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ON SOME ANISOTROPIC UNILATERAL ELLIPTIC PROBLEMS WITH MEASURE DATA

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Abstract. Here, we are concerned with the existence results for some non-linear unilateral elliptic problem in generalized Sobolev space.

1. **Introduction and Basic Assumptions**

Let Ω be a bounded open subset of $\mathbb{R}^N(N \geq 2)$. Boccardo et al have studied in [12] an elliptic problem modeled by

$$\begin{cases} B\varpi = f & \text{in } \Omega \\ \varpi = 0 & \text{on } \partial\Omega \end{cases}$$

where $B\varpi = -\operatorname{div} b(x, \varpi, \nabla\varpi)$, and the data f supposed be a bounded Radon measure on Ω. They have established the existence and some regularity result of solutions $\varpi \in W_0^{1,q}(\Omega)$ for all $1 < q < \bar{q} = \frac{N(p-1)}{N-1}$. On the other hand, Boccardo in [11] has considered the quasilinear elliptic problem of the type

$$\begin{cases} B\varpi = f - \operatorname{div} \phi(\varpi) & \text{in } \Omega \\ \varpi = 0 & \text{on } \partial\Omega \end{cases}$$

with $B\varpi = -\operatorname{div} b(x, \varpi, \nabla\varpi)$, $f \in L^1(\Omega)$ and $\phi(\cdot) \in C^0\left(\mathbb{R}, \mathbb{R}^N\right)$. He has proved the existence and some regularity of solutions.

In the framework of Orlicz Sobolev spaces, Aharouch and Bennouna [1] have been shown the existence and uniqueness of entropy solutions $W_0^1 L_M(\Omega)$ without any restriction on the N-function M of the Orlicz spaces for some elliptic unilateral problem as

$$\begin{cases} -\operatorname{div}(b(x, \nabla\varpi)) = f & \text{in } \Omega, \\ \varpi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$.

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In the Sobolev spaces with variable exponents, Bendahmane and Wittbold have established the existence and uniqueness solution in their work [8] some elliptic equation of the form $-\operatorname{div}\left(|\nabla \varpi|^{p(x)-2}\nabla\varpi\right) = f$ in Ω where $f \in L^1(\Omega)$.

Our aim in the present paper, is to study the anisotropic unilateral problem looks like

$$\begin{cases} \sum_{i=1}^N D^i b_i(x,\varpi,\nabla\varpi) + \Phi(x,\varpi,\nabla\varpi) + \Psi(x,,\varpi,\nabla\varpi) = f - \operatorname{div} F \text{ in } \Omega \\ \\ \varpi = 0 \text{ on } \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega), F \in \prod_{i=1}^N L^{p_i'}(\Omega)$. The function $\Psi(x,,s,\xi)$ satisfy only the hypothesis

$$\Psi(x,,s,\xi)| \leq f_0(x) + d(|s|)\sum_{i=1}^N |\xi|^{p_i(x)},$$

with $f_0(\cdot) \in L^1(\Omega)$ and $d(\cdot) \in L^1(\mathbb{R})$, without any sign condition.

The mathematical researches dealing the existence of solutions to some problems parabolic and elliptic under a different assumptions is massive; we refer the reader to ([3, 10, 6, 7, 4, 5, 15, 16] for more details).

The great difficulty in this study lies in the fact that the term $B\varpi$ is not coercive in generalized space $W_0^{1,p_i}(\Omega)$, More precisely in the anisotropic Sobolev space. to control this difficulty, we will use a penalization term $\frac{1}{n}|\varpi|^{p_0-2}\varpi$ in the approximations (4.2) to show our desired result.

The outline of this note is as follows. After giving some auxiliary results on anisotropic Sobolev space, we state in Section 3 some essential assumptions which are necessary to have an existence solution, finally section 4 will be devoted to give our major Conclusions.

2. **Preliminary**

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ with smooth boundary and, let $1 < p_1, \ldots, p_N < \infty$ be N real numbers,

$$p^+ = \max\{p_1,\ldots,p_N\}, p^- = \min\{p_1,\ldots,p_N\} \text{ and } \vec{p} = (p_1,\ldots,p_N).$$

We denote

$$\partial_i = \frac{\partial}{\partial x_i},$$

We set

$$\underline{\vec{p}} = \min\{p_0,p_1,p_2,\ldots,p_N\} \quad \text{then} \quad \underline{\vec{p}} > 1. \tag{2.1}$$

We define the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ like :

$$W^{1,\vec{p}}(\Omega) = \left\{\varpi \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i\varpi \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1,2,\ldots,N\right\},$$

endowed with the norm

$$\|\varpi\|_{1,\vec{p}} = \|\varpi\|_{1,1} + \sum_{i=1}^N \|D^i\varpi\|_{L^{p_i}(\Omega)}. \tag{2.2}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|\varpi\|_{1,\vec{p}}\right)$ is a reflexive Banach (separable) space.

The closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ symbolized by $W_0^{1,\vec{p}}(\Omega)$.

Lemma 2.1. *(see. [13, 17])*
Let $\varpi \in W_0^{1,\vec{p}}(\Omega)$, we have
***(i)** : there exists $C_p > 0$ / $\|\varpi\|_{L^{p_i}(\Omega)} \leq C_p \sum_{i=1}^N \|D^i\varpi\|_{L^{p_i}(\Omega)}$ for any $i = 1, \dots, N$.*
***(ii)** : there exists $C_s > 0$ / $\|\varpi\|_{L^q(\Omega)} \leq \frac{C_s}{N} \sum_{i=1}^N \left\|\frac{\partial\varpi}{\partial x_i}\right\|_{L^{p_i}(\Omega)}$, where*

$$\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N-\overline{p}} & \text{if} \quad \overline{p} < N \\ q \in [1, +\infty[& \text{if} \quad \overline{p} \geq N \end{cases}$$

Lemma 2.2. *Let Ω be a bounded open set in $I\!R^N$ $(N \geq 2)$, we set*

$$s = \max(q, \max_{1\leq i\leq N} p_i),$$

therefore, the embedding listed below holds :

- *if $\overline{p} < N$ so $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, s[$,*
- *if $\overline{p} = N$ therefore $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, +\infty[$,*
- *if $\overline{p} > N$ we have $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ is compact.*

The proof of the above result (lemma 2.2) depends to the lemma 2.1.

Definition 2.3. For $k > 0$, we give the following truncation $T_k(\cdot) : I\!R \longmapsto I\!R$, that will be used latter

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we put

$$\mathcal{T}_0^{1,\vec{p}}(\Omega) := \{\varpi : \Omega \mapsto I\!R \text{ measurable } /T_k(\varpi) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Lemma 2.4. *Let $\varpi \in \mathcal{T}_0^{1,\vec{p}}(\Omega)$, there exists one function $v_i : \Omega \mapsto \mathbb{R}$ measurable with $i \in \{1, \dots, N\}$, such that*

$$\forall k > 0 \qquad D^i T_k(\varpi) = v_i.\chi_{\{|\varpi|<k\}} \qquad a.e. \quad x \in \Omega,$$

with χ_A be a characteristic function of a measurable set A. v_i define the weak partial derivatives of ϖ denoted by $D^i\varpi$. Therefore, if $\varpi \in W_0^{1,1}(\Omega)$, then ($v_i = D^i\varpi$.)

Lemma 2.5. [14] *Let $(\varpi_n)_n \longrightarrow \varpi \in L^1(\Omega)$ with*

(i): *$\varpi_n \to \varpi$ a.e. in Ω,*
(ii): *$\varpi_n \geq 0$ and $\varpi \geq 0$ a.e. in Ω,*
(iii): $\displaystyle\int_\Omega \varpi_n\,dx \to \int_\Omega \varpi\,dx$,

then $\varpi_n \to \varpi$ in $L^1(\Omega)$.

For the proof we use the seem methods developed in [9] in Sobolev spaces.

3. Essential assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, and consider the vector $\vec{p} = (p_0, p_1, \ldots, p_N)$ such that $1 < p_i < \infty$ for $i = 0, 1, \ldots, N$.

