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STABILITY OF KREIN-VON NEUMANN SELF-ADJOINT OPERATOR EXTENSION UNDER UNBOUNDED PERTURBATIONS

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Abstract. We have considered a fourth order difference operator defined on the Hilbert space of square summable sequences on $\mathbb{N}$. We investigated the stability of existence of Krein-von Neumann self-adjoint extension of difference operators under bounded and unbounded coefficients. Using asymptotic summation based on discretised Levinson's theorem and appropriate smoothness and decay conditions, we have shown that unlike the case of deficiency indices and discrete spectrum, the existence of positive self-adjoint operator extensions is stable under unbounded perturbations. These results now exhaustively characterise the spectral and structural properties of difference operators with unbounded coefficients.

1. Introduction and preliminaries

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be fourth and second order symmetric difference maps defined by

$$\mathcal{L}_1 y(t) = \Delta^2\{p_2(t)\Delta^2 y(t-2)\} - \Delta\{p_1(t)\Delta y(t-1)\} + p_0(t)y(t) \tag{1.1}$$

and

$$\mathcal{L}_2 y(t) = -\Delta\{q_1(t)\Delta y(t-1)\} + q_0(t)y(t). \tag{1.2}$$

Here, the coefficients $p_2(t)$, $p_1(t)$, $p_0(t)$, $q_1(t)$ and $q_0(t)$ are real valued functions and are allowed to be unbounded. In this case, $\Delta f(t)$ is a forward difference operator defined by $\Delta f(t) = f(t+1) - f(t)$ where $f(t) = p_k(t), q_j(t)$. For $f(t), t \in \mathbb{N}$, we assume higher order differences exist, $\Delta^k f(t)$, $k = 1, 2, ...$ and that $\Delta^2 f(t)$ and $(\Delta f(t))^2$ tend to zero as $t \to \infty$. The maps $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined on $\ell^2_\omega(\mathbb{N})$ which is a weighted Hilbert space with $\omega = \omega(t) > 0$, weight function. In this case, the scalar product for any two vectors $y_1(t), y_2(t) \in \ell^2_\omega(\mathbb{N})$ is defined by

$$\langle y_1, y_2\rangle = \textstyle\sum_{t=t_0}^{\infty} y_1(t)\omega(t)\overline{y_2(t)} < \infty,\ y_1 = y_1(t) \text{ and } y_2 = y_2(t)$$

for some fixed left regular end point $t_0 \in \mathbb{N}$. In order to simplify the computations, we will assume that $\omega(t) = 1$, for all $t \in \mathbb{N}$. This results into the Hilbert

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space $\ell^2(\mathbb{N})$ which is isomorphic to $\ell^2_\omega(\mathbb{N})$. We have considered difference operators generated by the difference maps $\mathcal{L}_1$ and $\mathcal{L} = \mathcal{L}_1 - z\mathcal{L}_2$, where $\mathcal{L}$ is obtained by perturbing $\mathcal{L}_1$ by a scalar multiple of $\mathcal{L}_2$, $z \in \mathbb{R}$. Our main interest in this case was the stability of the existence of Krein-von Neumann self-adjoint operator extensions of the minimal operators generated by the difference maps $\mathcal{L}$ and $\mathcal{L}_1$ on $\ell^2(\mathbb{N})$ under various growth and decay conditions imposed on the coefficients of $\mathcal{L}_1$ and $\mathcal{L}_2$. The stability analysis has also been extended to absolutely continuous spectrum of the self-adjoint operator extension of the minimal operators generated by $\mathcal{L}$ and $\mathcal{L}_1$. In the case of bounded operators, stability analysis of deficiency indices and some components of the spectrum have been analysed in the past. These analyses were done when the perturbing term is bounded or compact. However, nothing has been done to check on the stability of the existence property of Krein-von Neumann self-adjoint operator extension of minimal difference operators when the coefficients are allowed to be unbounded. The Krein-von Neumann self-adjoint operator extension property is the property of obtaining positive self-adjoint operator extension of positive symmetric operators. The method applied in this case is asymptotic summation based on the discretized version of Levinson's theorem to establish the equality of deficiency indices and to determine the existence of Krein-von Neumann positive self-adjoint operator extension of the minimal difference operators. First, we have determined the growth conditions on the coefficients that guarantees that $\mathcal{L}_1$ and $\mathcal{L}$ are positive. Asymptotic summation method involves solving the equations $\mathcal{L}y = wy$ or $\mathcal{L}_1 y = wy$, where w is a spectral parameter. The method is based on the discretised version of Levinson theorem. Here below, we have stated the spectral parameter version of the theorem.

Theorem 1.1 *Let* $\Lambda(t,w) = diag\{\lambda_1(t,w), ..., \lambda_{2n}(t,w)\}$ *for* $t \geq t_0$. *Assume*

(i) $\lambda_i(t,w) \neq 0$ *for all* $1 \leq i \leq 2n$ *and* $t \geq t_0$

(ii) $R(t,w)$ *is a* $2n \times 2n$ *matrix defined for all* $t \geq t_0$, *satisfying* $\Sigma_{t=0}^{\infty}|\frac{1}{\lambda_i(t,z)}|\|R(t,z)\| < \infty$, *for all* $i = 1, 2, ..., 2n$

(iii) $\Lambda(t,w)$ *satisfies the following uniform dichotomy condition. For any pair of indices* i *and* j, *such that* $i \neq j$, *assume there exists* δ *with* $0 < \delta < 1$ *such that* $|\lambda_i(t,w)| \geq \delta$ *for all* $t \geq t_0$. *Then, either* $|\frac{\lambda_i(t)}{\lambda_j(t)}| \geq 1$ *or* $|\frac{\lambda_i(t)}{\lambda_j(t)}| \leq 1$ *for a large* t.

Then the linear system

$$Y(t+1,w) = [\Lambda(t,w) + R(t,w)]Y(t,w) \tag{1.3}$$

has a fundamental matrix satisfying,

$$Y(t,w) = [I + o(1)]\prod_{l=t_0}^{t-1}\Lambda(l,w) \ \text{as}\ t \to \infty.$$

Explicitly for the eigensolutions, we have

$$y_k(t,w) = (e_k(t,w) + r_k(t,w))\prod_{l=t_0}^{t-1}(\lambda_k(l,w)). \tag{1.4}$$

Here, $e_k(t,w)$ is the normalized eigenvector and $r_k(t,w) \to 0$ as $t \to \infty$. For more details, see [4].

In order to apply the discretised version of Levinson's theorem to our systems, we begin by systems formulation so that the equations $\mathcal{L}_1 y(t) = wy(t)$ and $\mathcal{L}y(t) = wy(t)$ can be converted into their first order systems. Thus for $\mathcal{L}_1 y(t) = wy(t)$ we require the following quasidifferences [5]: Let

$$\begin{aligned} x_1(t) &= y(t-1) \\ x_2(t) &= \Delta y(t-2) \\ u_1(t) &= p_1 \Delta y(t-1) - \Delta(p_2 \Delta^2 y(t-2)) \\ u_2(t) &= p_2 \Delta^2 y(t-2). \end{aligned}$$

We define the vector valued functions $x(t)$, $u(t)$ and $Y_1(t)$ by

$$\begin{aligned} x(t) &= (x_1(t), x_2(t)) \\ u(t) &= (u_1(t), u_2(t)) \\ Y_1(t) &= (x(t), u(t))^{tr} \end{aligned}$$

and the partial shift operator $R(Y_1)(t)$ by $(RY_1)(t) = (x(t+1), u(t))^{tr}$, the superscript tr denotes the transpose. The maximal difference operator $L_1^* y(t)$ generated by $\mathcal{L}_1$ is defined via its domain by

$$D(L_1^*) = \{y(t) \in \ell^2(\mathbb{N}) : \forall f(t) \in \ell^2(\mathbb{N}) \text{ such that } \\ \mathcal{J}\Delta Y_1(t) - P_1(t)(RY_1)(t) = F(t), t \in \mathbb{N}\}$$

and $L_1^* y(t) = f(t)$, if $\mathcal{J}\Delta Y_1(t) - P_1(t)(RY_1)(t) = F(t)$ where $\mathcal{J}$ is a simplectic matrix given by $\mathcal{J} = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$, $F(t) = \begin{bmatrix} f(t), & 0, & 0, & 0 \end{bmatrix}^{tr}$ and $P_1(t) = \begin{pmatrix} -C_1 & A_1^* \\ A_1 & B_1 \end{pmatrix}$ for which the non-zero elements of the 2×2 matrices A_1, B_1, C_1 are given by $(A_1)_{j,j+1} = 1, (B_1)_{2,2} = p_2^{-1}$ and $(C_1)_{j,j} = p_{j-1}$, for $j = 1, 2$. The pre-minimal operator $L_1^{'}$ is defined by

$$D(L_1^{'}) = \{y(t) \in D(L_1^*) : \text{there exists } n \in \mathbb{N} \text{ such that } y(0) = y(t) = 0, \text{ for all } \\ t \geq n+1\},\ L_1^* y = L_1^{'} y.$$

