2) &= (a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_1))\nabla(u_1-u_2)\\ &+ \underbrace{(a(x,u_2,\nabla u_1)-a(x,u_2,\nabla u_2))\nabla(u_1-u_2)}_{\geq 0}.\end{aligned}$$

One has

$$I = I_k + \int_\Omega \big(a(x,u_2,\nabla u_1)-a(x,u_2,\nabla u_2)\big)\frac{1}{k}\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx,$$

where

$$I_k = \int_\Omega \big(a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_1)\big)\frac{1}{k}\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx.$$

Let's show that $I_k \to 0$ as $k \to 0$. Since u_1 is bounded, then u_2 is also bounded on the set $[0<|u_1-u_2|<k]$. Thus, using (3.64), one obtains

$$\begin{aligned}|I_k| &\leq \frac{1}{k}\int_{[0<|u_1-u_2|<k]} |a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_1)||\nabla u_1-\nabla u_2|dx\\ &\leq \frac{1}{k}\int_{[0<|u_1-u_2|<k]} C\big(||u_1||_{L^\infty(\Omega)},||\nabla u_1||_{L^\infty(\Omega)}\big)|u_1-u_2||\nabla u_1-\nabla u_2|dx\\ &\leq C\big(||u_1||_{L^\infty(\Omega)},||\nabla u_1||_{L^\infty(\Omega)}\big)\int_{[0<|u_1-u_2|<k]} |\nabla u_1-\nabla u_2|dx \to 0, \text{ as } k\to 0.\end{aligned} \tag{3.70}$$

Notice that $\lim_{k\to 0} meas([0<|u_1-u_2|<k]) = 0$ and $|\nabla u_1-\nabla u_2| \in L^1(\Omega)$.
For the second term of the left hand side of (3.69), one has

$$\begin{aligned}\lim_{k\to 0}\int_\Omega (w_1-w_2)\frac{1}{k}T_k(u_1-u_2)dx &= \int_\Omega (w_1-w_2)sign_0(u_1-u_2)dx\\ &= \int_\Omega |w_1-w_2|dx.\end{aligned} \tag{3.71}$$

For the third term of the left hand side of (3.69), one has

$$\begin{aligned}\lim_{k\to 0}\int_\Omega \frac{1}{k}T_k(u_1-u_2)(d\mu_1-d\mu_2) &= \int_\Omega sign_0(u_1-u_2)(d\mu_1-d\mu_2)\\ &= \int_\Omega |d\mu_1-d\mu_2|\\ &= \int_\Omega |d(\mu_1-\mu_2)|.\end{aligned} \tag{3.72}$$

For the last term of the left hand side of (3.69), one obtains

$$\begin{aligned}&\lim_{k\to 0}\int_{\partial\Omega}\frac{1}{k}T_k(u_1-u_2)(|u_1|^{r(.,u_1)-2}u_1-|u_2|^{r(.,u_2)-2}u_2)d\sigma\\ &= \int_\Omega \Big||u_1|^{r(.,u_1)-2}u_1-|u_2|^{r(.,u_2)-2}u_2\Big|d\sigma.\end{aligned} \tag{3.73}$$

Finally, one lets k goes to 0 in (3.69) and taking into account inequalities (3.70), (3.71), (3.72) and (3.74), it follows that

$$\lim_{k\to 0}\int_\Omega \big(a(x,u_2,\nabla u_1)-a(x,u_2,\nabla u_2)\big)\frac{1}{k}\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx$$
$$+\int_\Omega |w_1-w_2|dx+\int_\Omega |d(\mu_1-\mu_2)|+\int_{\partial\Omega}\big||u_1|^{r(.,u_1)-2}u_1-|u_2|^{r(.,u_2)-2}u_2\big|d\sigma=0. \tag{3.74}$$

Since all the terms of equality (3.74) are nonnegative, one deduces that

$$\int_\Omega |w_1-w_2|dx=0,\ \int_\Omega |d(\mu_1-\mu_2)|=0 \text{ and } \int_{\partial\Omega}\big||u_1|^{r(.,u_1)-2}u_1-|u_2|^{r(.,u_2)-2}u_2\big|d\sigma=0.$$

Hence,

$$w_1=w_2 \ \text{ a.e. in } \ \Omega,\ \ \mu_1=\mu_2 \text{ and } |u_1|^{r(.,u_1)-2}u_1=|u_2|^{r(.,u_2)-2}u_2 \text{ a.e. on } \ \partial\Omega.$$

From assumption (H) and the fact that $|u_1|^{r(.,u_1)-2}u_1=|u_2|^{r(.,u_2)-2}u_2$ a.e. on $\partial\Omega$, one gets $u_1=u_2$ a.e. on $\partial\Omega$.

In summary, one gets

$$\begin{cases} u_1=u_2 \text{ a.e. on } \partial\Omega,\\ w_1=w_2 \text{ a.e. in } \Omega\\ \text{ and}\\ \mu_1=\mu_2.\end{cases}$$

□

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