Taking ϖ as a measurable function on Ω with values in $\overline{\mathbb{R}}$, such that

$$\varpi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

let us consider the following convex $K_\varpi = \left\{v \in W_0^{1,p_i}(\Omega) \text{ such that } v \geq \varpi \text{ a.e. in } \Omega\right\}$

Now, we define the operator B acted from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $W^{-1,\vec{p}'(\cdot)}(\Omega)$, by

$$B\varpi = -\sum_{i=1}^{N} \partial^i b_i(x, \varpi, \nabla\varpi),$$

with $b_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}$ satisfy the assumptions listed bellow as follows

$$|b_i(x, s, \xi)| \leq \beta \left(R_i(x) + |s|^{\underline{p}-1} + |\xi_i|^{p_i-1}\right) \text{ for any } i = 1, \ldots, N, \tag{3.1}$$

$$b_i(x, s, \xi)\xi_i \geq \alpha\, |\xi_i|^{p_i} \quad \text{ for any } i = 1, \ldots, N, \tag{3.2}$$

for all $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi'_1, \ldots, \xi'_N)$, we have

$$[b_i(x, s, \xi) - b_i\,(x, s, \xi')]\,(\xi_i - \xi'_i) > 0 \quad \text{ for } \quad \xi_i \neq \xi'_i, \tag{3.3}$$

for a.e. $x \in \Omega$, and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $R_i(x) > 0$ lying in $L^{p'_i(\cdot)}(\Omega)$ and $\alpha, \beta > 0$

By using the continuity of $b_i(x, s, \cdot)$ with respect to ξ, and thanks to (3.2) we may obtain

$$b_i(x, s, 0) = 0.$$

The nonlinear terms $\Phi(x, s, \xi)$ and $\Psi(x, , s, \xi)$ are two Carathéodory such that

$$\Phi(x, s, \xi)s \geq 0, \tag{3.4}$$

$$|\Phi(x, s, \xi)| \leq b(|s|) \left(c(x) + \sum_{i=1}^{N} |\xi_i|^{p_i}\right), \tag{3.5}$$

and

$$|\Psi(x, , s, \xi)| \leq f_0(x) + d(|s|) \sum_{i=1}^{N} |\xi_i|^{p_i}\,, \tag{3.6}$$

with $b(\cdot)$ and $d(\cdot) : \mathbb{R} \to \mathbb{R}^+$ are continuous positive function, such that $d(\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\Omega)$, we assume also that $f_0(\cdot) \in L^1(\Omega)$ and $0 \leq c(\cdot) \in L^1(\Omega)$.

Let us consider the following problem

$$\begin{cases} B\varpi + \Phi(x, \varpi, \nabla\varpi) + \Psi(x, , \varpi, \nabla\varpi) = f - \operatorname{div} F & \text{in } \Omega \\ \varpi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where

$$f \in L^1(\Omega) \quad \text{ and } \quad F \in \prod_{i=1}^{N} L^{p'_i}(\Omega). \tag{3.8}$$

Remark 3.1. It's important to see that the function $\Psi(x,,s,\xi)$ does not satisfy the assumption (3.5), since $b(\cdot)$ is assumed only to be continuous.

Remark 3.2. The hypothesis (3.1) is necessary to entails that $|b_i(x,\varpi,\nabla\varpi)| \in L^{p'_i(\cdot)}(\Omega)$. In the case of $B\varpi = -\sum_{i=1}^{N} D^i b_i(x,\nabla\varpi)$, the condition (3.1) will be seen as

$$|b_i(x,\xi)| \leq \beta\left(R_i(x) + \sum_{i=1}^{N} |\xi_i|^{p_i-1}\right).$$

Lemma 3.3. *Suppose that* (3.1)-(3.3) *holds, and let* $(\varpi_n)_{n\in N}$ *be a sequence in* $W_0^{1,\vec{p}(\cdot)}(\Omega)$ *such that* $\varpi_n \rightharpoonup \varpi$ *in* $W_0^{1,\vec{p}(\cdot)}(\Omega)$ *and*

$$\begin{aligned}&\int_\Omega \left(|\varpi_n|^{p-2}\,\varpi_n - |\varpi|^{p-2}\varpi\right)(\varpi_n-\varpi)\,dx\\ &\quad+\sum_{i=1}^{N}\int_\Omega \left(b_i\left(x,\varpi_n,\nabla\varpi_n\right) - b_i\left(x,\varpi_n,\nabla\varpi\right)\right)\left(D^i\varpi_n - D^i\varpi\right)dx \to 0,\end{aligned}$$

then $\varpi_n \to \varpi$ *in* $W_0^{1,\vec{p}(\cdot)}(\Omega)$ *for a subsequence.*

4. The Existence of an Entropy Solution

We first introduce the notion of entropy solutions as follows.

Definition 4.1. A function ϖ(*measurable*) is said entropy solution of the problem (3.7) if $T_k(\varpi) \in K_\varpi$ and satisfy

$$\begin{cases} \displaystyle\sum_{i=1}^{N}\int_\Omega b_i(x,\varpi,\nabla\varpi)D^iT_k(\varpi-v)dx + \int_\Omega \Phi(x,\varpi,\nabla\varpi)T_k(\varpi-v)dx\\ \displaystyle+\int_\Omega \Psi(x,,\varpi,\nabla\varpi)T_k(\varpi-v)dx \leq \int_\Omega fT_k(\varpi-v)dx + \sum_{i=1}^{N}\int_\Omega F_iD^iT_k(\varpi-v)dx,\end{cases} \tag{4.1}$$

for any $v \in K_\varpi \cap L^\infty(\Omega)$.

Our purpose is to establish the following theorem

Theorem 4.2. *Assume that* (3.1)-(3.6) *and* (3.8) *holds, then the problem* (3.7) *admits one solution in the sense of the definition 4.1.*

4.1. Proof of Theorem 4.2.

4.1.1. *Step 1: Penalized approximate problem.*

Let $(f_n)_{n\in N}$ be a sequence in $W^{-1,p'_i}(\Omega) \cap L^1(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$ (for example $f_n = T_n(f)$). We consider the penalized approximation.

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} b_i\left(x, T_n\left(\varpi_n\right), \nabla \varpi_n\right) D^i\left(\varpi_n-v\right) dx+\frac{1}{n}\int_{\Omega}\left|\varpi_n\right|^{p_0-2} \varpi_n\left(\varpi_n-v\right) dx \\
&+\int_{\Omega} \Phi_n\left(x, \varpi_n, \nabla \varpi_n\right)\left(\varpi_n-v\right) dx+\int_{\Omega} \Psi_n\left(x, \varpi_n, \nabla \varpi_n\right)\left(\varpi_n-v\right) dx \\
&\leq \int_{\Omega} f_n\left(\varpi_n-v\right) dx+\sum_{i=1}^{N}\int_{\Omega} F_i D^i\left(\varpi_n-v\right) dx,
\end{aligned} \tag{4.2}$$

for any $v \in K_{\varpi}$, where

$$\Phi_n(x, s, \xi)=T_n(\Phi(x, s, \xi)),$$

and

$$\Psi_n(x, s, \xi)=T_n(\Psi(x,, s, \xi)).$$

We can show, similarly as in [2], that the problem (4.2) admits at least a weak solution $\varpi_n \in W_0^{1,\vec{p}(x)}(\Omega)$.