$L_1^{'}$ is symmetric and densely defined but not necessarily closed. The closure of the pre-minimal operator, $L_1^{'}$, is the minimal difference operator, L_1. Thus L_1 and L_1^* are symmetric, $L_1 \subset L_1^*$ and $L_1 = L_1^{**}$ as required. We thus obtain a first order forward difference of

$$\Delta \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} -C_1 & A_1^* \\ A_1 & B_1 \end{pmatrix} \begin{pmatrix} x(t+1) \\ u(t) \end{pmatrix}.$$

The first order system

$$\begin{pmatrix} x(t+1) \\ u(t+1) \end{pmatrix} = \begin{pmatrix} E_1 & E_1 B_1 \\ C_1 E_1 & I - A_1^* + C_1 E_1 B_1 \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$$

where $E_1 = (I - A_1)^{-1}$ can be written as

$$Y_1(t+1, z, w) = S_1(t, z, w) Y_1(t, z, w) \tag{1.5}$$

for which $Y_1(t) = [x(t), u(t)]^{tr}, x(t) = [x_1(t), x_2(t)]^{tr}$ and $u(t) = [u_1(t), u_2(t)]^{tr}$. Thus

$$S_1(t, z, w) = \begin{pmatrix} 1 & 1 & 0 & p_2^{-1} \\ 0 & 1 & 0 & p_2^{-1} \\ (p_0 - w) & (p_0 - w) & 1 & (p_0 - w)p_2^{-1} \\ 0 & p_1 & -1 & 1 + p_1 p_2^{-1} \end{pmatrix}.$$

In order to proceed with transformation of equation (1.5), we require the eigenvalues and eigenvectors of $S_1(t, z, w)$. The associated characteristic polynomial is $\mathcal{P}_1(t, \lambda, w) = det(S_1(t, z, w) - \lambda \cdot I_4)$, which we multiply by $\frac{p_2}{\lambda^2}$ to obtain the Fourier characteristic equation

$$\mathcal{F}_1(t, \lambda, w) = p_2(1-\lambda)^2(1-\lambda^{-1})^2 + p_1(1-\lambda)(1-\lambda^{-1}) + (p_0 - w). \quad (1.6)$$

We let $\zeta = (1-\lambda)(1-\lambda^{-1})$ and so we obtain the quadratic equation

$$p_2\zeta^2 + p_1\zeta + (p_0 - w) = 0. \quad (1.7)$$

Analysis of the transformations required and eigenvectors shall be done in Section 2.

On the other hand, for $\mathcal{L}y(t) = wy(t)$, we require the following quasidifferences [5]: Let

$$\begin{aligned} x_1(t) &= y(t-1) \\ x_2(t) &= \Delta y(t-2) \\ \tilde{u}_1(t) &= (p_1 - zq_1)\Delta y(t-1) - \Delta(p_2\Delta^2 y(t-2)) \\ \tilde{u}_2(t) &= p_2\Delta^2 y(t-2). \end{aligned}$$

Then define the vector valued functions $x(t)$, $\tilde{u}(t)$ and $Y(t)$ by

$$\begin{aligned} x(t) &= (x_1(t), x_2(t)) \\ \tilde{u}(t) &= (\tilde{u}_1(t), \tilde{u}_2(t)) \\ Y(t) &= (x(t), \tilde{u}(t))^{tr} \end{aligned}$$

and the partial shift operator $R(Y)(t)$ by $(RY)(t) = (x(t+1), \tilde{u}(t))^{tr}$, the superscript tr denoting the transpose. Let $\ell^2(\mathbb{N})$ be as defined earlier. One therefore defines, the maximal and minimal difference operators L^* and L respectively generated by $\mathcal{L}$. The maximal difference operator L^* generated by $\mathcal{L}$ on $\ell^2(\mathbb{N})$ where

$$\mathcal{L}y(t) = (\mathcal{L}_1 - z\mathcal{L}_2)y(t) = wy(t) \quad (1.8)$$

is defined via its domain

$$D(L^*) = \{y(t) \in \ell^2(\mathbb{N}) : \forall f(t) \in \ell^2(\mathbb{N}) \text{ such that } \mathcal{J}\Delta Y(t) - P(t)(RY)(t) = F(t), t \in \mathbb{N}\}$$

and $L^*y(t) = f(t)$, if $\mathcal{J}\Delta Y(t) - P(t)(RY)(t) = F(t)$ where $F(t) = \begin{bmatrix} f(t), & 0, & 0, & 0 \end{bmatrix}^{tr}$. One defines the pre-minimal operator generated by (1.8) by

$$D(L') = \{y(t) \in D(L^*) : \text{there exists } n \in \mathbb{N} \text{ such that } y(0) = y(t) = 0, \text{ for all } t \geq n+1\},\ L^*y = L'y.$$

L' is symmetric and densely defined but not necessarily closed. The closure of the pre-minimal operator, L', is the minimal difference operator generated by (1.8), meaning that the minimal difference operator is a restriction of the maximal difference operator by further Dirichlet boundary conditions at 0. In this case, we shall denote this minimal difference operator by L. It therefore follows that L and L^* are symmetric, $L \subset L^*$ and $L = L^{**}$ as required. Equation (1.8) can therefore be written in its linear discrete form

$$\mathcal{J}\Delta Y(t) \;=\; [wI_4 + P(t)]R(Y)(t) \tag{1.9}$$

where $t \in \mathbb{N}$, I_2 and $P(t)$ are square symmetric matrices and $\mathcal{J}$ is a canonical symplectic matrix i.e

$$\mathcal{J} = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \text{ and } P(t) = \begin{pmatrix} -C & A^* \\ A & B \end{pmatrix}.$$

The non-zero elements of the 2×2 matrices A, B, C are given by $A_{j,j+1} = 1, B_{2,2} = p_2^{-1}$ and $c_{j,j} = p_{j-1} - zq_{j-1}$, $j = 1, 2$ and $p_0 - zq_0$ interpreted as $p_0 - zq_0 - w$. Thus the fourth order difference equation (1.8) has a first order form given by

$$\Delta \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \;=\; \begin{pmatrix} -C & A^* \\ A & B \end{pmatrix} \begin{pmatrix} x(t+1) \\ u(t) \end{pmatrix}$$

The direct formulation to convert the higher order system to first order system results to

$$\begin{pmatrix} x(t+1) \\ u(t+1) \end{pmatrix} \;=\; \begin{pmatrix} E & EB \\ CE & I - A^* + CEB \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$$

where $E = (I - A)^{-1}$. Since $Y(t) = (x(t), u(t))^{tr}$, we have that

$$Y(t+1, z, w) = S(t, z, w)Y(t, z, w) \tag{1.10}$$

with an explicit calculation of $S(t, z, w)$ giving

$$S(t, z, w) = \begin{pmatrix} 1 & 1 & 0 & p_2^{-1} \\ 0 & 1 & 0 & p_2^{-1} \\ (p_0 - zq_0 - w) & (p_0 - zq_0 - w) & 1 & (p_0 - zq_0 - w)p_2^{-1} \\ 0 & (p_1 - zq_1) & -1 & 1 + (p_1 - zq_1)p_2^{-1} \end{pmatrix}.$$

Computation of the associated characteristic polynomial gives

$$\mathcal{P}(t, \lambda, w) = det(S(t, z, w) - \lambda \cdot I_4),$$

which when multiplied by $\frac{p_2}{\lambda^2}$ one obtains

$$\mathcal{P}(t, \lambda, w) = p_2(1 - \lambda)^2(1 - \lambda^{-1})^2 + (p_1 - zq_1)(1 - \lambda)(1 - \lambda^{-1}) + (p_0 - zq_0 - w)$$