4.1.2. ***Step 2: A priori estimates***.
Suppose that $k \geq \max\left(1,\left\|\varpi^{+}\right\|_{\infty}\right)$, we put $v=\varpi_n-\eta T_k\left(\varpi_n-\varpi^{+}\right) \in W_0^{1,p_i}(\Omega)\cap L^{\infty}(\Omega)$, for η small enough we get $v \in K_{\Psi}$, therefore v is an admissible test function in (4.2), and we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} b_i\left(x, T_n\left(\varpi_n\right), \nabla \varpi_n\right) D^i T_k\left(\varpi_n-\varpi^{+}\right) dx+\frac{1}{n}\int_{\Omega}\left|\varpi_n\right|^{p_0-2} \varpi_n T_k\left(\varpi_n-\varpi^{+}\right) dx \\
&+\int_{\Omega} \Phi_n\left(x, \varpi_n, \nabla \varpi_n\right) T_k\left(\varpi_n-\varpi^{+}\right) dx+\int_{\Omega} \Psi_n\left(x, \varpi_n, \nabla \varpi_n\right) T_k\left(\varpi_n-\varpi^{+}\right) dx \\
&\leq \int_{\Omega} f_n T_k\left(\varpi_n-\varpi^{+}\right) dx+\sum_{i=1}^{N}\int_{\Omega} F_i D^i T_k\left(\varpi_n-\varpi^{+}\right) dx.
\end{aligned} \tag{4.3}$$

Using (3.4) and (3.6)), we have

$$\int_{\Omega} \Phi_n\left(x, \varpi_n, \nabla \varpi_n\right) T_k\left(\varpi_n-\varpi^{+}\right) dx \geq 0, \tag{4.4}$$

and

$$\begin{aligned}
&\left|\int_{\Omega} \Psi_n\left(x, \varpi_n, \nabla \varpi_n\right) T_k\left(\varpi_n-\varpi^{+}\right) dx\right| \leq k\int_{\Omega}\left|f_0(x)\right| dx \\
&+\sum_{i=1}^{N}\int_{\Omega} d\left(\left|\varpi_n\right|\right)\left|D^i \varpi_n\right|^{p_i}\left|T_k\left(\varpi_n-\varpi^{+}\right)\right| dx \\
&\leq k\int_{\Omega}\left|f_0(x)\right| dx.
\end{aligned} \tag{4.5}$$

According to (4.3)–(4.5), we may have

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} b_i\left(x, T_n\left(\varpi_n\right), \nabla \varpi_n\right) D^i T_k\left(\varpi_n-\varpi^{+}\right) dx+\frac{1}{n}\int_{\Omega}\left|\varpi_n\right|^{p_0-2} \varpi_n T_k\left(\varpi_n-\varpi^{+}\right) dx \\
&\leq k\int_{\Omega}(|f(x)|+|f_0(x)|)\, dx+\sum_{i=1}^{N}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}} F_i D^i\left(\varpi_n-\varpi^{+}\right) dx \\
&+\sum_{i=1}^{N}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}} b_i\left(x, T_n\left(\varpi_n\right), \nabla \varpi_n\right) D^i \varpi^{+} dx,
\end{aligned} \tag{4.6}
$$

Since $k \geq \|\varpi^{+}\|_{\infty}$, then $T_k\left(\varpi_n-\varpi^{+}\right)$ have the same sign as ϖ_n on the set $\{|\varpi_n-\varpi^{+}|>k\}$, this implies that

$$
\begin{aligned}
\frac{1}{n}\int_{\Omega}\left|\varpi_n\right|^{p_0-2} \varpi_n T_k\left(\varpi_n-\varpi^{+}\right) dx &= \frac{1}{n}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}}\left|\varpi_n\right|^{p_0-2} \varpi_n\left(\varpi_n-\varpi^{+}\right) dx \\
&\quad+\frac{1}{n}\int_{\{|\varpi_n-\varpi^{+}|>k\}}\left|\varpi_n\right|^{p_0-2} \varpi_n T_k\left(\varpi_n-\varpi^{+}\right) dx \\
&\geq \frac{1}{n}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}}\left|\varpi_n\right|^{p_0} dx-\frac{1}{n}\int_{\left\{|\varpi_n-\varpi^{+}|\leq\, k\right\}}\left|\varpi_n\right|^{p_0-1}\left|\varpi^{+}\right| dx
\end{aligned}
$$

taking into account (3.2), we get

$$
\begin{aligned}
&\alpha\sum_{i=1}^{N}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}}\left|D^i \varpi_n\right|^{p_i} dx+\frac{1}{n}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}}\left|\varpi_n\right|^{p_0} dx \\
&\leq \sum_{i=1}^{N}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}} b_i\left(x, T_n\left(\varpi_n\right), \nabla \varpi_n\right) D^i \varpi_n dx+\frac{1}{n}\int_{\Omega}\left|\varpi_n\right|^{p_0-2} \varpi_n T_k\left(\varpi_n-\varpi^{+}\right) dx \\
&\quad+\frac{1}{n}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}}\left|\varpi_n\right|^{p_0-1}\left|\varpi^{+}\right| dx \leq k\int_{\Omega}(|f(x)|+|f_0(x)|)\, dx \\
&+\sum_{i=1}^{N}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}}\left|b_i\left(x, T_n\left(\varpi_n\right), \nabla \varpi_n\right)\right|\left|D^i \varpi^{+}\right| dx+\frac{1}{n}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}}\left|\varpi_n\right|^{p_0-1}\left|\varpi^{+}\right| dx
\end{aligned} \tag{4.7}
$$

By using Young's inequality we obtain

$$
\int_{\{|\varpi_n-\varpi^{+}|\leq k\}}\left|\varpi_n\right|^{p_0-1}\left|\varpi^{+}\right| dx \leq \frac{1}{2}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}}\left|\varpi_n\right|^{p_0} dx+C_1, \tag{4.8}
$$

and

$$
\sum_{i=1}^{N}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}} F_i\left(D^i \varpi_n-D^i \varpi^{+}\right) dx=\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{|\varpi_n-\varpi^{+}|\leq k\}}\left|D^i \varpi_n\right|^{p_i} dx+C_4. \tag{4.9}
$$

Furthermore, by taking into account of (3.1) and Young's inequality, one has

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|\varpi_n-\varpi^+|\leq k\}} |b_i(x, T_n(\varpi_n), \nabla \varpi_n)| \left|D^i \varpi^+\right| dx \\
&\leq \beta \sum_{i=1}^{N} \int_{\{|\varpi_n-\varpi^+|\leq k\}} R_i(x) \left|D^i \varpi^+\right| dx + \beta \sum_{i=1}^{N} \int_{\{|\varpi_n-\varpi^+|\leq k\}} |\varpi_n|^{\underline{p}-1} \left|D^i \varpi^+\right| dx \\
&+ \beta \sum_{i=1}^{N} \int_{\{|\varpi_n-\varpi^+|\leq k\}} \left|D^i \varpi_n\right|^{p_i-1} \left|D^i \varpi^+\right| dx \\
&\leq C_2 + \varepsilon \sum_{i=1}^{N} \int_{\{|\varpi_n-\varpi^+|\leq k\}} |\varpi_n|^{\underline{p}} dx + \frac{\alpha}{4} \sum_{i=1}^{N} \int_{\{|\varpi_n-\varpi^+|\leq k\}} \left|D^i \varpi_n\right|^{p_i} dx \\
&+ \frac{1}{\varepsilon^{\underline{p}-1}} \sum_{i=1}^{N} \int_{\{|\varpi_n-\varpi^+|\leq k\}} \left|D^i \varpi^+\right|^{\underline{p}} dx + C_3 \sum_{i=1}^{N} \int_{\{|\varpi_n-\varpi^+|\leq k\}} \left|D^i \varpi^+\right|^{p_i} dx
\end{aligned}
\tag{4.10}
$$

By combining (4.6)-(4.10), we conclude that

$$
\begin{aligned}
&\frac{\alpha}{4} \sum_{i=1}^{N} \int_{\{|\varpi_n-\varpi^+|\leq k\}} \left|D^i \varpi_n\right|^{p_i} dx + \frac{1}{2n} \int_{\{|\varpi_n-\varpi^+|\leq k\}} |\varpi_n|^{p_0} dx \\
&\quad \leq kC + N\varepsilon \int_{\{|\varpi_n-\varpi^+|\leq k\}} |\varpi_n| \frac{p}{d} dx + C_4(\varepsilon)
\end{aligned}
\tag{4.11}
$$

Therefore, there exists a constant $C_5(k, \varepsilon)$ such that

$$
\frac{\alpha}{4} \sum_{i=1}^{N} \int_{\{|\varpi_n-\varpi^+|\leq k\}} \left|D^i \varpi_n\right|^{p_i} dx \leq C_5(k, \varepsilon). \tag{4.12}
$$

and since

$$
\{x \in \Omega, |\varpi_n| \leq k\} \subset \left\{x \in \Omega, \left|\varpi_n - \varpi^+\right| \leq k + \left\|\varpi^+\right\|_\infty\right\}
$$

Thus

$$
\begin{aligned}
\sum_{i=0}^{N} \int_{\Omega} \left|D^i T_k(\varpi_n)\right|^{p_i} dx &= \sum_{i=1}^{N} \int_{\{|\varpi_n|\leq k\}} \left|D^i \varpi_n\right|^{p_i} dx + \int_{\Omega} |T_k(\varpi_n)|^{p_0} dx \\
&\leq \sum_{i=1}^{N} \int_{\left\{|\varpi_n-\varpi^+|\leq k+\|\varpi^+\|_\infty\right\}} \left|D^i \varpi_n\right|^{p_i} dx + k^{p_0}|\Omega| \\
&\leq C_6\left(k, \left\|\varpi^+\right\|_\infty, \varepsilon\right)
\end{aligned}
$$

and we have

$$
\|T_k(\varpi_n)\|_{1,p_i} \leq C_7\left(k, \left\|\varpi^+\right\|_\infty, \varepsilon\right),
$$

where $C_7 > 0$ does not depend on n. Thus, the sequence $(T_k(\varpi_n))_n$ is bounded in $W_0^{1,p_i}(\Omega)$ uniformly in n, then there exists a subsequence still denoted $(T_k(\varpi_n))_{n\in N}$ and a function $v_k \in W_0^{1,p_i}(\Omega)$ such that

$$
\begin{cases}
T_k(\varpi_n) \rightharpoonup v_k & \text{weakly in } W_0^{1,p_i}(\Omega) \\
T_k(\varpi_n) \rightarrow v_k & \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e in } \Omega.
\end{cases}
\tag{4.13}
$$

Now, using (4.11) and Poincaré inequality, one has

$$\begin{aligned}
\|\nabla T_k(\varpi_n)\|_{\underline{p}}^{\underline{p}} &= \sum_{i=1}^{N} \int_\Omega \left|D^i T_k(\varpi_n)\right| \underline{p} dx \\
&\leq \sum_{i=1}^{N} \int_\Omega \left|D^i T_k(\varpi_n)\right|^{p_i} dx + N|\Omega| \\
&\leq \frac{4k}{\alpha} C + \frac{4N\varepsilon}{\alpha} \|T_k(\varpi_n)\| \frac{p}{\underline{p}} + C_8(\varepsilon) \\
&\leq \frac{4k}{\alpha} C + C' \frac{4N\varepsilon}{\alpha} \|\nabla T_k(\varpi_n)\|_{\underline{p}}^{\underline{p}} + C_8(\varepsilon).
\end{aligned}$$

Now, we choose ε small enough $\left(C' \frac{4N\varepsilon}{\alpha} \leq \frac{1}{2}\right)$, there exists a constant C_9 that does not depend on k and n, such that

$$\|\nabla T_k(\varpi_n)\|_{\underline{p}} \leq C_9 k^{\frac{1}{\underline{p}}} \quad \text{for} \quad k \geq 1,$$

and we arrive at

$$\begin{aligned}
k \operatorname{meas}\{|\varpi_n| > k\} &= \int_{\{|\varpi_n| > k\}} |T_k(\varpi_n)|\, dx \leq \int_\Omega |T_k(\varpi_n)|\, dx \\
&\leq \|1\|_{p'} \|T_k(\varpi_n)\|_{\underline{p}} \\
&\leq C \|\nabla T_k(\varpi_n)\|_{\underline{p}} \\
&\leq C_{10} k^{\frac{1}{\underline{p}}},
\end{aligned} \tag{4.14}$$

which yields.

$$\operatorname{meas}\{|\varpi_n| > k\} \leq C_{13} \frac{1}{k^{1-\frac{1}{\underline{p}}}} \longrightarrow 0 \text{ as } k \to \infty. \tag{4.15}$$

Now, we have for every $\delta > 0$

$$\begin{aligned}
&\operatorname{meas}\{|\varpi_n - \varpi_m| > \delta\} \leq \operatorname{meas}\{|\varpi_n| > k\} + \operatorname{meas}\{|\varpi_m| > k\} \\
&+ \operatorname{meas}\{|T_k(\varpi_n) - T_k(\varpi_m)| > \delta\}
\end{aligned}$$

suppose that $\varepsilon > 0$, and thanks to (4.15) we can take $k = k(\varepsilon)$ large enough such that

$$\operatorname{meas}\{|\varpi_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \operatorname{meas}\{|\varpi_m| > k\} \leq \frac{\varepsilon}{3}. \tag{4.16}$$

On the other hand, by (4.13) we obtain

$$T_k(\varpi_n) \longrightarrow \eta_k \text{ in } L^{\underline{p}}(\Omega) \text{ and a.e.in } \Omega.$$

Thus $(T_k(\varpi_n))_{n\in N}$ is a Cauchy sequence in measure, and for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\operatorname{meas}\{|T_k(\varpi_n) - T_k(\varpi_m)| > \delta\} \leq \frac{\varepsilon}{3} \quad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{4.17}$$

According to (4.16) and (4.17), we deduce that : for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$

$$\operatorname{meas}\{|\varpi_n - \varpi_m| > \delta\} \leq \varepsilon \quad \text{ for any } n, m \geq n_0.$$

Then $(\varpi_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function ϖ. By using (4.13) one has

$$\begin{cases} T_k(\varpi_n) \rightharpoonup T_k(\varpi) & \text{in } W_0^{1,p_i}(\Omega) \\ T_k(\varpi_n) \rightarrow T_k(\varpi) & \text{in } L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{4.18}$$

4.1.3. ***Step 3: Convergence of the gradient.***

Here, we denote by $\varepsilon_i(n), i = 1, 2, \ldots$ various real-valued functions of real variables that converge to 0 as $n \longrightarrow \infty$. For $h > k > 0$, we define

$$\mathbb{M} := 4k + h \quad \text{ and } \quad \omega_n := T_{2k}(\varpi_n - T_h(\varpi_n) + T_k(\varpi_n) - T_k(\varpi)).$$

Taking $v = \varpi_n - \eta\omega_n$, we have $v \geq \varpi$ for η small enough, thus v is an admissible test function in (4.2), and we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega b_i(x, T_n(\varpi_n), \nabla\varpi_n) D^i\omega_n dx + \frac{1}{n}\int_\Omega |\varpi_n|^{p_0-2}\varpi_n\omega_n dx \\ &+ \int_\Omega \Psi_n(x, \varpi_n, \nabla\varpi_n)\omega_n dx + \int_\Omega \Phi_n(x, \varpi_n, \nabla\varpi_n)\omega_n dx \\ &\leq \int_\Omega f_n\omega_n dx + \sum_{i=1}^N \int_\Omega F_i D^i\omega_n dx. \end{aligned}$$

It is clear to remark that ω_n have the same sign as ϖ_n on the set $\{|\varpi_n| > k\}$ and $\omega_n = T_k(\varpi_n) - T_k(\varpi)$ on the set $\{|\varpi_n| \leq k\}$, also we have $D^i\omega_n = 0$ on $\{|\varpi_n| > \mathbb{M}\}$ then,

$$\begin{aligned} &\sum_{i=1}^N \int_{\{|\varpi_n| \leq \mathbb{M}\}} b_i(x, T_{\mathbb{M}}(\varpi_n), \nabla T_{\mathbb{M}}(\varpi_n)) D^i\omega_n dx \\ &+ \frac{1}{n}\int_{\{|\varpi_n| \leq k\}} |\varpi_n|^{p_0-2}\varpi_n(T_k(\varpi_n) - T_k(\varpi))dx + \int_{\{|\varpi_n| \leq k\}} \Phi_n(x, \varpi_n, \nabla\varpi_n)\omega_n dx \\ &\leq \int_\Omega (|f_n(x)| + |f_0(x)|)|\omega_n| dx + \sum_{i=1}^N \int_\Omega d(|\varpi_n|)\left|D^i\varpi_n\right|^p |\omega_n| dx \\ &+ \sum_{i=1}^N \int_{\{|\varpi_n| \leq M\}} F_i D^i\omega_n dx. \end{aligned} \tag{4.19}$$