In order to solve for the values of λ in the polynomial, we let $(1 - \lambda)(1 - \lambda^{-1})$ to be ζ so that the polynomial becomes

$$\mathcal{F}(t, \lambda, w) = p_2\zeta^2 + (p_1 - zq_1)\zeta + (p_0 - zq_0 - w) \tag{1.11}$$

In this case after computing the eigenvalues of $S(t, z, w)$, the roots of (1.11), the eigenvectors $v_k, k = 1, 2, 3, 4$ are computed as follows: Let λ_k be the root of (1.11), then in the quasidufferences, we substitute $y(t+r)$ by λ^r and Δ^r by $(\lambda_k-1)^r$ so that

$$v_k = [\lambda_k^{-1}, (\lambda_k-1)\lambda_k^{-2}, (p_1-zq_1)(\lambda_k-1)\lambda_k^{-1}-(\lambda_k-1)p_2(\lambda_k-1)^2\lambda_k^{-2}, p_2(\lambda_k-1)^2\lambda_k^{-2}]^{tr}.$$

The matrix $T(t,z,w)=[v_1,v_2,v_3,v_4]$ is now used to diagonalise equation (1.10). A similar approach is used to obtain the transforming matrix $T_1(t,z,w)$ for equation (1.5). We make the transformation $\chi(t,z,w)=T(t,z,w)Y(t,z,w)$ which results into

$$\begin{aligned}\chi(t+1,z,w) &= T^{-1}(t+1,z,w)S(t,z,w)T(t,z,w)\chi(t,z,w)\\ &= [\Lambda(t,z,w)+\mathfrak{R}(t,z,w)]\chi(t,z,w)\end{aligned}$$

where,

$$\mathfrak{R}(t,z,w) = -T^{-1}(t+1,z,w)\Delta T(t,z,w)\Lambda(t,z,w) \tag{1.12}$$

and

$$\Lambda(t,z,w) = diag\,(\lambda_k(t,z,w)),\ k=1,...,4.$$

Here, $\mathfrak{R}(t,z,w)$ consists of terms in $\ell^2(\mathbb{N})$ and $\ell^1(\mathbb{N})$. The correction terms as a result of the first transformation are given by $\mathfrak{R}_{kk}(t,z,w)$, $k=1,2$. In (1.12), $\Delta T(t,z,w)=T(t+1,z,w)-T(t,z,w)$. The second diagonalisation is done by use of the eigenvectors of the matrix $[\Lambda(t,w)+\mathfrak{R}(t,z,w)]$. Using the results of Behncke [1], a matrix of the form $[I+B(t,z,w)]$ with

$$B_{kk}(t,z,w)=0, \qquad B_{kj}(t,z,w)=(\lambda_j-\lambda_k)^{-1}\mathfrak{R}_{kj}, \tag{1.13}$$

$$k\neq j, \qquad k,j=1,..,4, \qquad t\geq t_0.$$

will be required for the second diagonalisation. The second diagonalisation results into correction terms added to the diagonals given by $(\Lambda_2)_{kk}=diag((\mathfrak{R}B)_{kk})$. The second diagonalisation is thus done using the transformation

$$\mathcal{W}(t,z,w)=[I+B(t,z,w)]\chi(t,z,w) \tag{1.14}$$

and which results into a system of the form

$$\mathcal{W}(t+1,w) = \{[\Lambda(t,z,w)+\Lambda_2(t,w)]+[I+B(t+1,z,w)]^{-1}\mathfrak{R}(t,w)[I+ B(t,z,w)]\}\mathcal{W}(t,z,w) \tag{1.15}$$

where $(\Lambda_2)=diag(\mathfrak{R}B)_{kk}$.

A similar transformation can be carried out for the case of (1.5). One therefore obtains Levinson-Benzaid-Lutz (LBL) form of

$$\mathcal{W}(t+1,z,w)=[\Lambda(t,z,w)+R(t,z,w)]\mathcal{W}(t,z,w)$$

to which we apply Theorem 1.1 to obtain the solutions of the form

$$Y(t,w)=T(t,w)[I+B(t,w)][E(t,w)+\circ(1)]\prod_{t_0}^{t-1}\Lambda(t,w)$$

if we apply backward transformations. Note that the matrices $T(t,w)[I+B(t,w)]$ shall be bounded and the asymptotics of the eigensolutions shall depend on the matrix $\prod_{t_0}^{t-1}\Lambda(t,w)$. Explicitly, we have

$$y_k(t,z,w)=\rho_k(t,z,w)[e_k(t,z,w)+r_{kk}(t,z,w)]\prod_{t_0}^{t-1}\lambda_k(t,z,w)$$

where $\rho_k(t,z,w)$ is a bounded function from $T(t,z,w)[I+B(t,z,w)]$, $e_k(t,z,w)$ is a normalised vector while $r_{kk}(t,z,w)\in\ell^1(\mathbb{N})$ and tends to zero as $t\to\infty$. The remaining analyses are done in Section 2.
The deficiency indices, $defL$, is defined as the pair

$$defL=(dimN_{L^*-w},dimN_{L^*+w}),$$

where w is the spectral parameter. N_{L^*-w} and N_{L^*+w} are the null spaces of L^*-wI and L^*+wI respectively.

Our main results state that if the coefficient $p_2(t)$ is positive definite and unbounded while $p_1(t)$, $p_0(t)$, $q_1(t)$ and $q_0(t)$ are bounded, then the minimal operators generated by the difference maps $\mathcal{L}$ and $\mathcal{L}_1$ are positive, have similar deficiency indices and possess Krein-von Neumann positive self-adjoint operator extensions under similar growth and decay conditions. On the other hand, suppose all the coefficients of $\mathcal{L}_1$ and $\mathcal{L}_2$ are bounded except $q_1(t)$ which is unbounded negative definite function and $p_2(t)$ a decreasing positive definite function, then the minimal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}$ have Krein-von Neumann Self-adjoint operator extensions but the deficiency indices, spectral multiplicities and location of absolutely continuous spectra are not similar.

This paper is divided into three sections, namely: 1. Introduction, 2. Self-Adjoint Operator Extensions and 3. Power Coefficients.

2. Self-Adjoint Operator Extensions

In Lemma 2.1 below we have established conditions for positivity of the maps $\mathcal{L}_1$ and $\mathcal{L}$ while in Theorem 2.2 and Theorem 2.3 we have analysed the stability of existence of Krein-von Neumann self-adjoint operator extension for a bounded perturbation. Thus assume that

$$p_2(t)>0,\qquad p_0>zq_0,\qquad p_2(t)\nearrow\infty,\qquad p_0,p_1,q_1,q_0=o(p_2),\tag{2.1}$$

for every $t\in\mathbb{N}$.
Moreover, for the decay conditions, we shall assume that

$$\Delta g(t),\ \ p_2^{-1}\Delta(g(t))\in\ell^2(\mathbb{N}),\qquad g=p_0,p_1,q_1,q_0,\tag{2.2}$$
$$\Delta^2(g(t)),\ \ (p_2^{-1}\Delta(g(t)))^2,\ \ p_2^{-1}\Delta^2(g(t)),\ \ (\Delta g(t))^2\in\ell^1(\mathbb{N}).$$

We then introduce the following symbols: H shall be assumed to be the self-adjoint operator extension of the minimal operator generated by $\mathcal{L}$, $\sigma(H)$ the spectrum of H, $\sigma_{ac}(H)$ absolutely continuous spectrum of H and $\sigma_{ac}(H,k)$ absolutely continuous spectrum of H of multiplicity k.

Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator. Then we define T as positive operator if for all $y \in \mathcal{H}$, we have that $\langle Ty, y\rangle > 0$. As a result of this definition we have the following result:

Lemma 2.1 *Suppose that (2.1) is satisfied in addition to*

$$(p_2(t) - p_2(t+2)) + (p_2(t) - p_0(t)) > z(q_1(t) + q_0(t)). \tag{2.3}$$

Then the minimal difference operators L and L_1 are positive.

Proof. Let $Ly(t)$ be the difference expression generated by $\mathcal{L}$ acting on $y(t)$, where $y(t)$ is a summable sequence of real valued functions on $\ell^2(\mathbb{N})$. Then L is symmetric since

$$D(L) \subseteq D(L^*) \text{ and}$$

$$\begin{aligned} \langle Ly(t), y(t)\rangle &= \sum_{t=1}^{\infty} Ly(t) \cdot \overline{y(t)} \\ &= \sum_{t=1}^{\infty} y(t) \cdot (\overline{Ly(t)}) \\ &= \langle y(t), Ly(t)\rangle. \end{aligned}$$

By construction, we have that $L = L_1 - zL_2$ and for any $y(t) \in \ell^2(\mathbb{N})$, it suffices to show that $\langle Ly(t), y(t)\rangle > 0$ for all $t \in \mathbb{N}$ and $y(t) \in D(L)$. However, the expansion of the inner product leads to

$$\begin{aligned} \langle Ly(t), y(t)\rangle &= \langle (L_1 - zL_2)y(t), y(t)\rangle \\ &= \langle L_1 y(t), y(t)\rangle - \langle zL_2 y(t), y(t)\rangle \end{aligned}$$

and this inner product is positive if

$$\langle L_1 y(t), y(t)\rangle > \langle zL_2 y(t), y(t)\rangle.$$

Hence this condition implies that

$$\begin{gathered} \{\Delta^2[p_2(t)\Delta^2 y(t-2)] - \Delta[p_1(t)\Delta y(t-1)] + p_0(t)y(t)\}y(t) > \\ \{-\Delta[q_1(t)\Delta y(t-1) + q_0(t)y(t)]\}y(t). \end{gathered}$$

An explicit expansion of both sides of this inequality leads to

$$2p_2(t) - p_2(t+2) - p_0(t) > z(q_1(t) + q_0(t))$$

which we can rewrite as

$$(p_2(t) - p_2(t+2)) + (p_2(t) - p_0(t)) > z(q_1(t) + q_0(t)).$$

As $t \to \infty$, the summand $(p_2(t) - p_2(t+2))$ goes to zero. With the assumptions in (2.1) where $p_2(t)$ dominates $p_1(t)$, $p_0(t)$, $q_1(t)$ and $q_0(t)$, the inequality (2.3) is true for any $t \in \mathbb{N}$. Therefore, the summation $\Sigma_{t=t_0}^{\infty} Ly(t) \cdot \overline{y(t)} > 0$ implies that $\langle Ly(t), y(t)\rangle > 0$ as required. In a similar way one can show that L_1 is positive with these assumptions. To see this we have for any $y(t) \in \ell^2(\mathbb{N})$, $L_1 y(t)$ is positive if

$$\langle L_1 y(t), y(t)\rangle > 0. \tag{2.4}$$

An explicit expansion shows that the inequality (2.4) holds if

$$p_2(t) > p_0(t). \tag{2.5}$$

Given that $p_2(t)$ dominates $p_1(t)$ and $p_0(t)$, the inequality (2.5) is true for any $t \in \mathbb{N}$. Therefore, the summation $\Sigma_{t=t_0}^{\infty}(L_1 y(t) \cdot \overline{y(t)}) > 0$ and hence $\langle L_1 y(t), y(t)\rangle > 0$. □

Theorem 2.2 *Let L_1 be the minimal operator generated by $\mathcal{L}_1$ (1.1) on $\ell^2(\mathbb{N})$. Assume conditions (2.1) and (2.2) are satisfied.*

(i) *As $t \to \infty$ the operator L_1 is positive.*

(ii) *Moreover if $|\frac{g}{p_2}|^{\frac{1}{4}}$, $g = p_0, p_1$ is absolutely summable, then L_1 is in limit circle case, that is, $def L_1 = (4,4)$ and has positive self-adjoint operator extension (Krein-von Neumann extension) and spectrum of its self-adjoint operator extension is discrete. On the other hand if $|\frac{g}{p_2}|^{\frac{1}{4}}$ is not absolutely summable, then $def L_1 = (2,2)$ and $\sigma_{ac}(H) \subseteq [\overline{s}, \infty)$ where $\overline{s} = \limsup \frac{p_0(t)}{p_2(t)}$ of spectral multiplicity* 2.

Proof. (i.) This follows immediately from Lemma 2.1.

(ii.) We now solve $L_1 y(t) = wy(t)$ so as to determine the deficiency indices and spectral multiplicity. In this case w is our spectral parameter. We solve for the zeros of equation (1.7) up to the term $O(p_2^{-3})$ to obtain

$$\zeta_{\pm} \approx \pm i|\frac{p_0}{p_2}|^{\frac{1}{2}} - \frac{p_1}{p_2} \mp \frac{1}{8} i|\frac{p_0}{p_2}|^{\frac{1}{2}} \frac{(p_1)^2}{p_2(p_0)} \mp \frac{1}{128} i|\frac{p_0}{p_2}|^{\frac{1}{2}} \frac{(p_1)^4}{p_2^2(p_0)^2} + O(p_2^{-3})$$

with p_0 interpreted as $p_0 - w$. This leads to the following roots:

$$\begin{aligned}
\lambda_1 &\approx 1 + (\frac{1-i}{\sqrt{2}})|\frac{p_0 - w}{p_2}|^{\frac{1}{4}} - i|\frac{p_0 - w}{p_2}|^{\frac{1}{2}} + O(p_2^{-1}) \\
\lambda_2 &\approx 1 - (\frac{1-i}{\sqrt{2}})|\frac{p_0 - w}{p_2}|^{\frac{1}{4}} - i|\frac{p_0 - w}{p_2}|^{\frac{1}{2}} + O(p_2^{-1}) \\
\lambda_3 &\approx 1 + (\frac{1+i}{\sqrt{2}})|\frac{p_0 - w}{p_2}|^{\frac{1}{4}} + i|\frac{p_0 - w}{p_2}|^{\frac{1}{2}} + O(p_2^{-1}) \\
\lambda_4 &\approx 1 - (\frac{1+i}{\sqrt{2}})|\frac{p_0 - w}{p_2}|^{\frac{1}{4}} + i|\frac{p_0 - w}{p_2}|^{\frac{1}{2}} + O(p_2^{-1}).
\end{aligned}$$

The absolute value of λ_k tend to 1 as $t \to \infty$, in the limiting sense. The dichotomy conditions are established off the real axis by application of Levinson's-Benzaid Lutz theorem. Let $w = w_0 + i\eta$, $w_0, \eta \in \mathbb{R}$ and $\eta > 0$ and small. Then $Re\lambda_1 = O(1 + (\frac{\eta}{p_2})^{\frac{1}{4}})$ while $Re\lambda_2 = O(1 - (\frac{\eta}{p_2})^{\frac{1}{4}})$ and hence $|\lambda_1| > 1$ while $|\lambda_2| < 1$ as $t \to \infty$. Similarly, $|\lambda_3| > 1$ and $|\lambda_4| < 1$. This is the dichotomy condition required by Theorem 1.1. Two diagonalisations as outlined in Section 1 are enough to convert (1.5) into LBL-form because of (2.2). The solutions of $L_1 y(t) = wy(t)$ by application of LBL-theorem are of the form

$$y_k(t,w) = \rho_k(t,w)[e_k(t,w) + \circ(1)] \prod_{l=t_0}^{t-1} \lambda_k(l,w)$$

where $\rho_k(t,w)$ is bounded and obtained from $T_1(t,w)[I + B_1(t,w)]$. The asymptotics and summability of $y_k(t,w)$ thus depends on $\prod_{l=t_0}^{t-1} \lambda_k(l,w)$. We compute $\lim_{\eta\to 0}\langle y_k(t,w), y_k(t,w)\rangle \approx \prod_{l=t_0}^{t-1} |\lambda_k(l,w)|^2$ so that when $|\frac{g}{p_2}|^{\frac{1}{4}}$ is absolutely summable, then all the four solutions are square summable and $defL_1 = (4,4)$, limit circle case and hence the spectrum is discrete. However, if $|\frac{g}{p_2}|^{\frac{1}{4}}$ is not absolutely summable, then $y_1(t)$ and $y_3(t)$ lose their square summability while $y_2(t)$ and $y_4(t)$ stay square summable as $\eta \to 0^+$. The M-matrix can be shown to be finite and bounded using Euler's formula. To see this, we have

$$\begin{aligned} ImM(t,w) &= \lim_{\eta\to 0^+} \langle y_j(t,w), y_j(t,w)\rangle, j = 2,4 \\ &= \ln \prod_{l=t_0}^{t-1} |\lambda_j(l)|^2 \\ &\approx -2\eta \sum_{t_0}^{t-1} |p_0(l)|^{\frac{1}{4}} |p_2(l)|^{-\frac{1}{4}} \\ &\approx -2\eta \int_{t_0}^{t-1} |p_0(l)|^{\frac{1}{4}} |p_2(l)|^{-\frac{1}{4}} dl < \infty. \end{aligned}$$