Furthermore, the first term on the left-hand side of (4.19) is giving by

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|\varpi_n| \le M\}} b_i\left(x, T_{\mathbb{M}}\left(\varpi_n\right), \nabla T_{\mathbb{M}}\left(\varpi_n\right)\right) D^i \omega_n dx \\
&= \sum_{i=1}^{N} \int_{\{|\varpi_n| \le k\}} b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k\left(\varpi_n\right)\right)\left(D^i T_k\left(\varpi_n\right) - D^i T_k(\varpi)\right) dx \\
&\quad + \sum_{i=1}^{N} \int_{\{k<|\varpi_n| \le M\}} b_i\left(x, T_{\mathbb{M}}\left(\varpi_n\right), \nabla T_{\mathbb{M}}\left(\varpi_n\right)\right) D^i \omega_n dx \\
&= \sum_{i=1}^{N} \int_{\Omega} \left(b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k\left(\varpi_n\right)\right) - b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k(\varpi)\right)\right) \\
&\qquad \times \left(D^i T_k\left(\varpi_n\right) - D^i T_k(\varpi)\right) dx \\
&\quad + \sum_{i=1}^{N} \int_{\Omega} b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k(\varpi)\right)\left(D^i T_k\left(\varpi_n\right) - D^i T_k(\varpi)\right) dx \\
&\quad + \sum_{i=1}^{N} \int_{\{|\varpi_n| > k\}} b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k\left(\varpi_n\right)\right) D^i T_k(\varpi) dx \\
&\quad + \sum_{i=1}^{N} \int_{\{k<|\varpi_n| \le M\}} b_i\left(x, T_{\mathbb{M}}\left(\varpi_n\right), \nabla T_{\mathbb{M}}\left(\varpi_n\right)\right) D^i \omega_n dx .
\end{aligned}
\tag{4.20}
$$

For the second term on the right-hand side of (4.20), Lebesgue's dominated convergence theorem entails that $T_k\left(\varpi_n\right) \rightarrow T_k(\varpi)$ in $L^{p_i}(\Omega)$, then

$$
b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k(\varpi)\right) \rightarrow b_i\left(x, T_k(\varpi), \nabla T_k(\varpi)\right) \quad \text{in} \quad L^{p_i'}(\Omega),
$$

and since $D^i T_k\left(\varpi_n\right) \rightharpoonup D^i T_k(\varpi)$ in $L^{p_i}(\Omega)$, we may get

$$
\varepsilon_1(n) = \sum_{i=1}^{N} \int_{\Omega} b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k(\varpi)\right)\left(D^i T_k\left(\varpi_n\right) - D^i T_k(\varpi)\right) dx \longrightarrow 0 \text{ as } n \rightarrow \infty.
\tag{4.21}
$$

For the third term on the right-hand side of (4.20), by using (3.2) and the continuity of $b(x, s, \xi)$ with respect to ξ, we have $b(x, s, 0) = 0$, we can get

$$
\int_{\{|\varpi_n| > k\}} b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k\left(\varpi_n\right)\right) D^i T_k(\varpi) dx = \int_{\{|\varpi_n| > k\}} b_i\left(x, T_k\left(\varpi_n\right), 0\right) D^i T_k(\varpi) dx = 0.
\tag{4.22}
$$

For the last term on the right-hand side of (4.20), taking $z_n = \varpi_n - T_h(\varpi_n) + T_k(\varpi_n) - T_k(\varpi)$, then

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|\varpi_n|>k\}} b_i\left(x, T_{\mathbb{M}}(\varpi_n), \nabla T_{\mathbb{M}}(\varpi_n)\right) D^i\omega_n dx\\ &=\sum_{i=1}^{N}\int_{\{|\varpi_n|>k\}\cap\{|z_n|\leq 2k\}} b_i\left(x, T_{\mathbb{M}}(\varpi_n), \nabla T_{\mathbb{M}}(\varpi_n)\right) D^i\left(\varpi_n - T_h(\varpi_n) + T_k(\varpi_n) - T_k(\varpi)\right) dx\\ &\geq -\sum_{i=1}^{N}\int_{\{|\varpi_n|>k\}} \left|b_i\left(x, T_{\mathbb{M}}(\varpi_n), \nabla T_{\mathbb{M}}(\varpi_n)\right)\right| \left|D^i T_k(\varpi)\right| dx\end{aligned}$$

We have $\left(b_i\left(x, T_{\mathbb{M}}(\varpi_n), \nabla T_{\mathbb{M}}(\varpi_n)\right)\right)_{n\in N}$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\vartheta_i \in L^{p_i'}(\Omega)$ such that $b_i\left(x, T_{\mathbb{M}}(\varpi_n), \nabla T_{\mathbb{M}}(\varpi_n)\right) \rightharpoonup \vartheta_i$ in $L^{p_i'}(\Omega)$. Therefore,

$$\begin{aligned}\varepsilon_2(n) &= \sum_{i=1}^{N}\int_{\{|\varpi_n|>k\}} b_i\left(x, T_{\mathbb{M}}(\varpi_n), \nabla T_{\mathbb{M}}(\varpi_n)\right) D^i T_k(\varpi) dx\\ &\longrightarrow \sum_{i=1}^{N}\int_{\{|\varpi|>k\}} \vartheta_i D^i T_k(\varpi) dx = 0,\end{aligned}$$

this implies that

$$\sum_{i=1}^{N}\int_{\{|\varpi_n|>k\}} b_i\left(x, T_{\mathbb{M}}(\varpi_n), \nabla T_{\mathbb{M}}(\varpi_n)\right) D^i\omega_n dx \geq \varepsilon_2(n). \tag{4.23}$$

According to (4.20)-(4.23), we may obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}\left(b_i\left(x, T_k(\varpi_n), \nabla T_k(\varpi_n)\right) - b_i\left(x, T_k(\varpi_n), \nabla T_k(\varpi)\right)\right)\left(D^i T_k(\varpi_n) - D^i T_k(\varpi)\right) dx\\ &\quad\leq \sum_{i=1}^{N}\int_{\Omega} b_i\left(x, T_{\mathbb{M}}(\varpi_n), \nabla T_{\mathbb{M}}(\varpi_n)\right) D^i\omega_n dx + \varepsilon_3(n).\end{aligned} \tag{4.24}$$

For the second term on the left-hand side of (4.19),in view of (4.39) we have

$$\varepsilon_4(n) = \frac{1}{n}\left|\int_{\{|\varpi_n|\leq k\}}\left|\varpi_n\right|^{p_0-2}\varpi_n\left(T_k(\varpi_n) - T_k(\varpi)\right) dx\ \right| \leq \frac{2k}{n}\int_{\{|\varpi_n|\leq k\}}\left|\varpi_n\right|^{p_0-1} dx \longrightarrow 0 \text{ as } n\to\infty. \tag{4.25}$$

In view of (4.19) and (4.24)-(4.25), we may have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}\left(b_i\left(x,T_k\left(\varpi_n\right),\nabla T_k\left(\varpi_n\right)\right)-b_i\left(x,T_k\left(\varpi_n\right),\nabla T_k(\varpi)\right)\right)\left(D^iT_k\left(\varpi_n\right)-D^iT_k(\varpi)\right)dx\\
&+\int_{\{|\varpi_n|\leq k\}}\varPhi_n\left(x,T_k\left(\varpi_n\right),\nabla T_k\left(\varpi_n\right)\right)\omega_n dx\\
&\leq\int_{\Omega}\left(|f_n(x)|+|f_0(x)|\right)|\omega_n|\,dx+\sum_{i=1}^{N}\int_{\{|\varpi_n|\leq M\}}\phi_{n,i}\left(T_{\mathbb{M}}\left(\varpi_n\right)\right)D^i\omega_n dx+\sum_{i=1}^{N}\int_{\Omega}F_iD^i\omega_n dx+\varepsilon_5(n).
\end{aligned} \tag{4.26}$$

We have $f_n\to f$ in $L^1(\Omega)$ and $\omega_n\to T_{2k}\left(\varpi-T_h(\varpi)\right)$ weak-* in $L^\infty(\Omega)$, then

$$\int_{\Omega}(f_n+f_0)\omega_n dx=\int_{\Omega}(f+f_0)T_{2k}\left(\varpi-T_h(\varpi)\right)dx+\varepsilon_6(n). \tag{4.27}$$

and

$$\sum_{i=1}^{N}\int_{\Omega}F_iD^i\omega_n dx=\sum_{i=1}^{N}\int_{\{h<|\varpi|\leq 2k+h\}}F_iD^i\varpi dx+\varepsilon_8(n). \tag{4.28}$$

In view of (3.5), it's clear that

$$\begin{aligned}
&\left|\int_{\{|\varpi_n|\leq k\}}\varPhi_n\left(x,T_k\left(\varpi_n\right),\nabla T_k\left(\varpi_n\right)\right)\omega_n dx\right|\\
&\leq b_k\int_{\{|\varpi_n|\leq k\}}\left(c(x)+\sum_{i=1}^{N}\left|D^iT_k\left(\varpi_n\right)\right|^{p_i}\right)|\omega_n|\,dx\\
&\leq b_k\int_{\{|\varpi_n|\leq k\}}c(x)\,|\omega_n|\,dx\\
&+\frac{b_k}{\alpha}\sum_{i=1}^{N}\int_{\Omega}\left(b_i\left(x,T_k\left(\varpi_n\right),\nabla T_k\left(\varpi_n\right)\right)-b_i\left(x,T_k\left(\varpi_n\right),\nabla T_k(\varpi)\right)\right)\\
&\times\left(D^iT_k\left(\varpi_n\right)-D^iT_k(\varpi)\right)|\omega_n|\,dx\\
&+\frac{b_k}{\alpha}\sum_{i=1}^{N}\int_{\Omega}b_i\left(x,T_k\left(\varpi_n\right),\nabla T_k(\varpi)\right)\cdot\left(D^iT_k\left(\varpi_n\right)-D^iT_k(\varpi)\right)|\omega_n|\,dx\\
&+\frac{b_k}{\alpha}\sum_{i=1}^{N}\int_{\Omega}b_i\left(x,T_k\left(\varpi_n\right),\nabla T_k\left(\varpi_n\right)\right)\cdot D^iT_k(\varpi)\,|\omega_n|\,dx.
\end{aligned} \tag{4.29}$$

Since $T_k\left(\varpi_n\right)-T_k(\varpi)\rightharpoonup 0$ weak-$\star$ in $L^\infty(\Omega)$, then

$$\int_{\{|\varpi_n|\leq k\}}c(x)\,|\omega_n|\,dx=\int_{\{|\varpi_n|\leq k\}}c(x)\left|T_k\left(\varpi_n\right)-T_k(\varpi)\right|dx\longrightarrow 0. \tag{4.30}$$

and thanks to (4.26)-(4.30), one has

$$
\begin{aligned}
&\frac{b_k}{\alpha}\sum_{i=1}^{N}\int_{\Omega}\left(b_i\left(x,T_k\left(\varpi_n\right),\nabla T_k\left(\varpi_n\right)\right)-b_i\left(x,T_k\left(\varpi_n\right),\nabla T_k(\varpi)\right)\right)\\
&\times\left(D^iT_k\left(\varpi_n\right)-D^iT_k(\varpi)\right)|\omega_n|\,dx\\
&\geq\left|\int_{\{|\varpi_n|\leq k\}}\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\omega_n dx\right|+\varepsilon_8(n)+\int_{\Omega}(f_n+f_0)T_{2k}\left(\varpi-T_h(\varpi)\right)dx\\
&+\sum_{i=1}^{N}\int_{\{h<|\varpi|\leq 2k+h\}}F_iD^i\varpi dx+\varepsilon_9(n).
\end{aligned}
\tag{4.31}
$$

For the first term on the right-hand side of (4.31), it's easy to remark that

$$
\int_{\Omega}(f_n+f_0)T_{2k}\left(\varpi-T_h(\varpi)\right)dx\longrightarrow 0\quad\text{ as }\quad h\to\infty. \tag{4.32}
$$

Finally, choosing $v=\varpi_n-\eta T_{2k}\left(\varpi_n-T_h\left(\varpi_n\right)\right)$ for η small enough, thus v is an admissible test function in (3.7), we get

$$
\begin{aligned}
\sum_{i=1}^{N}\int_{\Omega}b_i\left(x,T_n\left(\varpi_n\right),\nabla\varpi_n\right)D^iT_{2k}\left(\varpi_n-T_h\left(\varpi_n\right)\right)dx+\frac{1}{n}\int_{\Omega}|\varpi_n|^{p_0-2}\varpi_nT_{2k}\left(\varpi_n-T_h\left(\varpi_n\right)\right)dx\\
+\int_{\Omega}\Phi(x,\varpi,\nabla\varpi)T_{2k}\left(\varpi_n-T_h\left(\varpi_n\right)\right)dx+\int_{\Omega}\Psi(x,,\varpi,\nabla\varpi)T_{2k}\left(\varpi_n-T_h\left(\varpi_n\right)\right)dx\\
=\int_{\Omega}f_nT_{2k}\left(\varpi_n-T_h\left(\varpi_n\right)\right)dx+\sum_{i=1}^{N}\int_{\Omega}F_iD^iT_{2k}\left(\varpi_n-T_h\left(\varpi_n\right)\right)dx.
\end{aligned}
\tag{4.33}
$$

According to (3.2),(3.4),(3.6) and Young's inequality, we get

$$
\begin{aligned}
\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{h<|\varpi_n|\leq 2k+h\}}\left|D^i\varpi_n\right|^{p_i}dx\leq\int_{\Omega}(f_n+f_0)T_{2k}\left(\varpi_n-T_h\left(\varpi_n\right)\right)dx\\
+C_{15}\sum_{i=1}^{N}\int_{\{h<|\varpi_n|\leq 2k+h\}}|F_i|^{p_i'}dx
\end{aligned}
\tag{4.34}
$$

this entails that

$$
\begin{aligned}
\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{h<|\varpi|\leq 2k+h]}\left|D^i\varpi\right|^{p_i}dx\leq\frac{\alpha}{2}\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{[h<|\varpi_n|\leq 2k+h|}\left|D^i\varpi_n\right|^{p_i}dx\\
\leq\lim_{n\to\infty}\int_{\Omega}(f_n+f_0)T_{2k}\left(\varpi_n-T_h\left(\varpi_n\right)\right)dx+\lim_{n\to\infty}C_{15}\sum_{i=1}^{N}\int_{\langle h<|\varpi_n|\leq 2k+h\}}|F_i|^{p_i'}dx\\
=\int_{\Omega}(f_n+f_0)T_{2k}\left(\varpi-T_h(\varpi)\right)dx+C_{15}\sum_{i=1}^{N}\int_{[h<|\varpi|\leq 2k+h]}|F_i|^{p_i'}dx.
\end{aligned}
\tag{4.35}
$$

By letting $h \longrightarrow \infty$ in (4.35), we may have

$$\limsup_{h\to\infty} \int_{\{h<|\varpi|\leq 2k+h\}} \left|D^i \varpi\right|^{p_i} dx = 0$$

Therefore,

$$\lim_{h\to\infty} \int_{[h<|\varpi|\leq 2k+h\}} |F_i| \left|D^i \varpi\right| dx = 0 \tag{4.36}$$

According to (4.32)-(4.32) and (4.36), by letting $h, n \longrightarrow \infty$ in (4.31), we may obtain

$$\sum_{i=1}^{N} \int_\Omega \left(b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k\left(\varpi_n\right)\right) - b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k(\varpi)\right)\right)\left(D^i T_k\left(\varpi_n\right) - D^i T_k(\varpi)\right) dx \to 0,$$

and since $T_k\left(\varpi_n\right) \to T_k(\varpi)$ in $L^{p_0}(\Omega)$, then

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \left(b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k\left(\varpi_n\right)\right) - b_i\left(x, T_k\left(\varpi_n\right), \nabla T_k(\varpi)\right)\right)\left(D^i T_k\left(\varpi_n\right) - D^i T_k(\varpi)\right) dx \\
&+ \int_\Omega \left(\left|T_k\left(\varpi_n\right)\right|^{p_0-2} T_k\left(\varpi_n\right) - \left|T_k(\varpi)\right|^{p_0-2} T_k(\varpi)\right)\left(T_k\left(\varpi_n\right) - T_k(\varpi)\right) dx \longrightarrow 0 \text{ as } n \to \infty
\end{aligned} \tag{4.37}$$

By using the idea of Lemma 3.3 and (4.8), we can obtain

$$\begin{cases} T_k\left(\varpi_n\right) \to T_k(\varpi) & \text{strongly in } W_0^{1,\vec{p}(\cdot)}(\Omega), \\ D^i \varpi_n \to D^i \varpi & \text{a.e. in } \Omega \text{ for } i = 1, \ldots, N. \end{cases} \tag{4.38}$$