Hence the spectral measure is continuous and the deficiency indices of L_1, $defL_1 = (2,2)$. This implies that L_1 possesses positive self-adjoint operator extension, H_1, by application of von-Neumann theorem defined using boundary conditions at 0 imposed by 2×2 matrices, α_1 and α_2 given by

$$D(H_1) = \{y(t) \in D(L_1^*) : (\alpha_1, \alpha_2)y(0) = 0\}, L_1^* y(t) = H_1 y(t)$$

and $L_1 \subset H_1 = H_1^* \subset L_1^*$. α_1 and α_2 are 2×2 matrices such that

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_2, \qquad \alpha_1\alpha_2^* - \alpha_2\alpha_1^* = 0 \qquad \text{and } (\alpha_1,\alpha_2)y(0) = 0.$$

The leading term associated with spectral parameter w is $|\frac{p_0 - w}{p_2}|^{\frac{1}{4}}$ such that as $p_2(t) \nearrow \infty$ and $p_2(t) > 0$, $\overline{s} < w < \infty$ where $\overline{s} = \limsup(\frac{p_0}{p_2})$. The absolutely continuous spectrum is a subset of $[\overline{s}, \infty)$ and of spectral multiplicity 2.

$\square$

Theorem 2.3 *Assume the conditions (2.1) and (2.2) are satisfied with $z \in \mathbb{R}$ finite*

(i) *As $t \to \infty$ the minimal operator generated by L on $\ell^2(\mathbb{N})$ is symmetric and positive.*

(ii) *Moreover if $|\frac{g}{p_2}|^{\frac{1}{4}}$, $g = p_0, q_0, p_1, q_1$, is absolutely summable, then L is in limit circle case, that is, $defL = (4,4)$ and has positive self-adjoint* *operator extension (Krein-von Neumann extension) and spectrum of its self-adjoint operator extension is discrete. On the other hand, if* $|\frac{g}{p_2}|^{\frac{1}{4}}$ *is not absolutely summable, then* $def\, L = (2,2)$ *and* $\sigma_{ac}(H) \subseteq [\overline{r}, \infty)$ *where* $\overline{r} = \limsup(\frac{p_0 - zq_0}{p_2})(t)$ *of spectral multiplicity 2.*

Proof. (i) The proof follows immediately from Lemma 2.1.

(ii) For deficiency indices and spectral results, we solve the equation $Ly(t) = wy(t)$ where w is the spectral parameter. This now requires the construction of the minimal and maximal difference operators generated by $\mathcal{L}$, eigenvalues and performance of asymptotic summation based on Levinson-Benzaid-Lutz theorem [2,4,5]. Application of quadratic formula and binomial expansion on (1.11) up to the term $O(p_2^{-3})$ lead to values of ζ that can be approximated by

$$\zeta_{\pm} \approx \pm i|\frac{p_0}{p_2}|^{\frac{1}{2}} - \frac{p_1}{p_2} \mp \frac{1}{8} i |\frac{p_0}{p_2}|^{\frac{1}{2}} \frac{(p_1)^2}{p_2(p_0)} \mp \frac{1}{128} i |\frac{p_0}{p_2}|^{\frac{1}{2}} \frac{(p_1)^4}{p_2^2(p_0)^2} + O(p_2^{-3})$$

where p_0 and p_1 are interpreted as $p_0 - zq_0 - w$ and $p_1 - zq_1$ respectively. A backward substitution for the values of ζ in terms of λ, that is,

$$\lambda^2 + (\zeta - 2)\lambda + 1 = 0 \tag{2.6}$$

leads to $\lambda-$values that can be written in terms of ζ given by

$$\lambda_{\pm} \approx 1 \pm (-\zeta)^{\frac{1}{2}} - \frac{\zeta}{2} \mp \frac{1}{8}(-\zeta)^{\frac{1}{2}}\zeta \mp \frac{1}{128}(-\zeta)^{\frac{1}{2}}\zeta^2 + O(-\zeta^{-3}). \tag{2.7}$$

Explicitly, one has

$$\begin{aligned}
\lambda_1 &\approx 1 + (\frac{1-i}{\sqrt{2}})|\frac{p_0 - zq_0 - w}{p_2}|^{\frac{1}{4}} - i|\frac{p_0 - zq_0 - w}{p_2}|^{\frac{1}{2}} + O(p_2^{-1}) \\
\lambda_2 &\approx 1 - (\frac{1-i}{\sqrt{2}})|\frac{p_0 - zq_0 - w}{p_2}|^{\frac{1}{4}} - i|\frac{p_0 - zq_0 - w}{p_2}|^{\frac{1}{2}} + O(p_2^{-1}) \\
\lambda_3 &\approx 1 + (\frac{1+i}{\sqrt{2}})|\frac{p_0 - zq_0 - w}{p_2}|^{\frac{1}{4}} + i|\frac{p_0 - zq_0 - w}{p_2}|^{\frac{1}{2}} + O(p_2^{-1}) \\
\lambda_4 &\approx 1 - (\frac{1+i}{\sqrt{2}})|\frac{p_0 - zq_0 - w}{p_2}|^{\frac{1}{4}} + i|\frac{p_0 - zq_0 - w}{p_2}|^{\frac{1}{2}} + O(p_2^{-1}).
\end{aligned}$$

By replacing in quasi-differences $y(t+k)$ by λ^k and Δ by $\lambda - 1$ and if we normalize the first component through multiplication by λ, we have the eigenvectors given as

$$v_k = \{1, (\lambda_k - 1)\lambda_k^{-1}, (p_1 - zq_1)(\lambda_k - 1) - p_2(\lambda_k - 1)^3\lambda_k^{-1}, p_2(\lambda_k - 1)^2\lambda_k^{-1}\}^{tr}.$$

Thus approximating eigenvectors using only the leading terms, we obtain eigenvectors of the form:

$$v_1 = [1, (\frac{1-i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}}, p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}}, -ip_2|\frac{p_0}{p_2}|^{\frac{1}{2}}]^{tr}$$
$$v_2 = [1, -(\frac{1-i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}}, -p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}}, -ip_2|\frac{p_0}{p_2}|^{\frac{1}{2}}]^{tr}$$
$$v_3 = [1, (\frac{1+i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}}, p_1(\frac{1+i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}}, ip_2|\frac{p_0}{p_2}|^{\frac{1}{2}}]^{tr}$$
$$v_4 = [1, -(\frac{1+i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}}, -p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}}, ip_2|\frac{p_0}{p_2}|^{\frac{1}{2}}]^{tr}.$$

Hence the matrix $T(t, \lambda, w)$ is given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ (\frac{1-i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}} & (\frac{-1+i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}} & (\frac{1+i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}} & (\frac{-1-i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}} \\ p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}} & -p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}} & p_1(\frac{1+i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}} & -p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0}{p_2}|^{\frac{1}{4}} \\ -ip_2|\frac{p_0}{p_2}|^{\frac{1}{2}} & -ip_2|\frac{p_0}{p_2}|^{\frac{1}{2}} & ip_2|\frac{p_0}{p_2}|^{\frac{1}{2}} & ip_2|\frac{p_0}{p_2}|^{\frac{1}{2}} \end{pmatrix}.$$

As $t \to \infty$, that is, in the limiting sense, the absolute value of λ_k are tending to 1. By application of Levinson's-Benzaid-Lutz theorem, the dichotomy conditions can only be established off the real axis. Now let $w = w_0 + i\eta$, $w_0, \eta \in \mathbb{R}$, $\eta > 0$ and small. Then $Re\lambda_1 = O(1 + (\frac{\eta}{p_2})^{\frac{1}{4}})$ while $Re\lambda_2 = O(1 - (\frac{\eta}{p_2})^{\frac{1}{4}})$ and hence $|\lambda_1| > 1$ while $|\lambda_2| < 1$ as $t \to \infty$. A similar analysis is true for λ_3 and λ_4. This is the required dichotomy condition by Theorem 1.1 (see [2]). The two diagolisations outlined in Section 1 are enough to convert (1.10) into LBL-form because of (2.2). An explicit computation shows that

$$\lim_{\eta \to 0} \langle y_k(t, z, w), y_k(t, z, w) \rangle \approx \prod_{l=t_0}^{t-1} |\lambda_k(l, w)|^2$$

so that when $|\frac{g}{p_2}|^{\frac{1}{4}}$ is absolutely summable, all the four solutions are square summable and $defL = (4, 4)$, hence limit circle case and the spectrum is discrete. On the other hand if $|\frac{g}{p_2}|^{\frac{1}{4}}$ is not absolutely summable, then $y_1(t)$ and $y_3(t)$ lose their square summability as $\eta \to 0^+$. We need to show that the M-matrix is finite and bounded. We compute the M-matrix using the eigenvectors associated with $\lambda_2(t, z, w)$ and $\lambda_4(t, z, w)$ whose eigensolutions are of the form :

$$y_k(t, z, w) = \rho_k(t, z, w)[e_k(t, z, w) + \circ(1)] \prod_{l=t_0}^{t-1} \lambda_k(l) \tag{2.8}$$

where $\rho_k(t, z, w)$ are the diagonal terms of $T(t, z, w)[I + B(t, z, w)]$ which are bounded. Let $w = w_0 + i\eta$, $\eta > 0$, small and $w_0 \in \mathbb{R}$, then from the relation

$$ImM(t, z, w) = \lim_{\eta \to 0^+} \langle F(t, z, w), F(t, z, w) \rangle,$$

it follows that $y_2(t)$ remains square summable and hence by Euler's formula:

$$\ln \prod_{l=t_0}^{t-1} |\lambda_2(l)|^2 \approx -2\eta \sum_{t_0}^{t-1} \frac{1}{|p_2(l)|^{\frac{1}{4}}} \approx -2\eta \int_{t_0}^{t-1} |p_2(l)|^{-\frac{1}{4}} dl.$$

The one for $y_4(t)$ follows in a similar way. Therefore, we have that $defL = (2,2)$ and hence L possesses self-adjoint operator extensions. Now fix two 2×2 matrices α_1 and α_2, that is, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^{2\times 2}$, with

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_2,\ \alpha_1\alpha_2^* - \alpha_2\alpha_1^* = 0_2$$

and set the boundary condition for the left regular end point a as

$$(\alpha_1, \alpha_2)y(a) = 0.$$

With $a = 0$, the self-adjoint extension, H of L is defined by

$$D(H) = \{y(t) \in D(L^*) : (\alpha_1, \alpha_2)y(0) = 0\}, L^*y(t) = Hy(t)$$

and $L \subset H = H^* \subset L^*$. The absolutely continuous spectrum is contributed by those eigenfunctions that lose their square summability. Note that the leading term associated with spectral parameter w is given by $|\frac{p_0 - zq_o - w}{p_2}|^{\frac{1}{4}}$ so that as $p_2(t) \nearrow \infty$ and $p_2(t) > 0$, $\overline{r} < w < \infty$ where $\overline{r} = \limsup(\frac{p_0 - zq_0}{p_2})$. The absolutely continuous spectrum is a subset of $[\overline{r}, \infty)$ and of spectral multiplicity 2.

□

Corollary 2.4 *Assume the conditions (2.1), (2.2) and (2.3) are satisfied, then the minimal operators generated by $\mathcal{L}y(t)$ and $\mathcal{L}_1y(t)$ are positive, have positive Krein-von Neumann positive self-adjoint operator extensions with the same location of absolutely continuous spectrum and spectral multiplicity of respective self-adjoint operator extensions equal.*

Proof. The proof follows immediately from Lemma 2.1 and Theorems 2.2 and 2.3. □

The example below has been used to verify the results obtained in Theorems 2.2 and 2.3.

Example 2.5 We consider the minimal and maximal difference operators generated by $\mathcal{L}_1y(t)$ and $\mathcal{L}_2y(t)$ in (1.1) and (1.2) given by

$$p_2(t) = t,\ p_1(t) = t^{-1},\ p_0(t) = t^{-2},\ q_1(t) = e^{-t}, q_0(t) = t^{-4},\ z < \infty.$$

This implies that $p_0, p_1, q_0, q_1 = \circ(p_2)$, $p_2(t) > 0$ and $|p_2(t)| \nearrow \infty$ as required by (2.1). We compute the range of z as required by (2.3)

$$\begin{aligned} \{(p_2(t) - p_2(t+2)) + (p_2(t) - p_0(t))\} &> z(q_1(t) + q_0(t)) \\ z &< \frac{\{(p_2(t) - p_2(t+2)) + (p_2(t) - p_0(t))\}}{(q_1(t) + q_0(t))} \\ &= \frac{(t - (t+2)) + (t - t^{-4})}{e^{-t} + t^{-4}} \end{aligned}$$

As $t \to \infty$, $e^{-t} \to 0$ faster than t^{-4}. Thus we have $z < \frac{t^4(t-2-t^{-4})}{(1+e^{-t}t^4)}$ so that as $t \to \infty$, then $z < \infty$. This implies that L and L_1 are positive operators. We now solve the roots of the associated $\mathcal{F}(t,z,w)$ polynomial. In the case of L_1 we have $t\zeta^2 + t^{-1}\zeta + t^{-2} - w = 0$. Hence by use of quadratic formula we obtain that $\zeta_\pm \approx \pm it^{-\frac{3}{2}} + O(t^{-2})$. The corresponding $\lambda-$roots are obtained from (2.7) and hence

$$\begin{aligned}
\lambda_1(t,z,w) &\approx 1 + (\frac{-1+i}{\sqrt{2}})(t^{-\frac{3}{4}} - w) + O(t^{-\frac{3}{2}}) \\
\lambda_2(t,z,w) &\approx 1 + (\frac{1-i}{\sqrt{2}})(t^{-\frac{3}{4}} - w) + O(t^{-\frac{3}{2}}) \\
\lambda_3(t,z,w) &\approx 1 + (\frac{1+i}{\sqrt{2}})(t^{-\frac{3}{4}} - w) + O(t^{-\frac{3}{2}}) \\
\lambda_4(t,z,w) &\approx 1 + (\frac{-1-i}{\sqrt{2}})(t^{-\frac{3}{4}} - w) + O(t^{-\frac{3}{2}}).
\end{aligned}$$

Here, $(t^{-\frac{3}{4}} - w)$ are bounded and hence summable. However, the value of w can shift the magnitude of λ_k to either less or greater than 1. A careful analysis shows that if $|\lambda_1| < 1$ then $|\lambda_2| > 1$ and vice versa. Similarly, if $|\lambda_3| < 1$ then $|\lambda_4| > 1$ and vice versa. Thus $def L_1 = (2,2)$, L_1 has positive self-adjoint operator extension with absolutely continuous spectrum contained in $(0,\infty)$ of spectral multiplicity 2. A similar analysis also reveals that the eigenvalues of L are similar to those of L_1 approximated to $O(t^{-\frac{3}{2}})$ and hence similar results in deficiency indices, existence of positive self-adjoint operator extension and spectral analysis are obtained.

In our subsequent analysis in this Section, we show that unlike in the case of deficiency indices and spectral results, the Krein-von Neumann self-adjoint operator extension is stable under unbounded perturbations. For that case we assume the following growth conditions on the coefficients:

$$\begin{aligned}
&z < 0, \qquad p_2(t) > 0, \qquad q_1(t) < 0, \qquad |q_1(t)| \nearrow \infty \qquad as \qquad t \to \infty, \\
&z \neq 0 \qquad and \qquad z \in \mathbb{R}.
\end{aligned} \tag{2.9}$$

$$p_0, p_1, p_2, q_0 = o(q_1), \qquad p_2(t) > p_0(t), \forall t \in \mathbb{N} \qquad and \qquad \Delta p_2(t) < 0, \forall t \in \mathbb{N}.$$

Note that by assumption $p_2(t)$ is a decreasing function. Since we are solving the equation $L = L_1 - zL_2$, the above growth conditions in (2.9) ensures that L_1 and L are positive symmetric operators with L_1 perturbed by unbounded operator $-zL_2$ because $|q_1(t)| \nearrow \infty$ as $t \to \infty$. For the decay conditions, we shall assume that

$$\begin{aligned}
\Delta g(t),\ \ q_1^{-1}\Delta(g(t)) \in \ell^2(\mathbb{N}), \qquad g = p_0, p_1, p_2, q_0, \\
\Delta^2(g(t)),\ \ (q_1^{-1}\Delta(g(t)))^2,\ \ q_1^{-1}\Delta^2(g(t)),\ \ (\Delta g(t))^2 \in \ell^1(\mathbb{N}).
\end{aligned} \tag{2.10}$$

With these conditions we obtain the following results:

Theorem 2.6 *Suppose that L_1 is the minimal difference operator generated by $\mathcal{L}_1$ on $\ell^2(\mathbb{N})$. Assume that the coefficients $p_k(t)$, $k = 0, 1, 2$ are bounded and almost constant with the condition (2.3) then at limiting point L has positive self-adjoint operator H with the absolutely continuous spectrum coinciding with that of constant coefficient operator with boundary condition α at 0 and spectral multiplicity equal to half the eigenvalues with unit magnitude.*

Proof. Positivity of the operator L follows from Lemma 2.1 while the other claims are immediate from the results of Behncke and Nyamwala [3].