4.1.4. ***Step 4: The equi-integrability.***

i) The equi-integrability of $\left(\frac{1}{n}\left|\varpi_n\right|^{p_0-2} \varpi_n\right)_n$.
Here, we establish that

$$\frac{1}{n}\left|\varpi_n\right|^{p_0-2} \varpi_n \longrightarrow 0 \text{ strongly in } L^1(\Omega). \tag{4.39}$$

By using the approach of Vitali's Theorem, it is suffices to prove that $\left(\frac{1}{n}\left|\varpi_n\right|^{p_0-2} \varpi_n\right)_n$ is uniformly equiintegrable. Indeed, taking $v = \varpi_n - \eta T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right)$ as a test in (4.2) for η small enough. Using (3.2), (3.4) and using the idea Young's inequality, we may get

$$\begin{aligned}
&\alpha \sum_{i=1}^{N} \int_{\{h<|\varpi_n|\leq h+1\}} \left|D^i \varpi_n\right|^{p_i} dx + \frac{1}{n} \int_{\{h<|\varpi_n|\}} \left|\varpi_n\right|^{p_0-2} \varpi_n T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right) dx \\
&\qquad \leq \int_{\{h<|\varpi_n|\}} |f_n| \, dx + \int_{\{h<|\varpi_n|\}} |f_0(x)| \, dx,
\end{aligned}$$

this implies that

$$\frac{1}{n}\int_{\{h+1\le|\varpi_n|\}}|\varpi_n|^{p_0-1}\,dx \le \frac{1}{n}\int_{\{h<|\varpi_n|\}}|\varpi_n|^{p_0-2}\,\varpi_n T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right)dx$$
$$\le \int_{\{h<|\varpi_n|\}}|f_n|\,dx + \int_{\{h<|\varpi_n|\}}|f_0(x)|\,dx,$$

Let $\eta>0$ be fixed, then there exists $h(\eta)\ge 1$ such that

$$\frac{1}{n}\int_{\{h(\eta)<|\varpi_n|\}}|\varpi_n|^{p_0-1}\,dx \le \frac{\eta}{2}, \tag{4.40}$$

Now, for any measurable subset $E\subset\Omega$, we have

$$\frac{1}{n}\int_E|\varpi_n|^{p_0-1}\,dx \le \frac{1}{n}\int_E\left|T_{h(\eta)}\left(\varpi_n\right)\right|^{p_0-1}dx + \frac{1}{n}\int_{\{h(\eta)<|\varpi_n|\}}|\varpi_n|^{p_0-1}\,dx, \tag{4.41}$$

It's clear to remark that : there exists $\beta(\eta)>0$ such that, for all $E\subseteq\Omega$ with meas $(E)\le\beta(\eta)$, we have

$$\frac{1}{n}\int_E\left|T_{h(\eta)}\left(\varpi_n\right)\right|^{p_0-1}dx \le \frac{\eta}{2}. \tag{4.42}$$

Consequently, according to (4.40),(4.41) and (4.42), we may have

$$\frac{1}{n}\int_E|\varpi_n|^{p_0-1}\,dx \le \eta \text{ for all } E \text{ such that } meas(E)\le\beta(\eta). \tag{4.43}$$

We conclude that $\left(\frac{1}{n}|\varpi_n|^{p_0-2}\,\varpi_n\right)_n$ is uniformly equi-integrable, and since

$$\frac{1}{n}|\varpi_n|^{p_0-2}\,\varpi_n \longrightarrow 0 \text{ a.e in } \Omega.$$

Consequently, the convergence (4.39) is deduced (Vitali's Theorem).

ii) The equi-integrability of $\left(\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right)_n$ **and**$\left(\Psi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right)_n$

Here, we shall show that

$$\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\to\Phi(x,\varpi,\nabla\varpi) \text{ and } \Psi_n\left(x,\varpi_n,\nabla\varpi_n\right)\to\Psi(x,,\varpi,\nabla\varpi) \text{ in } L^1(\Omega). \tag{4.44}$$

By using Vitali's Theorem, it is suffices to show that $\left(\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right)_n$ and $\left(\Psi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right)_n$ are uniformly equi-integrable. We set

$$\bar{D}(s)=\frac{2}{\alpha}\int_0^s d(|\tau|)d\tau.$$

By taking $T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right) e^{\bar{D}(|\varpi_n|)}$ as a test function in (4.2), we have

$$\sum_{i=1}^{N}\int_{\Omega} b_i\left(x,\varpi_n,\nabla\varpi_n\right) D^i T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right) e^{\bar{D}(|\varpi_n|)} dx$$

$$+\frac{2}{\alpha}\sum_{i=1}^{N}\int_{\Omega} b_i\left(x,\varpi_n,\nabla\varpi_n\right) D^i\varpi_n d\left(|\varpi_n|\right)\left|T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right)\right| e^{\bar{D}(|\varpi_n|)} dx$$

$$\begin{aligned}&+\int_{\Omega} \Phi_n\left(x,\varpi_n,\nabla\varpi_n\right) T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right) e^{\bar{D}(|\varpi_n|)} dx\\ &+\int_{\Omega} \Psi_n\left(x,\varpi_n,\nabla\varpi_n\right) T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right) e^{\bar{D}(|\varpi_n|)} dx\\ &=\int_{\Omega} f_n T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right) e^{\bar{D}(|\varpi_n|)} dx.\end{aligned}$$

According to (3.2) and (3.6), we obtain

$$\begin{aligned}&\alpha\sum_{i=1}^{N}\int_{\{h<|\varpi_n|\leq h+1\}} b_i\left(x,\varpi_n,\nabla\varpi_n\right) D^i\varpi_n e^{\bar{D}(|\varpi_n|)} dx\\ &+2\sum_{i=1}^{N}\int_{\{h<|\varpi_n|\}} \left|D^i\varpi_n\right|^{p_i} d\left(|\varpi_n|\right)\left|T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right)\right| e^{\bar{D}(|\varpi_n|)} dx\\ &+\int_{\{h<|\varpi_n|\}} \Phi_n\left(x,\varpi_n,\nabla\varpi_n\right) T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right) e^{\bar{D}(|\varpi_n|)} dx\\ &\leq\int_{\{h<|\varpi_n|\}} \left(|f_n|+|f_0|\right) e^{\bar{D}(|\varpi_n|)} dx\\ &+\sum_{i=1}^{N}\int_{\{h<|\varpi_n|\}} \left|D^i\varpi_n\right|^{p_i(x)}\left|T_1\left(\varpi_n - T_h\left(\varpi_n\right)\right)\right| d\left(|\varpi_n|\right) e^{\bar{D}(|\varpi_n|)} dx.\end{aligned}\tag{4.45}$$

Consequently

$$\sum_{i=1}^{N}\int_{\{h+1<|\varpi_n|\}} d\left(|\varpi_n|\right)\left|D^i\varpi_n\right|^{p_i} dx+\int_{\{h+1<|\varpi_n|\}} \left|\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right| dx \leq e^{\bar{D}(\infty)}\int_{\{h<|\varpi_n|\}} \left(|f|+|f_0|\right) dx.$$

Thus, for all $\eta>0$, there exists $h(\eta)\geq 1$ such that

$$\sum_{i=1}^{N}\int_{\{h(\eta)<|\varpi_n|\}} d\left(|\varpi_n|\right)\left|D^i\varpi_n\right|^{p_i} dx+\int_{\{h(\eta)<|\varpi_n|\}} \left|\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right| dx \leq \frac{\eta}{2}.\tag{4.46}$$