□

Theorem 2.7 *Assume (2.3), (2.9) and (2.10) are satisfied with $z \in \mathbb{R}$ finite.*

(i) *As $t \to \infty$ the minimal operator generated by L on $\ell^2(\mathbb{N})$ is positive.*

(ii) *Moreover if $|\frac{g}{q_1}|^{\frac{1}{2}}$, $g = p_0, q_0, p_1, p_2$ is absolutely summable, then $def L = (3, 3)$ and has positive self-adjoint operator extension (Krein-von Neumann extension) and spectrum of its self-adjoint operator extension is discrete. On the other hand if $|\frac{g}{q_1}|^{\frac{1}{2}}$ is not absolutely summable, then $def L = (2, 2)$ and $\sigma_{ac}(\mathcal{H}) \subseteq [\overline{u}, \infty)$ where $\overline{u} = \limsup(\frac{p_0 - zq_0}{p_1 - zq_1})(t)$ of spectral multiplicity 1.*

Proof. (i.) A computation of the inner product $\langle Ly(t), y(t)\rangle > 0$ for any $y(t) \in \ell^2(\mathbb{N})$ leads to the condition $\langle Ly(t), y(t)\rangle > \langle zL_2y(t), y(t)\rangle$ and results into (2.3) which is assumed. But (2.3) is satisfied by conditions in (2.9). The rest of the proof then follow from Lemma 2.1.

(ii.) In order to compute the deficiency indices and hence establish spectral multiplicity, we solve the equation $Ly(t) = wy(t)$ where w is the spectral parameter. As before, we introduce quasidifferences as stated by Shi [5] and perform similar computations as before to obtain the characteristic equation

$$p_2(1-\lambda)^2(1-\lambda^{-1})^2 + (p_1 - zq_1)(1-\lambda)(1-\lambda^{-1}) + (p_0 - zq_0 - w) = 0.$$

Assume $\zeta = (1-\lambda)(1-\lambda^{-1})$ and we obtain a biquadratic equation which we solve and make backward substitution to $O(p_2^{-2})$ for simplicity with $w = w_0 + i\eta$, $w_0, \eta \in \mathbb{R}$ and $\eta > 0$. This results to the following roots:

$$\begin{aligned}
\lambda_{1/2} &\approx 1 \mp i(p_o - zq_o - w)^{\frac{1}{2}} z^{-\frac{1}{2}} q_1^{-\frac{1}{2}} + O(q_1^{-\frac{3}{2}}) \\
\lambda_3 &\approx -p_2 z^{-1} q_1^{-1} + z^{-2} q_1^{-2} p_1 p_2 + O(q_1^{-3}) \\
\lambda_4 &\approx -p_2^{-1} z q_1 + p_2^{-1} p_1 + O(1).
\end{aligned}$$

We note that $|\lambda_3| < 1$ and $|\lambda_4| > 1$. If $|\frac{g}{q_1}|^{\frac{1}{2}}$ is absolutely summable, then $def L = (3, 3)$ and the spectrum is discreet. As $t \to \infty$, in the limiting sense, the absolute value of $\lambda_{1/2}$ are tending to 1. The dichotomy conditions for the case of $\lambda_{1/2}$ are established off the real axis by application of Levinson's-Benzaid-Lutz theorem. Let $w = w_0 + i\eta$, $w_0, \eta \in \mathbb{R}$, $\eta > 0$ and small. Then $Re\lambda_1 = O(1 - (\frac{\eta}{q_1})^{\frac{1}{2}})$ while $Re\lambda_2 = O(1 + (\frac{\eta}{q_1})^{\frac{1}{2}})$. Hence,

$|\lambda_1| < 1$ while $|\lambda_2| > 1$ as $t \to \infty$. The solutions of $Ly(t) = wy(t)$ are of the form

$$y_k(t,z,w) = \rho_k(t,z,w)[e_k(t,z,w) + \circ(1)] \prod_{l=t_0}^{t-1} \lambda_k(l,z,w)$$

and as before,

$$\lim_{\eta\to 0}\langle y_k(t,z,w), y_k(t,z,w)\rangle \approx \prod_{l=t_0}^{t-1} |\lambda_k(l,z,w)|^2.$$

If $|\frac{g}{q_1}|^{\frac{1}{2}}$ is not absolutely summable, then $y_2(t,z,w)$ loses its square summability as $\eta \to 0^+$ and $defL = (2,2)$. This implies that L possesses self-adjoint operator extensions, which can be defined as before. By Euler's formula, the M-matrix is finite and bounded. The leading term associated with spectral parameter w is given by $|\frac{p_0 - zq_o - w}{p_2}|^{\frac{1}{2}}$ so that as $q_1(t) \nearrow \infty$, $q_1(t) > 0$, then $\overline{u} < w < \infty$ where $\overline{u} = \limsup(\frac{p_0 - zq_0}{p_1 - zq_1})$. Therefore the absolutely continuous spectrum is a subset of $[\overline{u}, \infty)$ of spectral multiplicity 1.

□

Hence unbounded perturbation leads to stability in the existence of Krein-von Neumann self-adjoint operator extension which is similar to the case of bounded perturbation. However, the deficiency indices, absolutely continuous spectrum and spectral multiplicities are not stable.

The following example reiterates the results of Theorem 2.6 and Theorem 2.7.
Example 2.8 Assume the following coefficients

$$z < 0, p_2(t) = t^{-1}, p_1(t) = e^{-t}, p_0(t) = t^{-2}, q_1(t) = \ln t, \text{ and } q_0(t) = c$$

for some constant c, $c > 0$, $t \in \mathbb{N}$ and $t \geq 2$. Then it follows that $p_k(t) > 0$ for all $t \in \mathbb{N}$ and hence L_1 is a positive map. Similarly, L is positive since by application of condition (2.3) we need that

$$z \leq (t^2 + 3t - 2)\{(\ln t + c)(t^3 + 2t^2)\}^{-1}$$

which tends to zero as $t \to \infty$. Because $z \neq 0$, since otherwise there will be no perturbation, the positivity of L implies that $z < 0$. This is the assumption made at the beginning. Thus (2.3) now implies that both L and L_1 are positive. We now proceed to perform the deficiency indices analysis to determine the stability of the existence of the positive self-adjoint operator extension. The analysis of the eigenvalues now lead to

$$\begin{aligned}
\lambda_1(t,w,z) &\approx 1 - i|c|^{\frac{1}{2}}(z\ln t)^{-\frac{1}{2}} + O((\ln t)^{-1}) \\
\lambda_2(t,w,z) &\approx 1 + i|c|^{\frac{1}{2}}(z\ln t)^{-\frac{1}{2}} + O((\ln t)^{-1}) \\
\lambda_3(t,w,z) &\approx -t^{-1}(z\ln t)^{-1} + O((\ln t)^{-2}) \\
\lambda_4(t,w,z) &\approx -tz\ln t + 2 + t^{-1}(\ln t)^{-1} + O((\ln t)^{-2}).
\end{aligned}$$