Now, we put

$$b_{h(\eta)} := \max\{b(s) : |s|\leq h(\eta)\} \quad \text{and} \quad d_{h(\eta)} := \max\{d(s) : |s|\leq h(\eta)\},$$

for any measurable subset $E\subseteq\Omega$, we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_E d\left(|\varpi_n|\right)\left|D^i\varpi_n\right|^{p_i}dx+\int_E\left|\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right|dx\\&\leq d_{h(\eta)}\sum_{i=1}^{N}\int_E\left|D^iT_{h(\eta)}\left(\varpi_n\right)\right|^{p_i}dx+b_{h(\eta)}\int_E\left(c(x)+\sum_{i=1}^{N}\left|D^iT_{h(\eta)}\left(\varpi_n\right)\right|^{p_i}\right)dx\\&\quad+\sum_{i=1}^{N}\int_{\{h(\eta)<|\varpi_n|\}}d\left(|\varpi_n|\right)\left|D^i\varpi_n\right|^{p_i}dx+\int_{\{h(\eta)<|\varpi_n|\}}\left|\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right|dx.\end{aligned}\tag{4.47}$$

According to (4.38), there exists $\tau(\eta)>0$ such that, for any meas $(E)\leq\tau(\eta)$ we have

$$d_{h(\eta)}\sum_{i=1}^{N}\int_E\left|D^iT_{h(\eta)}\left(\varpi_n\right)\right|^{p_i}dx+b_{h(\eta)}\int_E\left(c(x)+\sum_{i=1}^{N}\left|D^iT_{h(\eta)}\left(\varpi_n\right)\right|^{p_i}\right)dx\leq\frac{\eta}{2}.\tag{4.48}$$

Furthermore, according to (4.46), (4.47) and (4.48), we may get

$$\sum_{i=1}^{N}\int_E d\left(|\varpi_n|\right)\left|D^i\varpi_n\right|^{p_i}dx+\int_E\left|\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right|dx\leq\eta\quad\text{for all}\quad\text{meas}(E)\leq\tau(\eta).\tag{4.49}$$

Thanks to (3.6), we deduce that $\left(\Psi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right)_n$ and $\left(\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\right)_n$ are uniformly equi-integrable, and since

$$\Psi_n\left(x,\varpi_n,\nabla\varpi_n\right)\to\Psi(x,,\varpi,\nabla\varpi)\text{ and }\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)\to\Phi(x,\varpi,\nabla\varpi)\text{ a.e. in }\Omega.$$

By using the approach of Vitali's theorem, the convergence (4.44) was verified.

4.1.5. ***Step 5: Passing to the limit.***
Let $v\in K_\varpi\cap L^\infty(\Omega)$, by taking $\varpi_n-\eta T_k\left(\varpi_n-v\right)$ as a test in (4.2), with η smal enough, one has

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega b_i\left(x,T_n\left(\varpi_n\right),\nabla\varpi_n\right)D^iT_k\left(\varpi_n-v\right)dx+\int_\Omega\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)T_k\left(\varpi_n-v\right)dx\\&+\frac{1}{n}\int_\Omega|\varpi_n|^{p_0-2}\varpi_nT_k\left(\varpi_n-v\right)dx\leq\int_\Omega f_nT_k\left(\varpi_n-v\right)dx+\int_\Omega\Psi_n\left(x,\varpi_n,\nabla\varpi_n\right)T_k\left(\varpi_n-v\right)dx\\&\qquad+\sum_{i=1}^{N}\int_\Omega F_iD^iT_k\left(\varpi_n-v\right)dx.\end{aligned}\tag{4.50}$$

Now, let $\mathbb{M}=k+\|v\|_{\infty}$ then $\{|\varpi_n-v|\leq k\}\subseteq\{|\varpi_n|\leq\mathbb{M}\}$, and we obtain

$$\begin{aligned}
&\int_\Omega b_i\left(x,T_n\left(\varpi_n\right),\nabla\varpi_n\right)D^iT_k\left(\varpi_n-v\right)dx\\
&=\int_\Omega b_i\left(x,T_{\mathbb{M}}\left(\varpi_n\right),\nabla T_{\mathbb{M}}\left(\varpi_n\right)\right)\left(D^iT_{\mathbb{M}}\left(\varpi_n\right)-D^iv\right)\chi_{\{|\varpi_n-v|\leq k\}}dx\\
&=\int_\Omega\left(b_i\left(x,T_{\mathbb{M}}\left(\varpi_n\right),\nabla T_{\mathbb{M}}\left(\varpi_n\right)\right)-b_i\left(x,T_{\mathbb{M}}\left(\varpi_n\right),\nabla v\right)\right)\left(D^iT_{\mathbb{M}}\left(\varpi_n\right)-D^iv\right)\chi_{\{|\varpi_n-v|\leq k\}}dx\\
&\quad+\int_\Omega b_i\left(x,T_{\mathbb{M}}\left(\varpi_n\right),\nabla v\right)\left(D^iT_{\mathbb{M}}\left(\varpi_n\right)-D^iv\right)\chi_{\{|\varpi_n-v|\leq k\}}dx.
\end{aligned}$$

It's clear that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega b_i\left(x,T_{\mathbb{M}}\left(\varpi_n\right),\nabla v\right)\left(D^iT_{\mathbb{M}}\left(\varpi_n\right)-D^iv\right)\chi_{\{|\varpi_n-v|\leq k\}}dx\\
&\quad=\int_\Omega b_i\left(x,T_{\mathbb{M}}(\varpi),\nabla v\right)\left(D^iT_{\mathbb{M}}(\varpi)-D^iv\right)\chi\{|\varpi-v|\leq k\}dx.
\end{aligned}$$

Fatou's Lemma implies that

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^N\int_\Omega b_i\left(x,T_n\left(\varpi_n\right),\nabla\varpi_n\right)D^iT_k\left(\varpi_n-v\right)dx\\
&\geq\sum_{i=1}^N\int_\Omega\left(b_i\left(x,T_{\mathbb{M}}(\varpi),\nabla T_{\mathbb{M}}(\varpi)\right)-b_i\left(x,T_{\mathbb{M}}(\varpi),\nabla v\right)\right)\left(D^iT_{\mathbb{M}}(\varpi)-D^iv\right)\chi_{\{|\varpi-v|\leq k\}}dx\\
&\quad+\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega b_i\left(x,T_{\mathbb{M}}\left(\varpi_n\right),\nabla v\right)\left(D^iT_{\mathbb{M}}\left(\varpi_n\right)-D^iv\right)\chi_{\{|\varpi_n-v|\leq k\}}dx\\
&\quad=\sum_{i=1}^N\int_\Omega b_i\left(x,T_{\mathbb{M}}(\varpi),\nabla T_{\mathbb{M}}(\varpi)\right)\left(D^iT_{\mathbb{M}}(\varpi)-D^iv\right)\chi_{\{|\varpi-v|\leq k\}}dx\\
&\quad=\sum_{i=1}^N\int_\Omega b_i(x,\varpi,\nabla\varpi)D^iT_k(\varpi-v)dx.
\end{aligned}\tag{4.51}$$

Here, since $T_k\left(\varpi_n-v\right)\to T_k(\varpi-v)$ weak$-\star$ in $L^\infty(\Omega)$ and according to (4.39) we may obtain

$$\frac{1}{n}\int_\Omega\left|\varpi_n\right|^{p_0-2}\varpi_nT_k\left(\varpi_n-v\right)dx\longrightarrow 0,\tag{4.52}$$

and

$$\int_\Omega\Phi_n\left(x,\varpi_n,\nabla\varpi_n\right)T_k\left(\varpi_n-v\right)dx\longrightarrow\int_\Omega\Phi(x,\varpi,\nabla\varpi)T_k\left(\varpi_n-v\right)dx,\tag{4.53}$$

$$\int_\Omega\Psi_n\left(x,\varpi_n,\nabla\varpi_n\right)T_k\left(\varpi_n-v\right)dx\longrightarrow\int_\Omega\Psi(x,,\varpi,\nabla\varpi)T_k\left(\varpi_n-v\right)dx,\tag{4.54}$$

Now, as f_n tends to f in $L^1(\Omega)$ we get

$$\int_\Omega f_n T_k(\varpi_n - v)\, dx \longrightarrow \int_\Omega f T_k(\varpi - v) dx. \tag{4.55}$$

since $T_k(\varpi_n - v) \rightharpoonup T_k(\varpi - v)$ in $W_0^{1,p}(\Omega)$, then

$$\int_\Omega F_i D^i T_k(\varpi_n - v)\, dx \longrightarrow \int_\Omega F_i D^i T_k(\varpi - v) dx. \tag{4.56}$$

In view of (4.50)-(4.56), the proof of Theorem 4.2 is achieved.

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