Suppose that $\eta|c|^{\frac{1}{2}}(z\ln t)^{-\frac{1}{2}}$ is summable, then $\lambda_1(t,w,z), \lambda_2(t,w,z)$ and $\lambda_3(t,w,z)$ lead to eigensolutions which are square summable and hence $defL = (3,3)$ with discrete spectrum at most. On the other hand, if $\eta|c|^{\frac{1}{2}}(z\ln t)^{-\frac{1}{2}}$ is non-summable, then off the real axis, i.e taking $w = w_0 + i\eta$, $\eta > 0$ and small and $w, \eta \in \mathbb{R}$, then only two eigensolutions associated with $\lambda_1(t,w,z)$ and $\lambda_3(t,w,z)$ will be square summable leading to $defL = (2,2)$. The operator L has self-adjoint operator extension with absolutely continuous spectrum of multiplicity 1 contained in $[0,\infty)$. A similar analysis of the deficiency indices of L_1 leads to the following eigenvalues:

$$\begin{aligned}
\lambda_1(t,w) &\approx 1 + (\frac{i-1}{\sqrt{2}})t^{-\frac{1}{4}} - it^{-\frac{1}{2}} \\
\lambda_2(t,w) &\approx 1 - (\frac{i-1}{\sqrt{2}})t^{-\frac{1}{4}} - it^{-\frac{1}{2}} \\
\lambda_3(t,w) &\approx 1 + (\frac{i-1}{\sqrt{2}})t^{-\frac{1}{4}} + it^{-\frac{1}{2}} \\
\lambda_4(t,w) &\approx 1 - (\frac{i-1}{\sqrt{2}})t^{-\frac{1}{4}} + it^{-\frac{1}{2}}.
\end{aligned}$$

This shows that if $O(\eta t^{-\frac{1}{4}})$ is summable, then $defL_1 = (4,4)$. However, if $O(\eta t^{-\frac{1}{4}})$ is non-summable, then $defL_1 = (2,2)$ and has absolutely continuous spectrum of multiplicity 2 contained in $[0,\infty)$. Hence L_1 has positive self-adjoint operator extension.

3. Power Coefficients

In this section, we have discussed two examples of maps $\mathcal{L}$ and $\mathcal{L}_1$ with power coefficients as reinforcement of results in Section 2. Let $f(t) = ct^{\varepsilon}$, $t \in \mathbb{N}$ and c a non-zero constant. Then $f(t)$ is known as a power coefficient. $f(t)$ is bounded if $\varepsilon \leq 0$ otherwise it is unbounded.

Example 3.1 *Let $p_2(t) = t^{\alpha_2}$, $p_1(t) = t^{\alpha_1}$, $p_0(t) = t^{\alpha_0}$, $q_1(t) = t^{\beta_1}$ and $q_0(t) = t^{\beta_0}$ such that the following conditions are satisfied:*

$$\alpha_1, \alpha_0, \beta_1, \beta_0 < 0 \qquad \textit{and} \qquad \alpha_2 > 0 \qquad \textit{so} \qquad \textit{that} \qquad t^{\alpha_2} \to \infty \tag{3.1}$$
$$\text{as} \qquad t \to \infty.$$

We obtain the equation

$$Ly(t) = \Delta^2[t^{\alpha_2}\Delta^2 y(t-2)] - \Delta[(t^{\alpha_1} - zt^{\beta_1})\Delta y(t-1)] + [t^{\alpha_0} - zt^{\beta_0}]y(t) \tag{3.2}$$

Solving the equation $\mathcal{L}y(t) = wy(t)$ leads to the characteristic equation

$$t^{\alpha_2}(1-\lambda)^2(1-\lambda^{-1})^2 + (t^{\alpha_1} - zt^{\beta_1})(1-\lambda)(1-\lambda^{-1}) + (t^{\alpha_0} - zt^{\beta_0} - w) = 0 \tag{3.3}$$

Thus approximating the roots of (3.3) to $O(t^{-\alpha_2})$ leads to the following roots:

$$\begin{aligned}
\lambda_1 &\approx 1 + (\frac{1-i}{\sqrt{2}})t^{-\frac{1}{4}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{4}} - it^{-\frac{1}{2}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{2}} + O(t^{-\alpha_2}) \\
\lambda_2 &\approx 1 - (\frac{1-i}{\sqrt{2}})t^{-\frac{1}{4}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{4}} - it^{-\frac{1}{2}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{2}} + O(t^{-\alpha_2}) \\
\lambda_3 &\approx 1 + (\frac{1+i}{\sqrt{2}})t^{-\frac{1}{4}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{4}} + it^{-\frac{1}{2}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{2}} + O(t^{-\alpha_2}) \\
\lambda_4 &\approx 1 - (\frac{1+i}{\sqrt{2}})t^{-\frac{1}{4}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{4}} + it^{-\frac{1}{2}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - wt|^{\frac{1}{2}} + O(t^{-\alpha_2}).
\end{aligned}$$

The magnitudes of $\lambda_1(t,z,w)$ and $\lambda_3(t,z,w)$ are greater than 1 while the magnitudes of $\lambda_2(t,z,w)$ and $\lambda_4(t,z,w)$ are less than 1 as $t \to \infty$. Therefore $def L = (2,2)$ and the spectrum is absolutely continuous since two eigensolutions stay $w-$uniformly square summable. Therefore the minimal difference operator generated by L has a positive self-adjoint operator extension under bounded perturbation.

Example 3.2 *We consider Example 3.1 with the growth conditions*

$$\alpha_2, \alpha_1, \alpha_0, \beta_0 < 0, \qquad \beta_1 > 0 \quad , \qquad q_1(t) = -t^{\beta_1}, \qquad z \neq 0, \qquad \textit{and} \quad z \in \mathbb{R}. \tag{3.4}$$

These conditions imply that $p_2, p_1, p_0, q_0 = o(q_1)$ as $t \to \infty$. Further, $p_2(t) > 0$ while $\Delta p_2(t) < 0$, for all $t \in \mathbb{N}$. Thus by the results of Theorem 2.7, $\mathcal{L} = \mathcal{L}_1 - z\mathcal{L}_2$ is a positive map and hence generates a positive operator. It remains to check the nature of the roots associated with the difference operator $\mathcal{L} = \mathcal{L}_1 - z\mathcal{L}_2$ under these new conditions. In line with equation (3.3) we obtain a modified polynomial of the form

$$t^{\alpha_2}(1-\lambda)^2(1-\lambda^{-1})^2 + (t^{\alpha_1} + zt^{\beta_1})(1-\lambda)(1-\lambda^{-1}) + (t^{\alpha_0} - zt^{\beta_0} - w) = 0. \tag{3.5}$$

This yields the roots

$$\begin{aligned}
\lambda_{1/2}(t,z,w) &\approx 1 \mp (t^{\alpha_o} - zt^{\beta_o} - w)^{\frac{1}{2}} z^{-\frac{1}{2}} t^{-\frac{1}{2}\beta_1} + O(t^{-\frac{3}{2}\beta_1}) \\
\lambda_3(t,z) &\approx z^{-1}t^{\alpha_2-\beta_1} - z^{-2}t^{\alpha_2+\alpha_1-s\beta_1} + O(t^{-3\beta_1}) \\
\lambda_4(t,z) &\approx zt^{\beta_1-\alpha_2} + t^{\alpha_1-\alpha_2} + O(1).
\end{aligned}$$

The absolute value of λ_3 is less than one while that of λ_4 is greater than one. The analysis for λ_1 and λ_2 is done off the real axis using spectral parameter $w = w_o + i\eta$, $w_o, \eta \in \mathbb{R}$, $w_o > 0$ and $\eta > 0$ is small. Thus

$$\begin{aligned}
\lambda_1 &\approx 1 + \frac{1}{2}\eta|w_o|^{-\frac{1}{2}} t^{-\frac{1}{2}\beta_1} - i|w_o|^{\frac{1}{2}} t^{-\frac{1}{2}\beta_1} \\
\lambda_2 &\approx 1 - \frac{1}{2}\eta|w_o|^{-\frac{1}{2}} t^{-\frac{1}{2}\beta_1} + i|w_o|^{\frac{1}{2}} t^{-\frac{1}{2}\beta_1}
\end{aligned}$$

Computation of the absolute values of λ_1 and λ_2 now shows that $|\lambda_1| > 1$ while $|\lambda_2| < 1$ as $t \to \infty$, meaning that the solution $y_2(t,z,w) \approx c_2 \prod_{t=t_o}^{t-1} \lambda_2(t,z,w)$, where $c_2 = 1 \pm \eta|w_o|^{-\frac{1}{2}} t^{-\frac{1}{2}\beta_1}$, is absolutely summable. This implies that L_1 and L have similar deficiency indices, that is $(2,2)$ and therefore there exists positive self-adjoint operator extensions under these unbounded perturbation. The absolutely continuous spectrum is $(0,\infty)$ of spectral multiplicity one.

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